Nonlinear Systems Analysis

Lecture 2

Appendix A

Mathematical Preliminary

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Outline

- Space
- Vector & Matrix Norms
- Sequence
- Set
- Continuous Functions
- Differentiable Functions

Space

- n-dimensional Euclidean Space, R^n:
  - x_i is a real number

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$x + y = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

$$ax = \begin{bmatrix} ax_1 \\ ax_2 \\ \vdots \\ ax_n \end{bmatrix}$$

$$x^T y = \sum_{i=1}^{n} x_i y_i$$

Vector Norm - 1

- Vector Norm:
  - x is a vector
  - Norm $||x||$ is a real-valued function:
  - $||x|| \geq 0$ for all $x \in R^n$, with $||x|| = 0$ iff $x = 0$.
  - $||x + y|| \leq ||x|| + ||y||$, for all $x, y \in R^n$.
  - $||\alpha x|| = |\alpha| \; ||x||$, for all $\alpha \in R$ and $x \in R^n$.

- p-norm:

$$||x||_p = (|x_1|^p + \cdots + |x_n|^p)^{(1/p)}$$
$$1 \leq p < \infty$$
$$||x||_\infty = \max_i |x_i|$$

- 1-, 2-, ∞-norms are the three commonly used norms.
  2-norm is also called the Euclidean norm.

Vector Norm - 2

- All p-norms are equivalent in the sense that

$$c_1||x||_\alpha \le ||x||_\beta \le c_2||x||_\alpha$$

  $||\cdot||_\alpha$ and $||\cdot||_\beta$ are two different p-norms and c_1 and c_2 are positive constants.

- For 1-, 2-, ∞-norms, the inequalities take the form:

$$||x||_2 \le ||x||_1 \le \sqrt{n}||x||_2$$

$$||x||_\infty \le ||x||_2 \le \sqrt{n}||x||_\infty$$

$$||x||_\infty \le ||x||_1 \le n||x||_\infty$$

- The Hölder inequality:

$$|x^T y| \le ||x||_p \, ||y||_p, \quad \frac{1}{p} + \frac{1}{q} = 1$$

  for all $x \in R^n, y \in R^n$.

Matrix Norm - 1

- An $m \times n$ matrix A of real elements defines a linear mapping $y = Ax$ from R^n into R^m.

- The induced p-norm of A is defined by:

$$||A||_p = \sup_{x \ne 0} \frac{||Ax||_p}{||x||_p} = \max_{||x||_p = 1} ||Ax||_p$$

- For $p = 1, 2, \infty$:

$$||A||_1 = \max_j \sum_{i=1}^{m} |a_{ij}|$$

$$||A||_2 = \left[\lambda_{\max}(A^T A)\right]^{1/2}$$

$$||A||_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$$

  where $\lambda_{\max}(A^T A)$ is the maximum eigenvalue of $A^T A$.

Matrix Norm - 2

- Some useful properties

$$\frac{1}{\sqrt{n}}||A||_\infty \le ||A||_2 \le \sqrt{m}||A||_\infty$$

$$\frac{1}{\sqrt{m}}||A||_1 \le ||A||_2 \le \sqrt{n}||A||_1$$

$$||A||_2 \le \sqrt{||A||_1\ ||A||_\infty}$$

$$||AB||_p \le ||A||_p\ ||B||_p$$

Sequence Convergence - 1

- A sequence of vectors $x_0, x_1, ..., x_k, ...$ in R^n, denoted by $\{x_k\}$, is said to converge to a limit vector x if $||x_k - x|| \to 0$ as $k \to \infty$.

- Or, given any $\epsilon > 0$, there is an integer N such that $||x_k - x|| < \epsilon, \forall k \ge N$

- A vector x is an accumulation point of a sequence $\{x_k\}$ if there is a subsequence of $\{x_k\}$ that converges to x

- A bounded sequence $\{x_k\}$ in R^n has at least one accumulation point in R^n.

Sequence Convergence - 2

- An increasing (monotonically increasing or nondecreasing) sequence of real numbers $\{r_k\}$:
  if $r_k \leq r_{k+1}$
- A strictly increasing sequence $\{r_k\}$:
  if $r_k < r_{k+1}$
- Decreasing (monotonically decreasing or non-increasing) & strictly decreasing
- An increasing sequence of real numbers that is bounded from above converges to a real number.
- An decreasing sequence of real numbers that is bounded from below converges to a real number.

Set - 1

- A set $S \subset R^n$ is said to be open,
  if, for every vector $x \in S$,
  one can find an ϵ-neighborhood of x
  $N(x, \epsilon) = \{z \in R^n \mid ||z - x|| < \epsilon\}$
  such that $N(x, \epsilon) \subset S$.
- A set S is closed
  iff its complement in R^n is open.
- A set S is bounded
  if there is $r > 0$
  such that $||x|| \leq r$ for all $x \in S$.
- A set S is compact
  if it is closed and bounded.
- A point p is a boundary point of a set S
  if every neighborhood of p contains at least one point of S and one point not belonging to S.

Set - 2

- The set of all boundary points of S, denoted by ∂S, is called the boundary of S.
- A closed set contains all its boundary points.
- The interior of a set S is $S - \partial S$.
- An open set is equal to its interior.
- The closure of a set S, denoted by $\bar{S}$, is the union of S and its boundary.
- A closed set is equal to its closure.
- A open set S is connected if every pair of points in S can be joined by an arc lying in S.

Set - 3

- A set S is called region if it is the union of an open connected set with some, none, or all of its boundary points.
- If none of the boundary points are included, the region is called an open region or domain.
- A set S is convex if for every $x, y \in S$ and every real number θ, $0 < \theta < 1$, the point $\theta x + (1 - \theta) y \in S$.
- If $x \in X \subset R^n$ and $y \in Y \subset R^m$, we say that (x, y) belongs to the product set $X \times Y \subset R^n \times R^m$.

Continuous Functions - 1

- $f : S_1 \to S_2$ is that a function f maps a set S_1 into a set S_2.
- A function $f : R^n \to R^m$ is said to be continuous at a point x if $f(x_k) \to f(x)$ whenever $x_k \to x$.
- Equivalently, f is continuous at a x if, given $\epsilon > 0$, there is $\delta > 0$ such that $||x - y|| < \delta \Rightarrow ||f(x) - f(y)|| < \epsilon$.
- A function f is continuous on a set S if it is continuous at every point of S.
- f is uniformly continuous on S if, given $\epsilon > 0$, there is $\delta > 0$ (dependent only on ϵ) such that the inequality holds for all $x, y \in S$. That is, the same constant δ works for all points in the set.

Continuous Functions - 2

- If f is uniformly continuous on a set S, then it is continuous on S. The opposite statement is not true in general.
- However, if S is a compact set, then continuity and uniform continuity on S are equivalent.
- The function $(a_1 f_1 + a_2 f_2)(\cdot) = a_1 f_1(\cdot) + a_2 f_2(\cdot)$ is continuous for any two scalars a_1 and a_2 and any two continuous functions f_1 and f_2.
- If S_1, S_2, S_3 are any sets and $f_1 : S_1 \to S_2$ and $f_2 : S_2 \to S_3$ are functions, then the function $f_2 \circ f_1 : S_1 \to S_3$ is called the composition of f_1 and f_2 and defined by $(f_2 \circ f_1)(\cdot) = f_2(f_1(\cdot))$.

Continuous Functions - 3

- The composition of two continuous functions is continuous.

- If $S \subset R^n$ and $f : S \to R^m$, then the set of $f(x), x \in S$, is called the image of S under f and is denoted by $f(S)$.

- If f is a continuous function defined on a compact set S, then $f(S)$ is compact; hence, continuous functions on compact sets are bounded.

- If f is real valued, that is, $f : S \to R$, then there are points p and q in the compact set S such that $f(x) \leq f(p)$ and $f(x) \geq f(q)$, $\forall x \in S$.

Continuous Functions - 4

- If f is a continuous function defined on a connected set S, then $f(S)$ is connected.

- A function f defined on a set S is said to be one to one on S if whenever $x, y \in S$, and $x \neq y$, then $f(x) \neq f(y)$.

- If $f : S \to R^m$ is a continuous, one-to-one function on a compact set $S \subset R^n$, then f has a continuous inverse f^{-1} on $f(S)$.

- The composition of f and f^{-1} is identity, that is, $f^{-1}(f(x)) = x$.

Continuous Functions - 5

- A function $f : R \to R^n$ is said to be piecewise continuous on an interval $J \subset R$ if for every bounded subinterval $J_0 \subset J$, f is continuous for all $x \in J_0$, except possibly at a finite number of points where f may have discontinuities.

- At each point of discontinuity x_0, the right-side limit $\lim_{h \to 0} f(x_0 + h)$ and the left-side limit $\lim_{h \to 0} f(x_0 - h)$ exist; that is, the function has a finite jump at x_0.

Differentiable Functions - 1

- A function $f : R \to R$ is said to be differentiable at x if the limit $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exists.

- The limit $f'(x)$ is called the derivative of f at x.

- A function $f : R^n \to R^m$ is said to be continuously differentiable at x_0 if the partial derivatives $\partial f_i / \partial x_j$ exist and are continuous at x_0 for $1 \leq i \leq m, 1 \leq j \leq n$.

- A function f is continuously differentiable on a set S if it is continuously differentiable at every point of S.

Differentiable Functions - 2

- For a continuous differentiable function
  $f : R^n \to R$,
  define $\frac{\partial f}{\partial x} = \left[\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right]$.

- And, the gradient vector as $\nabla f(x) = \left[\frac{\partial f}{\partial x}\right]^T$.

- For a continuously differentiable function
  $f : R^n \to R^m$,
  the Jacobian matrix $\left[\frac{\partial f}{\partial x}\right]$ is $\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \frac{\partial f_i}{\partial x_j} & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$

Differentiable Functions - 3

- Suppose $S \subset R^n$ is open,
  f maps S into R^m,
  f is continuously differentiable at $x_0 \in S$,
  g maps an open set containing $f(S)$ into R^k, and
  g is continuously differentiable at $f(x_0)$

- Then the mapping of h of S into R^k,
  defined by $h(x) = g(f(x))$,
  is continuously differentiable at x_0 and
  its Jacobian matrix is given by
  the chain rule $\left.\frac{\partial h}{\partial x}\right|_{x=x_0} = \left.\frac{\partial g}{\partial f}\right|_{f=f(x_0)} \left.\frac{\partial f}{\partial x}\right|_{x=x_0}$