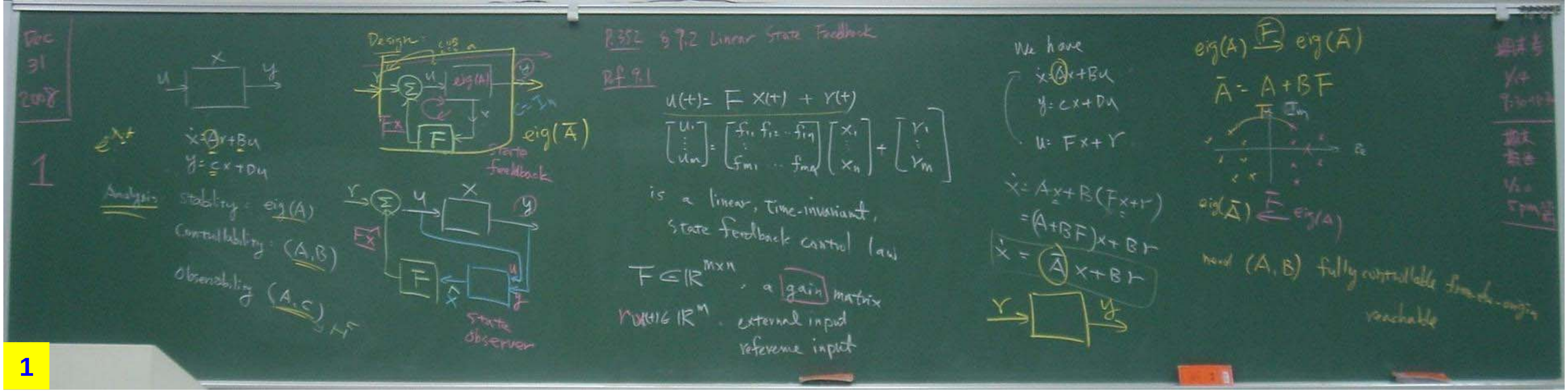
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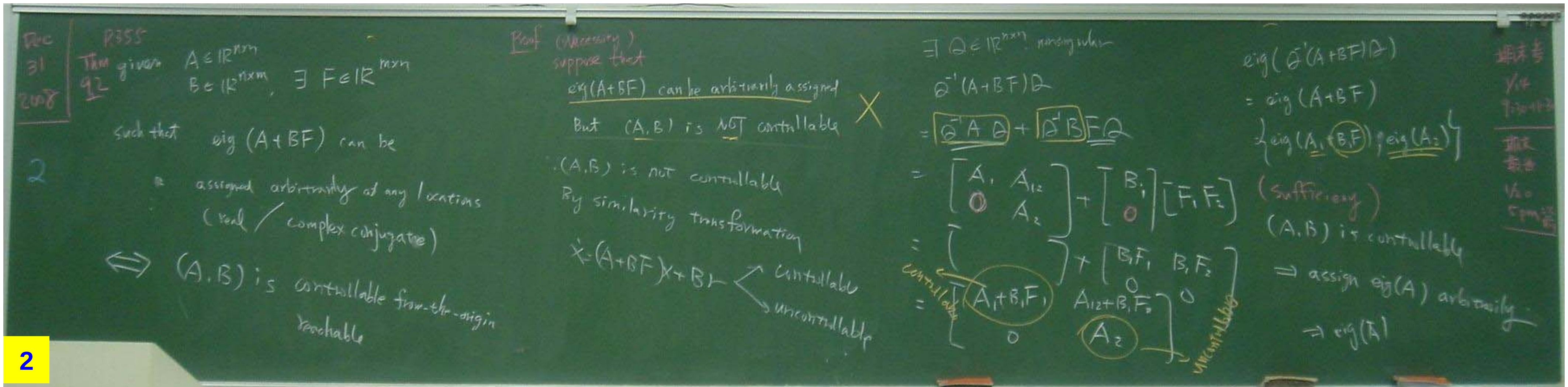
1
Design
Analysis
stability : eig(A)
Controllability : (A,B)
Observability (A,C)
state feedback
state observer
P.352 §9.2 Linear State Feedback
Df 9.1
u(t) = F x(t) + r(t)
is a linear, time-invariant, state feedback control law
F ∈ IR^{m×n}, a gain matrix
r(t) ∈ IR^m . external input
reference input
We have
ẋ = Ax + Bu
y = Cx + Du
u = Fx + r
ẋ = Ax + B(Fx + r)
= (A+BF)x + Br
ẋ = Ā x + Br
eig(A) → eig(Ā)
Ā = A + BF
eig(Ā) ← eig(A)
need (A,B) fully controllable from-the-origin
reachable
1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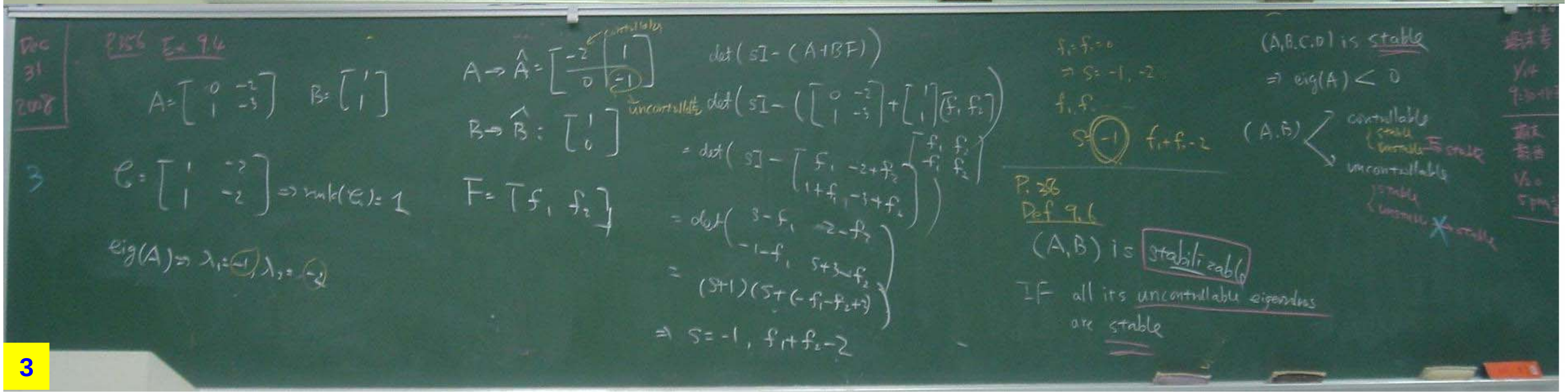
P.355
Thm 9.2
given A ∈ IR^{n×n}
B ∈ IR^{n×m}, ∃ F ∈ IR^{m×n}
such that eig(A+BF) can be
assigned arbitrarily at any locations
(real / complex conjugate)
⟺ (A,B) is controllable from-the-origin
reachable
Pf (Necessity)
suppose that
eig(A+BF) can be arbitrarily assigned
But (A,B) is NOT controllable
(A,B) is not controllable
By similarity transformation
ẋ = (A+BF)x + Br
controllable
uncontrollable
∃ Q ∈ IR^{n×n} nonsingular
Q^{-1}(A+BF)Q
eig(Q^{-1}(A+BF)Q)
= eig(A+BF)
(Sufficiency)
(A,B) is controllable
⇒ assign eig(A) arbitrarily
⇒ eig(Ā)
2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P.356 Ex 9.4
A = [0 -2; 1 -3]   B = [1; 1]
C = [1 -2; 1 -2] ⇒ rank(C) = 1
eig(A) ⇒ λ1 = -1, λ2 = -2
controllable
uncontrollable
F = [f1 f2]
det(sI − (A+BF))
⇒ s = -1, f1+f2-2
f1 = f2 = 0
⇒ s = -1, -2
(A,B,C,D) is stable
⇒ eig(A) < 0
P. 356
Def 9.6
(A,B) is stabilizable
IF all its uncontrollable eigenvalues are stable
3

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P.356 Controllability & Observability after state feedback

$\dot{x} = Ax + Bu$

$y = Cx + Du$

$u = Fx + r$

$\Rightarrow \begin{cases} \dot{x} = (A+BF)x + Br \\ y = (C+DF)x + Dr \end{cases}$

stability, observability, controllability rank

$$\begin{bmatrix} sI-A & B \\ -C & D \end{bmatrix}$$

$$\xrightarrow{F} \begin{bmatrix} sI-(A+BF) & B \\ -(C+DF) & D \end{bmatrix} = \begin{bmatrix} sI-A & B \\ -C & D \end{bmatrix} \begin{bmatrix} I & 0 \\ -F & I \end{bmatrix}$$

full rank

$\mathrm{rank}(sI-A \ \ B) = \mathrm{rank}(sI-(A+BF) \ \ B)$

$\mathrm{rank}\begin{pmatrix} sI-A \\ C \end{pmatrix} = \mathrm{rank}\begin{pmatrix} sI-(A+BF) \\ C+DF \end{pmatrix}$

$$\mathcal{C}_F = [B \ \ (A+BF)B \ \ (A+BF)^2B \ \cdots]$$

$$= \underbrace{[B \ \ AB \ \ A^2B \ \cdots]}_{\mathcal{C}} \begin{bmatrix} I & FB & F(A+BF)B \\ 0 & I & FB \\ 0 & 0 & I & \ddots \\ \vdots & \vdots & & & I \end{bmatrix}$$

$(A, C) \to ((A+BF), (C+DF))$

observable $\xrightarrow{F}$ observable / unobservable

unobservable $\xrightarrow{F}$ observable / unobservable

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$\mathrm{rank}(B) = m$

full rank or m linearly indep inputs

P.358 1 Direct Method

$$F = [f_{ij}] \in \mathbb{R}^{m \times n}$$

$$\det(sI-(A+BF)) = s^n + g_{n-1}(f_{ij})s^{n-1} + \cdots + g_1(f_{ij})s + g_0(f_{ij})$$

$\lambda_1^d, \lambda_2^d, \ldots, \lambda_n^d$

$$\Rightarrow (s-\lambda_1^d)(s-\lambda_2^d)\cdots(s-\lambda_n^d) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0$$

$$\Rightarrow g_k(f_{ij}) = \alpha_k^d, \quad k = 0, 1, \ldots, n-1$$

$m = 1$ $F = [f_1 \ f_2 \ \cdots \ f_n]$

$$\Rightarrow g_k(f_i) = \alpha_k, \quad k = 0 \ldots n-1$$

P.358 Ex 9.7

$A = \begin{bmatrix} \frac{1}{2} & 1 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\det(sI-A) = \cdots = s(s-\frac{5}{2})$

$\lambda^d_{1,2} = -1 \pm j$

$\alpha_d(s) = (s-(-1+j))(s-(-1-j)) = s^2 + 2s + 2$

$F = [f_1 \ f_2]$

$$\det(sI-(A+BF)) = \det\left(sI - \left(\begin{bmatrix} \frac{1}{2} & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}[f_1 \ f_2]\right)\right)$$

$$= \cdots = s^2 + s(-\tfrac{5}{2} - f_1 - f_2) + (f_1 - \tfrac{1}{2}f_2)$$

$$\Rightarrow \begin{cases} 2 = (-\frac{5}{2} - f_1 - f_2) \\ 2 = (f_1 - \frac{1}{2}f_2) \end{cases}$$

$$\Rightarrow \begin{cases} f_1 = -\frac{1}{6} \\ f_2 = -\frac{13}{3} \end{cases} \quad F = \left[-\tfrac{1}{6} \ \ -\tfrac{13}{3}\right]$$

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$\to \mathrm{eig}(A) = 0, \frac{5}{2}$

$x(t) = \bigcirc e^{0t} + \bigcirc e^{\frac{5}{2}t}$

negative control

$\bar{A} = A + BF$

$\mathrm{eig}(\bar{A}) = -1 \pm j$

$x(t) = \bigcirc e^{(-1+j)t} + \bigcirc e^{(-1-j)t}$

2 The Use of Controller Form

$$(A, B) \xrightarrow{P} (A_c, B_c)$$

controllable — Controller Form

$A_c = PAP^{-1}$

$B_c = PB$

$$\downarrow F$$

$$A+BF \xleftrightarrow{P^{-1}} A_c \xrightarrow{F_c} A_c + B_cF_c \to \alpha_d(s)$$

$$P(A+BF)P^{-1} = PAP^{-1} + PBFP^{-1} = A_c + B_c F_c$$

$m = 1$ (single input)

$F_c = [f_0 \ f_1 \ \cdots \ f_{n-1}]$

$$A_c = \begin{bmatrix} 0 & & I_{n-1} & \\ \vdots & & & \\ 0 & & & \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n-1} \end{bmatrix}$$

$$B_c = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$A_c + B_cF_c = \begin{bmatrix} & I_{n-1} \\ -\alpha_0 & -\alpha_1 \cdots -\alpha_{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}[f_0 \ f_1 \ \cdots \ f_{n-1}]$$

$$\begin{bmatrix} 0 \\ f_0 \ f_1 \ \cdots \ f_{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & I_{n-1} \\ -\alpha_0 + f_0 & -\alpha_1 + f_1 \ \cdots \ -\alpha_{n-1} + f_{n-1} \end{bmatrix}$$

$$\Rightarrow \alpha_F(s) = s^n + (\alpha_{n-1} - f_{n-1})s^{n-1} + (\alpha_{n-2} - f_{n-2})s^{n-2} + \cdots + (\alpha_0 - f_0)$$

$$\alpha_d(s) = s^n + \alpha^d_{n-1}s^{n-1} + \cdots + \alpha^d_0$$

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$d_i = \alpha_i - f_i$

$\Rightarrow f_i = \alpha_i - d_i$

$F_c = \begin{bmatrix} \alpha_0 - d_0 & \alpha_1 - d_1 & \cdots & \alpha_{n-1} - d_{n-1} \end{bmatrix}$

$F = F_c P$

P.360, Ex 9.8

$A = \begin{bmatrix} \frac{1}{2} & 1 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2} \\ 1 & 3 \end{bmatrix}$

$\mathcal{C}^{-1} = \frac{2}{3}\begin{bmatrix} 3 & -\frac{3}{2} \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -\frac{2}{3} & \frac{2}{3} \end{bmatrix} \rightarrow q$

$P = \begin{bmatrix} q \\ qA \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$

$P^{-1} = \cdots = \begin{bmatrix} -1 & 1 \\ \frac{1}{2} & 1 \end{bmatrix}$

$A_c = PAP^{-1} = \begin{bmatrix} 0 & 1 \\ 0 & \frac{5}{2} \end{bmatrix} \rightarrow s(s - \frac{5}{2}) = s^2 - \frac{5}{2}s + 0$

$B_c = PB = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\alpha_d(s) = s^2 + 2s + 2$

$F_c = \begin{bmatrix} f_0 & f_1 \end{bmatrix}$

$= \begin{bmatrix} \alpha_0 - d_0 & \alpha_1 - d_1 \end{bmatrix}$

$= \begin{bmatrix} 0 - 2 & -\frac{5}{2} - 2 \end{bmatrix}$

$= \begin{bmatrix} -2, & -\frac{9}{2} \end{bmatrix}$

$F = F_c P$

$= \begin{bmatrix} -2 & -\frac{9}{2} \end{bmatrix}\begin{bmatrix} -\frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$

$= \begin{bmatrix} -\frac{1}{6} & -\frac{13}{3} \end{bmatrix}$

$\Rightarrow F = -e_n^T \mathcal{C}^{-1} \alpha_d(A)$

$= -\begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & \frac{3}{2} \\ 1 & 3 \end{bmatrix}^{-1} \left(\begin{bmatrix} \frac{1}{2} & 1 \\ 1 & 2 \end{bmatrix}^2 + 2\begin{bmatrix} \frac{1}{2} & 1 \\ 1 & 2 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$

$= \cdots = \begin{bmatrix} -\frac{1}{6} & -\frac{13}{3} \end{bmatrix}$

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P.361 $m > 1$ (multiple inputs)

$A_{cF} = A_c + B_c F_c$

$n = 10$

$m = 3$

I_3 $\quad$ I_3 $\quad$ I_2 $\quad$ + $\quad$ $\begin{bmatrix} f_{1,1} & f_{1,2} & \cdots & f_{1,10} \\ f_{2,1} & & \cdots & f_{2,10} \\ f_{3,1} & & \cdots & f_{3,10} \end{bmatrix}$ $\quad$ 30

known

$\Rightarrow s^{10} + \alpha_9 s^9 + \cdots + \alpha_1 s + \alpha_0$

$\lambda_1^d, \lambda_2^d, \ldots, \lambda_{10}^d$

Ex 9.11

$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$

$\lambda_i^d = \{-2, -1 \pm j\}$

$\alpha_d(s) = (s+2)(s^2+2s+2)$

$= s^3 + 4s^2 + 6s + 4$

$\mathrm{eig}(\bar{A})$

$A_{c1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -6 & -4 \end{bmatrix}$

$A_{c2} = \begin{bmatrix} 0 & 1 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

$\Rightarrow \begin{cases} F_{c1} = \begin{bmatrix} 3 & 7 & 5 \\ -5 & -6 & -4 \end{bmatrix} \\ F_{c2} = \begin{bmatrix} -3 & -1 & 2 \\ -1 & 0 & -2 \end{bmatrix} \end{cases}$

$F_1 = \begin{bmatrix} 5 & 7 & -\frac{9}{2} \\ -4 & -6 & \frac{5}{2} \end{bmatrix}$

$F_2 = \begin{bmatrix} 2 & -1 & -2 \\ -2 & 0 & \frac{1}{2} \end{bmatrix}$

$\mathrm{eig}(A_c + B_c F_c) = \begin{cases} s^3 + 4s^2 + 6s + 4 \\ (s+2)(s^2+2s+2) \end{cases}$

$\mathrm{eig}(A + B F_i) = \bigcirc, \quad i = 1, 2$

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$\dot{x} = Ax$

$\mathrm{eig}(A): \lambda_i$

$\mathrm{eigvector}(A): v_i$

$\Rightarrow x(t) = \sum_{i=1}^{n} v_i e^{\lambda_i t}$

P.364 Assigning Eigenvalues & Eigenvectors

$\xrightarrow{F} A + BF$ $\quad \begin{cases} \lambda_i: \text{eigenvalue} \\ v_i: \text{eigenvector} \end{cases}$

$\Rightarrow \begin{bmatrix} \lambda_i I - (A+BF) \end{bmatrix} v_i = 0$

$\Rightarrow \begin{bmatrix} \lambda_i I - A & B \end{bmatrix} \begin{bmatrix} v_i \\ -Fv_i \end{bmatrix} = 0$

$\begin{bmatrix} \lambda_i I - A & B \end{bmatrix} \begin{bmatrix} M_i \\ -D_i \end{bmatrix} = 0$

null space

basis $\begin{bmatrix} M_i \\ -D_i \end{bmatrix}$

$\dim = (n+m) - \mathrm{rank}\begin{bmatrix} \lambda_i I - A & B \end{bmatrix}$

$= m \qquad\qquad \overset{\shortparallel}{n}$

$\begin{bmatrix} v_i \\ -Fv_i \end{bmatrix} = \begin{bmatrix} M_i \\ -D_i \end{bmatrix} a_i \qquad a_i \in \mathbb{R}^{m \times 1}$

basis $\quad$ coefficient

$\begin{cases} M_i a_i = v_i \\ D_i a_i = F v_i \end{cases}$

$\Rightarrow D_i a_i = F M_i a_i$

basis

$V \triangleq \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$

$= \begin{bmatrix} M_1 a_1 & M_2 a_2 & \cdots \end{bmatrix}$

$W \triangleq \begin{bmatrix} D_1 a_1 & D_2 a_2 & \cdots & D_n a_n \end{bmatrix}$

$\Rightarrow W = FV$

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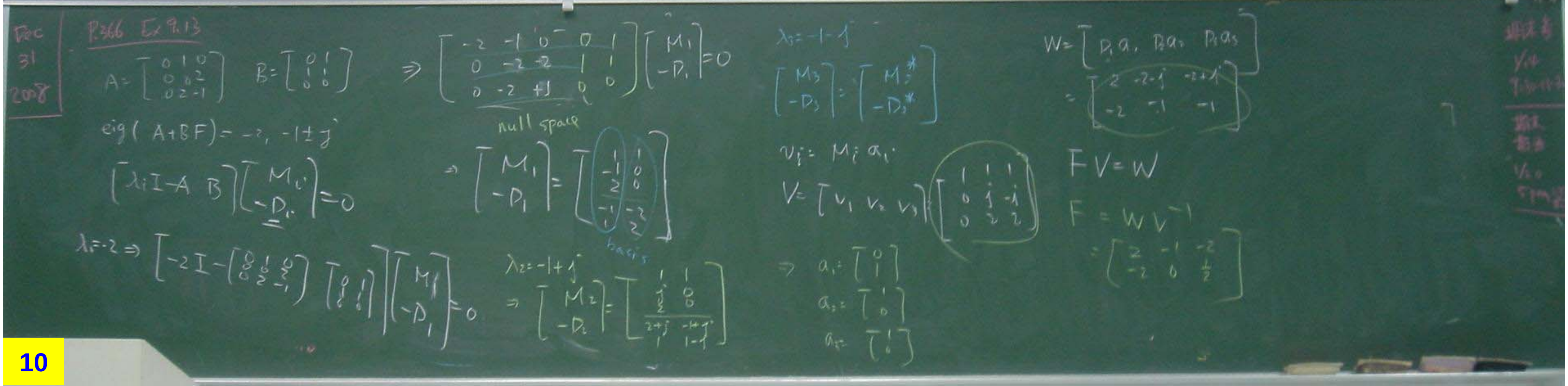

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P.366 Ex 9.13
eig(A+BF) = -2, -1±j
null space
basis
FV=W
F = WV^-1

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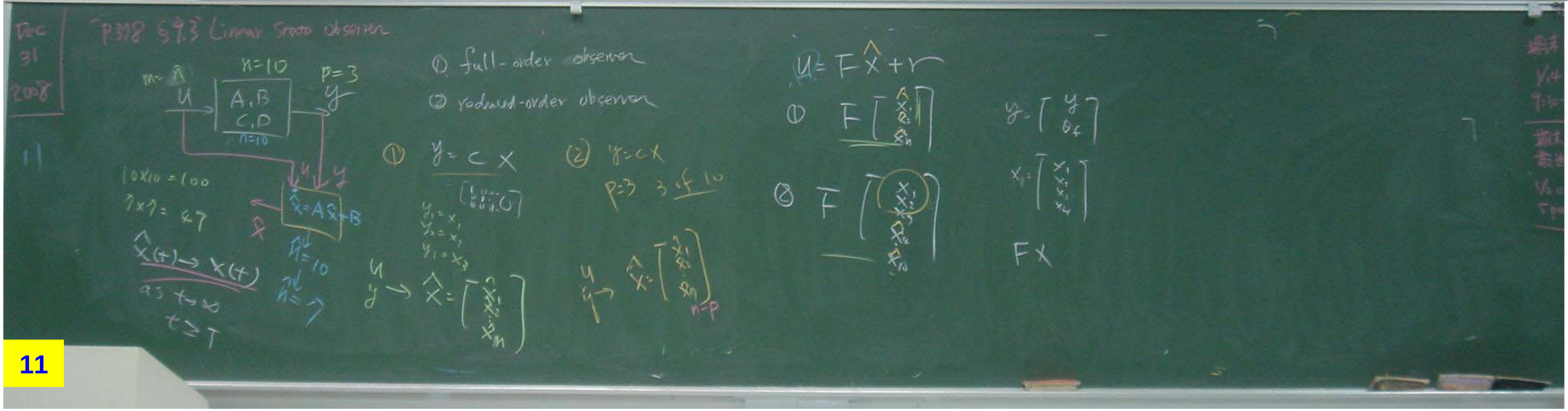

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P.378 §9.3 Linear State observer
① full-order observer
② reduced-order observer
n=10
p=3
A,B
C,D
10x10=100
① y=CX
② y=CX
FX

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