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P.320 The minimal realization is not unique!

Thm 8.10

Let $\{A, B, C, D\}$, $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ are two realizations of $H(s)$

"Both" $\{A, B, C, D\}$ and $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ are minimal realizations (IF / THEN)

$\Longleftrightarrow$ The two realizations are "equivalent"

i.e. $\exists\, P$ nonsingular such that $\bar{A} = PAP^{-1}$, $\bar{C} = CP^{-1}$, $\bar{B} = PB$, $\bar{D} = D$

proof

"$\Leftarrow$" $\{A, B, C, D\}$ & $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$: equivalent

Since $\{A, B, C, D\}$ is minimal

$\Rightarrow$ Controllable & Observable (A, B) (A, C)

$\because$ "equivalent" $\Rightarrow$ $(\bar{A}, \bar{B})$: controllable, $(\bar{A}, \bar{C})$: observable

$\Rightarrow$ $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$: minimal realization

"$\Rightarrow$" $H(s) = C(sI-A)^{-1}B + D$

$H(t,0) = Ce^{At}B + D = \bar{C}e^{\bar{A}t}\bar{B} + \bar{D}$ $\quad n \neq \bar{n}$

$\Rightarrow \begin{cases} D = \bar{D} \\ CA^kB = \bar{C}\bar{A}^k\bar{B}, \quad k = 0, 1, 2, \dots \end{cases}$

$\because$ minimal $\Rightarrow$ of order n $\Rightarrow$ controllable & observable

$\operatorname{rank}(\mathcal{C} / \mathcal{O} / \bar{\mathcal{C}} / \bar{\mathcal{O}}) = n$

$$\mathcal{O}\mathcal{C} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = \begin{bmatrix} CB & CAB & \cdots & CA^{n-1}B \\ CAB & & & \\ \vdots & & & \\ CA^{n-1}B & & \cdots & CA^{2n-2}B \end{bmatrix} = \begin{bmatrix} \bar{C}\bar{B} & & \cdots & \bar{C}\bar{A}^{n-1}\bar{B} \\ \vdots & & & \\ \bar{C}\bar{A}^{n-1}\bar{B} & & \cdots & \bar{C}\bar{A}^{2n-2}\bar{B} \end{bmatrix} = \begin{bmatrix} \bar{C} \\ \bar{C}\bar{A} \\ \vdots \end{bmatrix} \begin{bmatrix} \bar{B} & \bar{A}\bar{B} & \cdots \end{bmatrix} = \bar{\mathcal{O}}\bar{\mathcal{C}}$$

$\mathcal{O}A\mathcal{C} = \bar{\mathcal{O}}\bar{A}\bar{\mathcal{C}}$

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$\bar{\mathcal{O}}^T\mathcal{O}\mathcal{C} = \bar{\mathcal{O}}^T\bar{\mathcal{O}}\bar{\mathcal{C}}$ ($n \times n$ nonsingular)

$\operatorname{rank}(\bar{\mathcal{O}}^T\bar{\mathcal{O}}) = n$

By Sylvester's Rank Inequality

$\bar{\mathcal{C}} = \underbrace{(\bar{\mathcal{O}}^T\bar{\mathcal{O}})^{-1}\bar{\mathcal{O}}^T\mathcal{O}}_{\triangleq P}\mathcal{C}$

$\operatorname{rank}(P) = n$

$\boxed{\bar{\mathcal{C}} = P\mathcal{C}}$

$\mathcal{O}\mathcal{C}\mathcal{C}^T = \bar{\mathcal{O}}\bar{\mathcal{C}}\mathcal{C}^T$ $\quad$ $n \times mn$, $mn \times n$ $\Rightarrow \operatorname{rank}(\mathcal{C}\mathcal{C}^T) = n$

$\mathcal{O} = \bar{\mathcal{O}}\underbrace{\bar{\mathcal{C}}\mathcal{C}^T(\mathcal{C}\mathcal{C}^T)^{-1}}_{\triangleq \bar{P}}$

$\operatorname{rank}(\bar{P}) = n$

$\boxed{\mathcal{O} = \bar{\mathcal{O}}\bar{P}}$

$P = (\bar{\mathcal{O}}^T\bar{\mathcal{O}})^{-1}\bar{\mathcal{O}}^T\mathcal{O}$

$= (\bar{\mathcal{O}}^T\bar{\mathcal{O}})^{-1}\bar{\mathcal{O}}^T\bar{\mathcal{O}}\bar{P}$

$P = I\bar{P}$

$\Rightarrow P = \bar{P}$

(a) $\bar{\mathcal{O}}^T\mathcal{O}A\mathcal{C}\mathcal{C}^T = \bar{\mathcal{O}}^T\bar{\mathcal{O}}\bar{A}\bar{\mathcal{C}}\mathcal{C}^T$

$\underbrace{(\bar{\mathcal{O}}^T\bar{\mathcal{O}})^{-1}\bar{\mathcal{O}}^T\mathcal{O}}_{P}A = \bar{A}\underbrace{\bar{\mathcal{C}}\mathcal{C}^T(\mathcal{C}\mathcal{C}^T)^{-1}}_{\bar{P} = P}$

$\Rightarrow \bar{A} = PAP^{-1}$

(b) $\bar{\mathcal{C}} = P\mathcal{C}$ $\quad [\bar{B}\ \bar{A}\bar{B} \dots] \quad [B\ AB \dots] \rightarrow \bar{B} = PB$

(c) $\bar{\mathcal{O}} = \mathcal{O}\bar{P}^{-1} = \mathcal{O}P^{-1} \Rightarrow \bar{C} = CP^{-1}$

(d) $\bar{D} = D$

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P.324 §8.4 Realization Algorithms

$H(s) = \dfrac{n(s)}{d(s)}$ or $N(s)D^{-1}(s)$, $D^{-1}(s)N(s)$

$\rightarrow \{A, B, C, D\}$

① duality $\{A\ B\ C\ D\}$, $\{A^T\ C^T\ B^T\ D^T\}$

② minimal realization

$\Rightarrow$ controllable & observable

$\{A, B, C, D\} \rightarrow \{A_c, B_c, C_c, D_c\} \rightarrow \{A_{c(o)}, B_{c(o)}, C_{c(o)}, D_{c(o)}\}$

$\{A, B, C, D\} \rightarrow \{A_o, B_o, C_o, D_o\} \rightarrow \{A_{o(c)}, B_{o(c)}, C_{o(c)}, D_{o(c)}\}$

not minimal, not $\mathcal{C}$ or $\mathcal{O}$

Kalman's $\Rightarrow$ (A_{11}, B_1, C_1): Both $\mathcal{C}$ and $\mathcal{O}$

$H(s) = C(sI-A)^{-1}B + D = C_1(sI-A_{11})^{-1}B_1 + D$

③ $H(s) = \dfrac{n(s)}{d(s)} = \dfrac{\cdots}{s^n + a_1 s^{n-1} \cdots}$

$\Rightarrow$ controller form, observer form

$A: \left[\, 0 \mid I \,\right]$

④ A is diagonal

⑤ balanced realization

$\mathcal{C}$ $\mathcal{O}$

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§8.4.1 Realization Using Duality

$H(s) \Rightarrow \begin{cases} \dot{x} = Ax + Bu & x \in \mathbb{R}^n \\ y = Cx + Du & y \in \mathbb{R}^p,\ u \in \mathbb{R}^m \end{cases}$

$m \Rightarrow \overset{p\times m}{H(s)} = C(sI-A)^{-1}B + D$

$\overset{m\times p}{\tilde{H}(s)} = H^T(s)$

$\Rightarrow \dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\tilde{u}$

$\tilde{y} = \tilde{C}\tilde{x} + \tilde{D}\tilde{u}$

$\tilde{H}(s) = H^T(s)$

$= \left(C(sI-A)^{-1}B + D\right)^T$

$= \left(C(sI-A)^{-1}B\right)^T + D^T$

$= \left(B^T\left((sI-A)^{-1}\right)^T C^T + D^T\right)$

$= \left(B^T\left(sI-A^T\right)^{-1} C^T + D^T\right)$

$= \underset{\tilde{C}}{B^T}(sI-\underset{\tilde{A}}{A^T})^{-1}\underset{\tilde{B}}{C^T} + \underset{\tilde{D}}{D^T}$

$\{A,B,C,D\}$ $\{\tilde{A},\tilde{B},\tilde{C},\tilde{D}\}$

$\mathcal{C} \Longleftrightarrow \mathcal{O}$

$\mathcal{O} \Longleftrightarrow \mathcal{C}$

$H(s)$: $p\times m$ proper transfer function matrix

i.e. $\lim_{s\to\infty} H(s) < \infty$

$\hat{H}(s) \triangleq H(s) - \underbrace{\lim_{s\to\infty} H(s)}_{D}$

strictly proper

$H(s) \rightarrow \{A,B,C,D\}$ $\lim_{s\to\infty} H(s)$

$\hat{H}(s) \rightarrow \{A,B,C\}$

$\hat{H}(s) \rightarrow \{A_c, B_c, C_c\}$ controllable (may not be observable)

$\tilde{H}(s) = \hat{H}^T(s)$

$\tilde{H}(s) \rightarrow \{\tilde{A}, \tilde{B}, \tilde{C}\}$, $\tilde{A} = A_c^T$, $\tilde{B} = C_c^T$, $\tilde{C} = B_c^T$

observable (may not be controllable)

$\{A,B,C\}$ observable

$\rightarrow \left\{ \left[\begin{array}{c|c} A_1 & A_{12} \\ \hline 0 & A_2 \end{array}\right], \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, [C_1 \;\; \cdot] \right\}$

$\Rightarrow \{A_1, B_1, C_1\}$

$\Rightarrow \mathcal{C}\ \&\ \mathcal{O}$

$\Rightarrow$ minimal

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§8.4.2 Realization in Controller/observer forms

SISO $p = m = 1$

$H(s) = \dfrac{n(s)}{d(s)} = \dfrac{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} = b_n + \dfrac{b_{n-1}s^{n-1}}{s^n}$

$\begin{cases} b_n \neq 0 \Rightarrow \text{proper} \\ b_n = 0 \Rightarrow \text{strictly proper} \end{cases}$

$[b_{n-1}\ b_{n-2}\ \cdots\ b_0]\begin{bmatrix} 1 \\ s \\ \vdots \\ s^{n-1}\end{bmatrix}$

$\hat{y}(s) = \hat{H}(s)\hat{u}(s)$

$\Rightarrow y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1\dot{y}(t) + a_0$

$= b_n u^{(n)}(t) + \cdots + b_0$

Controller form realization

$A_c = \begin{bmatrix} 0 & 1 & & \\ 0 & & \ddots & \\ 0 & & & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix}$, $B_c = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

$C_c = [\ \times\ \times\ \times\ \cdots\ \times\ ] \in \mathbb{R}^{1\times n}$, $D_c = [\ *\ ]$

$S(s) \triangleq \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{n-1} \end{bmatrix}$

assume $\boxed{n(s) = C_c S(s) + D_c d(s)}$

$H(s) = \dfrac{n(s)}{d(s)} = \dfrac{C_c S(s) + D_c d(s)}{d(s)}$

$= C_c S(s) \dfrac{1}{d(s)} + D_c$

$\lim_{s\to\infty} H(s) = D_c = b_n$

$n(s) = C_c S(s) + b_n d(s)$

$\dfrac{n(s) - b_n d(s)}{s^n} = \dfrac{C_c S(s)}{s^{n-1}}$

exist!

① If $b_n = 0$

$C_c = [b_{n-1}\ b_{n-2}\ \cdots\ b_1\ b_0]$

② If $b_n \neq 0$

$n(s) - b_n d(s) = C_c S(s)$

$= [b_n s^n + b_{n-1}s^{n-1} \cdots b_0]$

$- b_n[s^n + a_{n-1}s^{n-1} \cdots a_0]$

$= (b_n - b_n)s^n + (b_{n-1} - b_n a_{n-1})s^{n-1} \cdots$

$\Rightarrow C_c = [b_{n-1} - b_n a_{n-1},\ b_{n-2} - b_n a_{n-2}, \cdots]$

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Lemma 8.19

The controller form realization is minimal.

proof

① It is a realization

② It is minimal

i.e. controllable

observable

of order n

$\dot{x} = A_c x + B_c u$

$y = C_c x + D_c u$

$\Rightarrow \boxed{\begin{matrix} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_{n-1} = x_n \end{matrix}}$

$\dot{x}_n = -a_0 x_1 - a_1 x_2 - \cdots + u$

$\ddot{x}_{n-1} =$

$\overset{(3)}{x}_{n-2} =$

$x_1^{(n)} = -a_0 x_1 - a_1 \dot{x}_1 \cdots + u$

$x_1^{(n)} = -a_0 x_1 - a_1 \dot{x}_1 - a_2 x_1^{(2)} \cdots - a_{n-1} x_1^{(n-1)} + u$

$x_1^{(n)} + a_{n-1}x_1^{(n-1)} \cdots + a_1 x_1^{(1)} + a_0 x_1 = u$

$q = \dfrac{d}{dt}$

$x_1 q^n + a_{n-1}x_1 q^{n-1} + \cdots + a_0 x_1 = u$

$\underbrace{(q^n + a_{n-1}q^{n-1} + \cdots + a_0)}_{d(q)} x_1 = u$

$d(q)\, x_1(t) = u(t)$

$\Rightarrow y(t) = C_c x + D_c u$

$= n(q)\, x_1 + b_n u(t)$

$d(q)\, x_1(t) = u(t)$

$\Rightarrow \hat{x}_1(s) = d(s)^{-1}\hat{u}(s)$

$\hat{x}(s) = \begin{bmatrix} \hat{x}_1(s) \\ \hat{x}_2(s) \\ \vdots \\ \hat{x}_n(s) \end{bmatrix} = \begin{bmatrix} 1 \\ s \\ s^2 \\ \vdots \\ s^{n-1} \end{bmatrix} \hat{x}_1(s)$

$= S(s)\, d(s)^{-1}\hat{u}(s)$

$\hat{x}(s) = (sI - A_c)^{-1} B_c \hat{u}(s)$

$S(s)\, d(s)^{-1}\hat{u}(s) = (sI - A_c)^{-1} B_c \hat{u}(s)$

$\Rightarrow (sI - A)^{-1} B_c = S(s)\, d(s)^{-1}$

$B_c\, d(s) = (sI - A)\, S(s)$

$\Rightarrow$

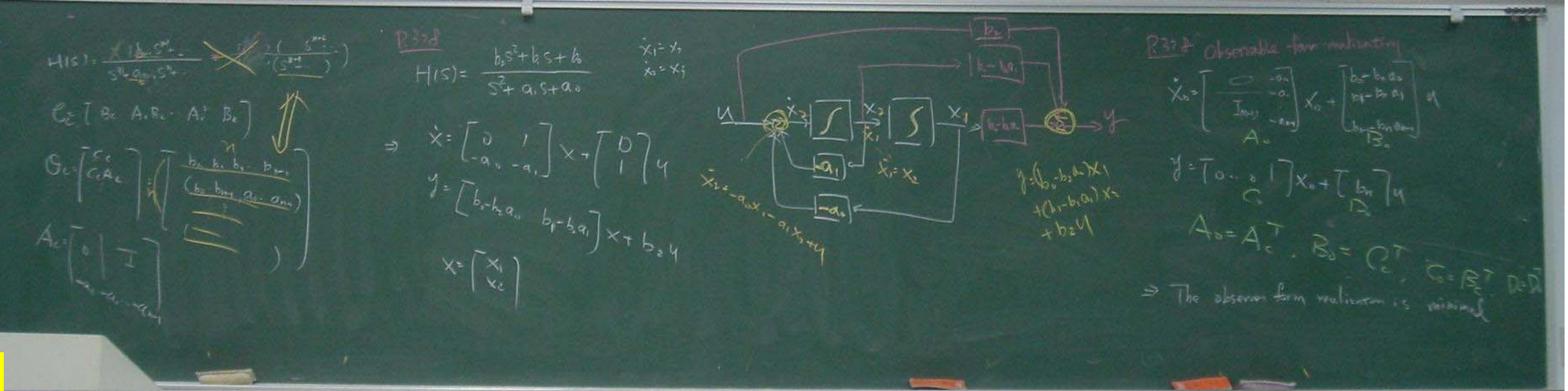
$C_c(sI - A_c)^{-1}B_c + D_c$

$= C_c S(s) d(s)^{-1} + D_c$

$= \underbrace{[C_c S(s) + D_c d(s)]}_{n(s)} d(s)^{-1}$

$\Rightarrow n(s) d(s)^{-1} = H(s)$

$\{A_c, B_c, C_c, D_c\}$ is a realization of $H(s)$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P.3-2-d
$H(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^2 + a_1 s + a_0}$
P.3-2-4 Observable form realization
$A_o = A_c^T$, $B_o = C_c^T$
⇒ The observer form realization is minimal

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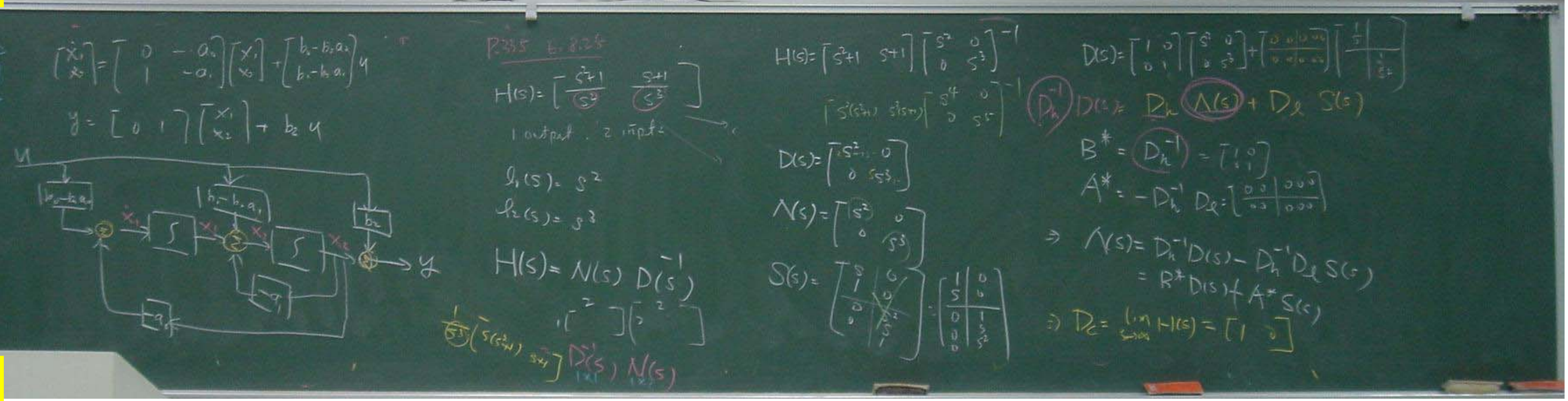
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P.3-3-5 E.8.2-8
1 output, 2 inputs
$l_1(s) = s^2$
$l_2(s) = s^3$
$H(s) = N(s) D^{-1}(s)$
$H(s) = [s^2+1 \quad s+1] \begin{bmatrix} s^2 & 0 \\ 0 & s^3 \end{bmatrix}^{-1}$
$D_c = \lim_{s\to\infty} H(s) = [1 \quad 0]$

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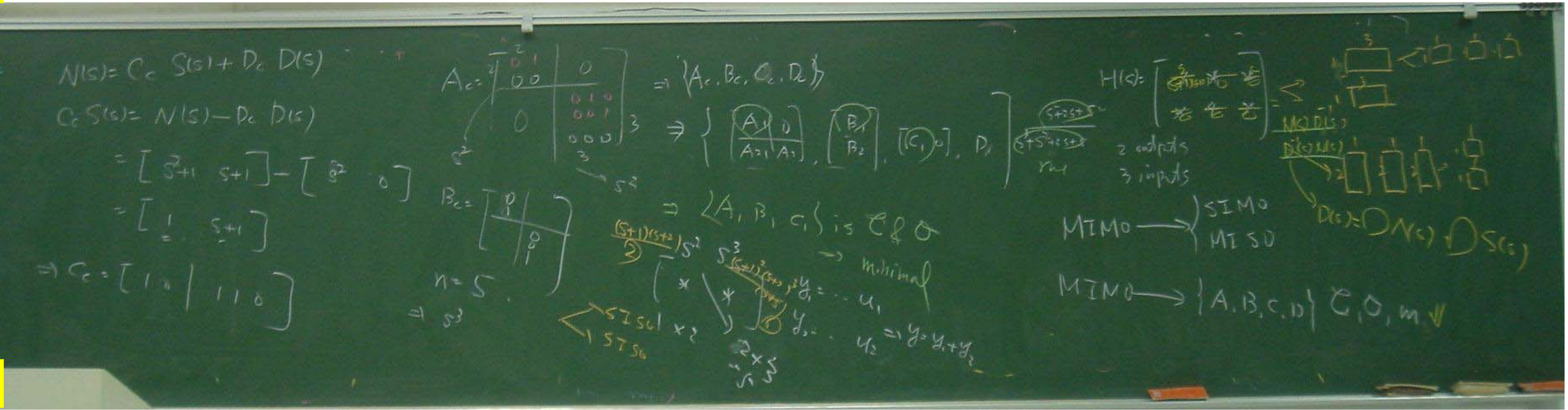
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$N(s) = C_c S(s) + D_c D(s)$
$C_c S(s) = N(s) - D_c D(s)$
$\Rightarrow C_c = [1 \; 0 \,|\, 1 \; 1 \; 0]$
$\Rightarrow \{A_c, B_c, C_c, D_c\}$
$\{A, B, C\}$ is C & O
→ minimal
2 outputs
3 inputs
MIMO → SIMO, MISO
MIMO → {A, B, C, D} C, O, m

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