

Special Forms

10 uncontrollable (unobservable) rank(B) = r1 < n rank(O) = r2 < n

08 $x \in \mathbb{R}^n$ rank(B) = r1 < n rank(O) = r2 < n

1 $A \Rightarrow \hat{A} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$ $\hat{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$ $\hat{C} = [C_1 \ 0]$

Kalman's decomposition for unC & unO

$\hat{A} = \begin{bmatrix} A_{11} & 0 & A_{13} & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} \\ 0 & 0 & A_{33} & 0 \\ 0 & 0 & A_{43} & A_{44} \end{bmatrix}$ $\hat{B} = \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix}$ $\hat{C} = [C_1 \ 0 \ C_3 \ 0]$

controllable rank(C) = n observable rank(O) = n

$\hat{A} = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ 0 & A_{22} & 0 & 0 \\ A_{31} & A_{32} & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix}$ $\hat{B} = \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix}$ $\hat{C} = [C_1 \ C_2 \ 0 \ 0]$

$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix}$

$\alpha(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \dots + \alpha_1\lambda + \alpha_0$

10 P. 50 & 6.4 Controller Canonical Form

08 $\dot{x} = Ax + Bu$ $A \in \mathbb{R}^{n \times n}$ $B \in \mathbb{R}^{n \times m}$ $y = Cx + Du$ $C \in \mathbb{R}^{p \times n}$ $D \in \mathbb{R}^{p \times m}$

2 (A, B) controllable $\Rightarrow \text{rank}(CB) = n$

rank(B) = m $\Rightarrow \begin{bmatrix} B \\ CB \\ \vdots \\ C^{n-1}B \end{bmatrix}$

single-input case (m=1) $\Rightarrow \text{rank}(B) = 1$

$\{A, B, C, D\} \xrightarrow{P} \{A_c, B_c, C_c, D_c\}$ $\alpha(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \dots + \alpha_1\lambda + \alpha_0$

$A_c = PAP^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & -\alpha_0 \end{bmatrix}$

$B_c = PB = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$C_c = CP^{-1}$

$D_c = D$

Block diagram for realization showing input u, integrators, and outputs y.

10 what is P?

08 $C = [B \ AB \ \dots \ A^{n-1}B]$ $n \times n$

$\Rightarrow \text{rank}(C) = n$

$\Rightarrow C$ nonsingular $n \times n$ matrix

$\Rightarrow C^{-1} = \begin{bmatrix} * \\ * \\ * \end{bmatrix}$

$P = \begin{bmatrix} I \\ qA \\ qA^2 \\ \vdots \\ qA^{n-1} \end{bmatrix}$ $C^{-1}C = I$

$PC = \begin{bmatrix} qA \\ qA^2 \\ \vdots \\ qA^n \end{bmatrix} [B \ AB \ \dots \ A^{n-1}B]$

$PA = \begin{bmatrix} qA \\ qA^2 \\ \vdots \\ qA^n \end{bmatrix} A = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} + \begin{bmatrix} qA \\ qA^2 \\ \vdots \\ qA^n \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$

$PA = A_c P$

$PB = \begin{bmatrix} qA \\ qA^2 \\ \vdots \\ qA^n \end{bmatrix} B = \begin{bmatrix} qAB \\ qA^2B \\ \vdots \\ qA^nB \end{bmatrix} = B_c$

10 $[B \ AB \ \dots \ A^{n-1}B]$

08 $= [PB \ PAP^{-1}PB \ (PAP^{-1})^2(PB) \ \dots]$

4 $= [PB \ PAB \ PA^2B \ \dots \ PA^{n-1}B]$

$= P[B \ AB \ \dots \ A^{n-1}B]$

$\begin{bmatrix} P \\ P \\ \vdots \\ P \end{bmatrix} C = C_c$

P. 254 $P_1 = \begin{bmatrix} qA^{n-1} \\ qA^{n-2} \\ \vdots \\ qA \end{bmatrix}$

$A_{c1} = \begin{bmatrix} -\alpha_0 & -\alpha_1 & \dots & -\alpha_{n-1} \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$ $B_{c1} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$P_2 = C^{-1} = [B \ AB \ \dots \ A^{n-1}B]^{-1}$ $B_{c2} = P_2 B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$P_3 = [A^{n-1}B \ A^{n-2}B \ \dots \ B]^{-1}$ $A_{c3} = \begin{bmatrix} -\alpha_0 & -\alpha_1 & \dots & -\alpha_{n-1} \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$ $B_{c3} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$

10 P. 263 & 6.4.2 observer Form

08 $\dot{x} = Ax + Bu$ $y = Cx + Du$

5 (A, C) observable i.e. rank(O) = n

rank(C) = p = n

Find $P \in \mathbb{R}^{n \times n}$ nonsingular

By duality $\Rightarrow \exists \tilde{P} \in \mathbb{R}^{n \times n}$ nonsingular

$A_0 = PAP^{-1}$ $\tilde{A}_c = \tilde{P}A\tilde{P}^{-1}$

$B_0 = PB$ $\tilde{B}_c = \tilde{P}B$

$C_0 = CP^{-1}$ $\tilde{C}_c = \tilde{P}C$

$D_0 = D$ $\tilde{D}_c = D$

(A_0, C_0) observer form $\Leftrightarrow (\tilde{A}_c, \tilde{B}_c)$ is controllable

(A, C) is observable $\Leftrightarrow (\tilde{A}_c, \tilde{B}_c)$ is controllable

$O^{-1} = \begin{bmatrix} P \\ P \\ \vdots \\ P \end{bmatrix} \begin{bmatrix} B \\ AB \\ \vdots \\ A^{n-1}B \end{bmatrix} = I$

10 $AQ = [A\hat{A} \ A\hat{A}^2 \ \dots \ A\hat{A}^{n-1}]$

08 $C_0 = CP^{-1}CQ = C[\hat{A} \ \hat{A}^2 \ \dots \ \hat{A}^{n-1}]$

6 $= [C\hat{A} \ C\hat{A}^2 \ \dots \ C\hat{A}^{n-1}]$

$\Rightarrow A_0 = Q^{-1}AQ = PAP^{-1}$

$\Rightarrow (A_0, C_0)$ is observer form

P. 213 chap 8 Realization Theory & Algorithms

$U \rightarrow \frac{1}{s} F(s) \rightarrow Y$ $G(s) = F(s)G(s)$

$x_1 \in \mathbb{R}^n$ $x_2 \in \mathbb{R}^n$ n_1, n_2

$\dot{x}_1 = A_1x_1 + B_1u$ $\dot{x}_2 = A_2x_2 + B_2u$

$y = C_1x_1 + D_1u$ $y = C_2x_2 + D_2u$

$A_1 \in \mathbb{R}^{n_1 \times n_1}$ $A_2 \in \mathbb{R}^{n_2 \times n_2}$

Block diagram showing two parallel realizations connected to a summing junction.

P314 S 2.1 CT systems

LTI

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

$H(s) = C(sI - A)^{-1}B + D$

Block diagram realization

Laurent series

$$H_1 = \lim_{s \rightarrow \infty} H(s)$$

$$H_1 = \lim_{s \rightarrow \infty} s(H(s) - H_1)$$

$$H_2 = \lim_{s \rightarrow \infty} s^2(H(s) - H_1 - H_2 s^{-1})$$

$H(s) = C(sI - A)^{-1}B + D$

$$H(s) = \frac{H_1 + H_2 s^{-1} + H_3 s^{-2} + \dots}{1}$$

P315 Thm 8.3

$\{A, B, C, D\}$ is a realization of $H(s)$

$(I - s^{-1}A)^{-1} = \frac{1}{1 - as^{-1}} = \sum_{k=0}^{\infty} A^k s^{-k-1}$

$H(s) = D + C s^{-1} \sum_{k=0}^{\infty} A^k s^{-k-1} B = D + C \sum_{k=0}^{\infty} A^k s^{-k-2} B$

$H_0 = D$

$H_i = CA^{i-1}B, i=1,2,3, \dots$

$H(s) = D + C(sI - A)^{-1}B = D + C s^{-1} (I - A s^{-1})^{-1} B$

P316 S 2.1 Existence of Realization

P317 Minimality

P318 Thm 8.5

$H(s)$ is realizable

$H(s)$ is a matrix of rational functions and $\lim_{s \rightarrow \infty} H(s) < \infty$

$\{A, B, C, D\}$ is a proper rational matrix

Properties of minimal realizations

- a realization is minimal $\{A, B, C, D\}$ if and only if it is both controllable and observable
- all minimal realizations are equivalent

$$\begin{cases} A_1 = P A_2 P^{-1} \\ B_1 = P B_2 \\ C_1 = C_2 P^{-1} \\ D_1 = D_2 \end{cases}$$

P317 Minimality

$\dot{x} = Ax + Bu$

$y = Cx + Du$

$x \in \mathbb{R}^n$

$u \in \mathbb{R}^m$

$y \in \mathbb{R}^p$

$\tilde{x} = Fz + Gu$

$z \in \mathbb{R}^n$

$\tilde{y} = [y; u]$

$\begin{bmatrix} \dot{x} \\ x \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B \\ G \end{bmatrix} u$

$y = [C \ 0] \begin{bmatrix} x \\ z \end{bmatrix} + Du$

Ex 8.8

$H(s) = \frac{1}{s+1}$

- $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [-1 \ 0], D = 0$
- $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, C = [0 \ 1], D = 0$
- $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [0 \ 1], D = 0$
- $A = -1, B = 1, C = 1, D = 0$

P318 Thm 8.9

An n -dim realization $\{A, B, C, D\}$ of $H(s)$ is minimal (irreducible of least order) if and only if it is both controllable and observable.

Proof: minimal, not (Both C & O)

$A, B, C, D \Rightarrow$ by Kalman's algorithm

$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$

$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$

$C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$

$n < n \Rightarrow$ NOT minimal

$\hat{C}(sI - \hat{A})^{-1} \hat{B} = C(sI - A)^{-1} B = C_1(sI - A_{11})^{-1} B_1$

$D = \bar{D} = \lim_{s \rightarrow \infty} H(s)$

$C^T B = \bar{C}^T \bar{B}, \bar{C} = [C_1^T \ C_2^T]$

Markov parameters

$C_n = [B \ AB \ A^2 B \ \dots \ A^{n-1} B]$ $\text{rank}(C_n) = n$

$O_n = \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^n \end{bmatrix}$ $\text{rank}(O_n) = n$

$\bar{O}_n \bar{C}_n = \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^n \end{bmatrix} \begin{bmatrix} C_1 & C_2 \end{bmatrix} = \begin{bmatrix} CA_1 C_1 & CA_1 C_2 \\ CA_2 C_1 & CA_2 C_2 \\ \vdots & \vdots \\ CA_n C_1 & CA_n C_2 \end{bmatrix}$

$\bar{O}_n \bar{C}_n = \begin{bmatrix} CA_1 C_1 & CA_1 C_2 \\ CA_2 C_1 & CA_2 C_2 \\ \vdots & \vdots \\ CA_n C_1 & CA_n C_2 \end{bmatrix}$

$MN = P, MNP \in \mathbb{R}^{n \times n}$

$\Rightarrow \text{rank}(M) + \text{rank}(N) = n$

$\text{rank}(M) \leq \text{rank}(P) \leq \min(\text{rank}(M), \text{rank}(N))$

$\text{rank}(M) + \text{rank}(N) \leq \text{rank}(M) + \text{rank}(N) \leq \min(\text{rank}(M), \text{rank}(N)) + \min(\text{rank}(M), \text{rank}(N))$

$\Rightarrow \text{rank}(M) = \text{rank}(N) = n$

$\Rightarrow \{A, B, C, D\}$ minimal realization