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P.237 Chap 6

Controllability & observability, special forms

	A	B	C	D
controllable				
observable				
uncontrollable	$\begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix}$	$\begin{bmatrix} B_1 \\ 0 \end{bmatrix}$	$[C_1 \mid C_2]$	
unobservable	$\begin{bmatrix} A_1 & 0 \\ A_{21} & A_2 \end{bmatrix}$	$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$	$[C_1 \mid 0]$	

§6.2 standard forms for uncontrollable & unobservable systems

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

① $\mathcal{C} \triangleq [B \ AB \ \cdots \ A^{n-1}B]$

② $\mathcal{O} \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$

uncontrollable

$\Rightarrow \operatorname{rank}(\mathcal{C}) = n_r < n$

$\dim(R_r) = \dim R(\mathcal{C}) = n_r$

unobservable

$\Rightarrow \operatorname{rank}(\mathcal{O}) = n_o < n$

$\dim(R_{\bar{o}}) = \dim(N(\mathcal{O})) = n - n_o$

$\Rightarrow \dim(R_o) = \dim(R(\mathcal{O})) = n_o$

$x: \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ → uncontrollable, controllable, uncontrollable, observable; n_r, n_o; n_r states; $n-n_r$; n_o states

n

$x_n \Rightarrow \begin{cases} c\ o \\ c\ \bar{o} \\ \bar{c}\ o \\ \bar{c}\ \bar{o} \end{cases}$

R_r: basis $\{v_1, v_2, \ldots, v_{n_r}\}$

$\mathcal{C} = [B \ AB \ \cdots \ A^{n-1}B]$

n_r linearly indep

$Q = [v_1 \ v_2 \ \cdots \ v_{n_r} \ Q_{n-n_r}]$

$n \times n$

$\Rightarrow$ nonsingular

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$Q \triangleq [q_1 \ q_2 \ \cdots \ q_n]$

$= [v_1 \ v_2 \ \cdots \ v_{n_r} \ Q_{n-n_r}]$

$AQ = A[v_1 \ v_2 \ \cdots \ v_{n_r} \ Q_{n-n_r}]$

$= [Av_1 \ Av_2 \ \cdots \ Av_{n_r} \ AQ_{n-n_r}]$

$Av_i = \sum_{k=1}^{n_r} \alpha_k^i v_k$

$= \alpha_1^i v_1 + \alpha_2^i v_2 + \cdots + \alpha_{n_r}^i v_{n_r}$

$AQ_{n-n_r} \notin \{v_1, v_2, \ldots, v_{n_r}\}$ linearly indep

$= 0v_1 + 0v_2 + \cdots + 0v_{n_r}$

$AQ = [Av_1 \ Av_2 \ \cdots \ Av_{n_r} \ AQ_{n-n_r}]$

$= [v_1 \ v_2 \ \cdots \ v_{n_r} \ Q_{n-n_r}] \begin{bmatrix} \alpha_1^1 & \alpha_1^2 & \cdots & \alpha_1^{n_r} & \ast \\ \vdots & & & \vdots & \ast \\ \alpha_{n_r}^1 & \alpha_{n_r}^2 & \cdots & \alpha_{n_r}^{n_r} & \ast \\ 0 & 0 & \cdots & 0 & \ast \end{bmatrix}$

$= Q \begin{array}{c|c} & n_r \quad n-n_r \\ \begin{matrix} n_r \\ n-n_r \end{matrix} & \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix} \end{array}$

$= Q\hat{A}$

$AQ = Q\hat{A}$

$\Rightarrow \boxed{\hat{A} = Q^{-1}AQ}$

equivalence transformation

$B = [b_1 \ b_2 \ \cdots \ b_m]$

$= [v_1 \ v_2 \ \cdots \ v_{n_r} \ Q_{n-n_r}] \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$

$B = Q\hat{B}$

$\boxed{\hat{B} = Q^{-1}B}$

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$A \Rightarrow \hat{A} = Q^{-1}AQ$

$B \Rightarrow \hat{B} = Q^{-1}B$

$C \Rightarrow \hat{C} = CQ$

$D \Rightarrow \hat{D} = D$

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

$$\begin{cases} \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u \\ y = \hat{C}\hat{x} + \hat{D}u \end{cases}$$

$x \rightarrow \hat{x}$

$\hat{A} = \begin{array}{c|c} & n_r \quad n-n_r \\ \begin{matrix} n_r \\ n-n_r \end{matrix} & \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix} \end{array}$

$\hat{B} = \begin{array}{c} \begin{matrix} n_r \\ n-n_r \end{matrix} \end{array} \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$

$\hat{C} = \begin{array}{c} n_r \quad n-n_r \\ [C_1 \mid C_2] \end{array}$

$\hat{D} = D$

(A_1, B_1) is controllable

$$\begin{matrix} n_r \\ n-n_r \end{matrix} \begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u$$

n_r: $\dot{\hat{x}}_1 = A_1\hat{x}_1 + A_{12}\hat{x}_2 + B_1u \Rightarrow$ controllable $(\hat{x}_1)$

$n-n_r$: $\dot{\hat{x}}_2 = A_2\hat{x}_2 \Rightarrow$ uncontrollable $(\hat{x}_2)$

$\mathcal{C}_1 = [B_1 \ A_1B_1 \ \cdots \ A_1^{n_r-1}B_1]$

$\Rightarrow \boxed{\operatorname{rank}(\mathcal{C}_1) = n_r}$

$\mathcal{C} = [B \ AB \ \cdots]$

$= \left[\begin{bmatrix} B_1 \\ 0 \end{bmatrix} \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \cdots \right]$

$= \begin{bmatrix} B_1 & A_1B_1 & A_1^2B_1 & \cdots & A_1^{n-1}B_1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{matrix} n_r \\ n-n_r \end{matrix}$

$\operatorname{rank}(\mathcal{C}) = n_r$

$\operatorname{rank}(\mathcal{C}) = \operatorname{rank}\, n_r[B_1 \ A_1B_1 \ \cdots \ A_1^{n_r-1}B_1] = n_r$

$A_1^{n_r-1}B_1$

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P.180 Ex 6.2

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\mathrm{eig}(A) = 0, -1, -2$$

$$J = \begin{bmatrix} 0 & & 0 \\ & -1 & \\ 0 & & -2 \end{bmatrix}$$

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$\mathrm{rank}(\mathcal{C}) = 2$

$$\Rightarrow \{v_1, v_2\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$$

$$Q \triangleq \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

$\Rightarrow Q$: nonsingular

$0, -1$

$$\hat{A} = Q^{-1}AQ = \cdots = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & ? \end{bmatrix} \quad A_1 \quad A_{12} \quad A_2$$

$$\hat{B} = Q^{-1}B = \cdots = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B_1$$

-2

$$\mathcal{C}_1 = \begin{bmatrix} B_1 & A_1B_1 & \cancel{A_1^2B_1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

$\Rightarrow (A_1, B_1)$ is controllable

$$x(t) = \begin{bmatrix} & \end{bmatrix} \begin{bmatrix} e^{-0t} \\ e^{-t} \\ e^{-2t} \end{bmatrix} \begin{matrix} \to \text{controllable} \\ \to \text{uncon.} \end{matrix}$$

$x(t) \to \hat{x}(t)$

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① $\dot{x}_1 = -x_1 + x_2$

$\dot{x}_2 = -2x_2$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} x = Ax$$

$$A \triangleq \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$$

$$\lambda(A) = \det(\lambda I - A) = \lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2)$$

$\mathrm{eig}(A) = -1, -2 \Rightarrow e^{-t}, e^{-2t}$

$$\lambda_1 = -1 \Rightarrow \cdots \ v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = -2 \Rightarrow \cdots \ v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\phi(t) = e^{At}x(0)$$

①

$$e^{At} = I + At + A^2\frac{t^2}{2!} + \cdots = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} t + \frac{t^2}{2!}\begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}^2 + \cdots$$

$$= \begin{bmatrix} 1 - t + \frac{1}{2!}t^2 - \frac{t^3}{3!} \cdots & t - 3\frac{t^2}{2!} + 7\frac{t^3}{3!} - \cdots \\ 0 & 1 - 2t + 4\frac{t^2}{2!} - 8\frac{t^3}{3!} \cdots \end{bmatrix}$$

e^{-t} $\quad e^{-2t}$

$\to ae^{-t} + be^{-2t}$

$e^{-t} - e^{-2t}$

$$\Rightarrow \phi(t) = \begin{bmatrix} e^{-t} & e^{-t} - e^{-2t} \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} -e^{-t} \\ 0 \end{bmatrix} \quad e^{-2t} \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$\phi(t) = \begin{bmatrix} x_1(0)e^{-t} + x_2(0)e^{-t} - x_2(0)e^{-2t} \\ x_2(0)e^{-2t} \end{bmatrix}$$

② $P \triangleq \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$

$$P^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$J = P^{-1}AP = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$A = PJP^{-1}$$

$$e^{At} = e^{PJP^{-1}t} = Pe^{Jt}P^{-1}$$

$$\Rightarrow e^{Jt} = I + Jt + J^2\frac{t^2}{2!} \cdots = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$$

$$\Rightarrow e^{At} = \begin{bmatrix} e^{-t} & e^{-t} - e^{-2t} \\ 0 & e^{-2t} \end{bmatrix}$$

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③ $\alpha(\lambda) = \lambda^2 + 3\lambda + 2$

$\Rightarrow A^2 + 3A + 2I = 0$

$$\Rightarrow e^{At} = \alpha_0(t) I + \alpha_1(t) A = \begin{bmatrix} \alpha_0(t) - \alpha_1(t) & \alpha_1(t) \\ 0 & \alpha_0(t) - 2\alpha_1(t) \end{bmatrix}$$

$t = 0$, $t = 1$, $t = 2$

$\frac{d}{dt}$, $t = 0$

$\frac{d^2}{dt^2}$, $t = 0$

$$\Rightarrow \begin{cases} \alpha_0(t) = 2e^{-t} - e^{-2t} \\ \alpha_1(t) = e^{-t} - e^{-2t} \end{cases}$$

$$e^{At} = \begin{bmatrix} e^{-t} & e^{-t} - e^{-2t} \\ 0 & e^{-2t} \end{bmatrix}$$

$$\phi(t) = e^{At}x(0)$$

④ $sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} s+1 & -1 \\ 0 & s+2 \end{bmatrix}$

$$\Phi(s) = (sI - A)^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+2 & 1 \\ 0 & s+1 \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix}$$

$$\Phi(t) = \mathcal{L}^{-1}\{\Phi(s)\} = \begin{bmatrix} e^{-t} & e^{-t} - e^{-2t} \\ 0 & e^{-2t} \end{bmatrix}$$

$$\phi(t) = e^{At}x(0)$$

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②

$\dot{x} = Ax \quad x=0$ is E.S.

$\Rightarrow \mathrm{Real}\{\mathrm{eig}(A)\} < 0$

I/O

$C(sI-A)^{-1}B = \hat{H}(s) = \dfrac{\hat{n}(s)}{\hat{d}(s)}$ ← zero

S.S. eig(A) $\supset$ (root $(\hat{d}(s))$)

$n \qquad \leq n$

Real$\{$Pole$\{\hat{H}(s)\}\} < 0$

$\Rightarrow$ BIBO Stable

OR $\left\| \int_0^t H(t,\tau)\, d\tau \right\|$

$\leq \int_0^t \| H(t,\tau) \| d\tau$

$= \int_0^t \| C e^{A(t-\tau)} B \| d\tau$

$\Phi(t,\tau)$

$\leq \|C\| \, \|B\| \int_0^t \| e^{A(t-\tau)} \| d\tau$

$\Phi(t,\tau)$

$\leq \|C\|\,\|B\| \int_0^t \frac{a}{c} e^{-b\tau} d\tau$

$(1 - \frac{a}{b})$

$\phi(t,t_0,x_0) = \Phi(t,t_0)\, x(t_0)$

$\| \phi(t,t_0,x_0) \| \leq a e^{-b(t-t_0)} \quad a, b > 0$

$\Rightarrow \| \Phi(t,t_0)\, x(t_0) \| \leq a e^{-b(t-t_0)}$

constant

$\Rightarrow \| \Phi(t,t_0) \| \leq \frac{a}{c} e^{-b(t-t_0)}$

②b

$A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$

eig(A) = -1, 2

$C(sI-A)^{-1}B = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+1 & 0 \\ 0 & s-2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$y = \frac{1}{s+1} u$

$\frac{1}{(s+1)(s-2)} \begin{bmatrix} s-2 & 0 \\ 0 & s+1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s-2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$= \left[\frac{1}{s+1} \right]$

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P.241 § 6.7.2

standard forms for unobservable systems

$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \qquad \mathcal{O} v_i = 0$

$\begin{bmatrix} C \\ CA \\ CA^{n-1} \end{bmatrix} v_i = 0$

rank$(\mathcal{O}) = n_0 < n \Rightarrow C v_i = 0$

unobservable subspace $R_{\bar{o}} = N(\mathcal{O})$

$\dim(R_{\bar{o}}) = n - n_0$

a basis $\{ v_1 \; v_2 \cdots v_{n-n_0} \}$

$Q \triangleq \left[\underbrace{Q_{n_0}}_{n_0} \; \underbrace{v_1 \; v_2 \cdots v_{n-n_0}}_{n-n_0} \right]$

$AQ = A \left[Q_{n_0} \; v_1 \; v_2 \cdots v_{n-n_0} \right]$

$= \left[AQ_{n_0} \; Av_1 \; Av_2 \cdots Av_{n-n_0} \right]$

$v_1 \; v_2 \quad v_{n-n_0}$

$= \left[Q_{n_0} \; v_1 \; v_2 \cdots v_{n-n_0} \right] \begin{bmatrix} * & 0 \\ * & \times \end{bmatrix} \begin{matrix} n_0 \\ n-n_0 \end{matrix}$

$= Q \begin{bmatrix} A_1 & 0 \\ A_{21} & A_2 \end{bmatrix}$

$\hat{A}$

$AQ = Q\hat{A}$

$\Rightarrow \boxed{\hat{A} = Q^{-1} A Q}$

$CQ = C \left[Q_{n_0} \; v_1 \cdots v_{n-n_0} \right]$

$= \left[CQ_{n_0} \; Cv_1 \; Cv_2 \cdots Cv_{n-n_0} \right]$

$= \left[C_1 \mid 0 \right] = \hat{C}$

$n_0 \quad n-n_0$

$\boxed{\hat{C} = CQ}$

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$\hat{\mathcal{O}} = \begin{bmatrix} \hat{C} \\ \hat{C}\hat{A} \\ \vdots \\ \hat{C}\hat{A}^{n-1} \end{bmatrix}$

$= \begin{bmatrix} C_1 \quad 0 \\ \begin{bmatrix} C_1 & 0 \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ A_{21} & A_2 \end{bmatrix} \\ \vdots \end{bmatrix}$

$= \begin{array}{cc} n_0 & n-n_0 \end{array}$

$\left[\begin{array}{c|c} C_1 & 0 \\ C_1 A_1 & 0 \\ \vdots & \\ C_1 A_1^{n-1} & 0 \end{array} \right]$

rank$(\mathcal{O})$ = rank$(\hat{\mathcal{O}}) = n_0$

$= \text{rank} \begin{bmatrix} C_1 \\ C_1 A_1 \\ \vdots \\ C_1 A_1^{n-1} \end{bmatrix} n$

$= \text{rank} \begin{bmatrix} C_1 \\ C_1 A_1 \\ C_1 A_1^{n_0-1} \end{bmatrix} n_0$

$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$

$\hat{x} = Q^{-1} x$

$\dot{\hat{x}} = Q^{-1} \dot{x}$

$= Q^{-1}(Ax + Bu)$

$= Q^{-1} A x + Q^{-1} B u$

$= \underbrace{Q^{-1} A Q}_{\hat{A}} \hat{x} + \underbrace{Q^{-1} B}_{\hat{B}} u$

$y = Cx + Du$

$= \underbrace{CQ}_{\hat{C}} \hat{x} + \underbrace{D}_{\hat{D}} u$

$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ A_{21} & A_2 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$

$y = \begin{bmatrix} C_1 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \hat{D} u$

$\dot{\hat{x}}_1 = A_1 \hat{x}_1 + B_1 u$

$\dot{\hat{x}}_2 = A_2 \hat{x}_2 + A_{21} \hat{x}_1 + B_2 u$

$y = C_1 \hat{x}_1 + \hat{D} u$

(A_1, C_1): observable

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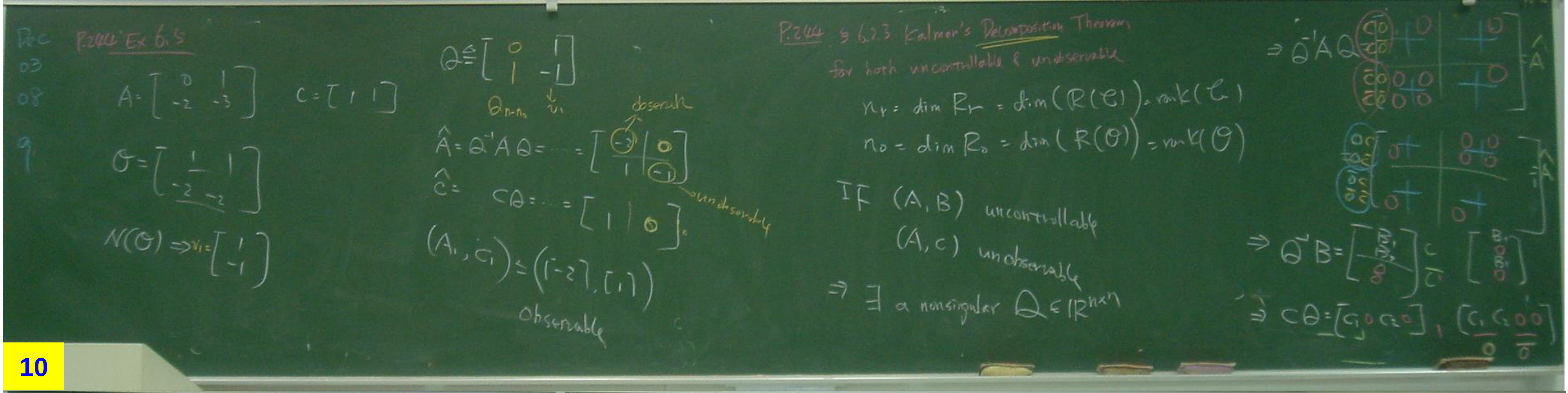
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P.244 Ex 6.5
observable
unobservable
Observable
P.244 §6.2.3 Kalman's Decomposition Theorem
for both uncontrollable & unobservable
IF (A, B) uncontrollable
(A, C) unobservable
⇒ ∃ a nonsingular Q

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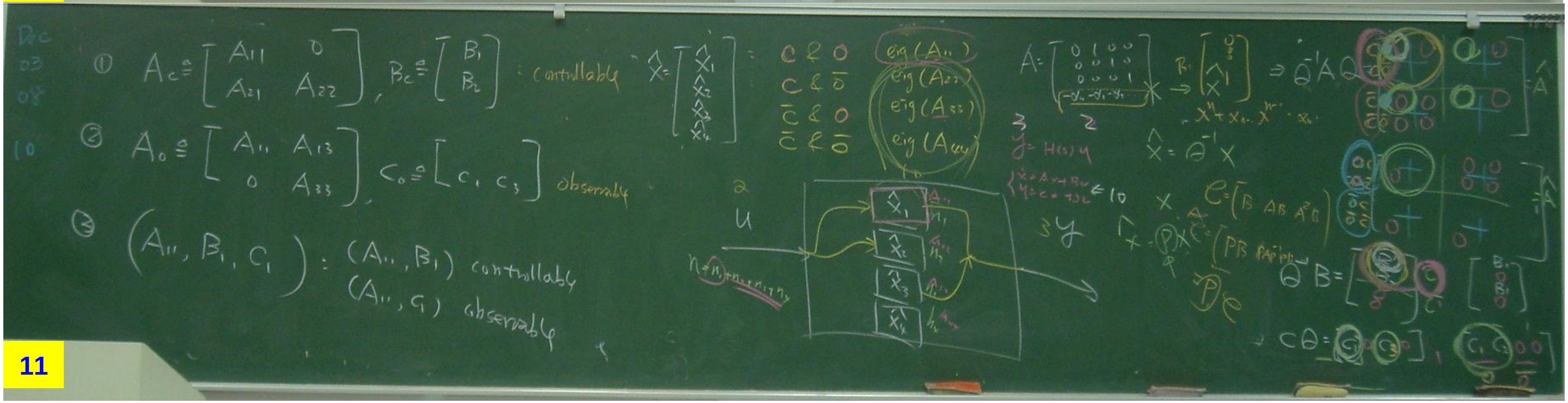
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controllable
observable
(A11, B1) controllable
(A11, C1) observable
eig (A11)
eig (A22)
eig (A33)
eig (A44)

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