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5.5.4 Observability: determine $x(t_0)$ from $\{u(t), y(t)\}$, $t \geq t_0$ (current, future)

Constructibility: determine $x(t_0)$ from $\{u(t), y(t)\}$, $t \leq t_0$ (current, past)

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \qquad x(0) = x_0$$

$$\Rightarrow \underbrace{y(t)}_{\text{Known}} = Ce^{At}\underbrace{x(0)}_{x_0}{}^{?} + \underbrace{\int_0^t Ce^{A(t-\tau)}B u(\tau)\, d\tau + D u(t)}_{\text{Known}}$$

$$\tilde{y}(t) \triangleq y(t) - \left[\int_0^t \cdots d\tau + Du(t)\right]$$

$$\tilde{y}(t) = Ce^{At} x_0 \overset{?}{=} 0$$

$p \times 1$ = $p \times n$, $n \times 1$; 2×1, 2×10, 10×1

$y = Ax \Rightarrow 0 = Ax$, x: ?

(5) $y(t_1) = e^{At_1}x$, $y(t_2) = e^{At_2}x$, ...

$$\begin{bmatrix} y(t_1) \\ \vdots \\ y(t_5) \end{bmatrix} = \begin{bmatrix} Ce^{At_1} \\ Ce^{At_2} \\ \vdots \\ Ce^{At_5} \end{bmatrix} x_0$$

10×1, 10×10, 10×1

$y = Ax$, $x = A^{-1}y$

P.219 Def 5.34

① X is unobservable IF the zero-input response is zero, $\forall t \geq 0$

i.e. $Ce^{At}x = 0$, $\forall t \geq 0$

$x \in Null(Ce^{At})$

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② unobservable subspace $R_{\bar{o}}$ = the set of all unobservable states

③ the system is (completely state) observable

(A, C) is observable

IF the only state that is unobservable is $x = 0$

i.e. $R_{\bar{o}} = \{0\}$, $0 \in \mathbb{R}^n$

P.220 Def 5.35

Observability Gramian

$$\underset{n\times n}{W_o(0,T)} \triangleq \int_0^T \underset{n\times n}{e^{A'\tau}}\, \underset{n\times p}{C'}\, \underset{p\times n}{C}\, \underset{n\times n}{e^{A\tau}}\, d\tau, \quad T > 0$$

① $W_o(0,T)$ symmetric, i.e. $W_o'(0,T) = W_o(0,T)$

② $W_o(0,T)$ positive semidefinite

$W_o(0,T) \geq 0$

$A \geq 0$

$\Rightarrow \forall x \in \mathbb{R}^n$

$x^T A x \geq 0$

$x^T \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} x$

$\lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_n x_n^2 \geq 0$

observability matrix

$$\mathcal{O} \triangleq \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}_{np \times n}$$

$\mathcal{N}(W_o(0,T))$: Null space of $W_o(0,T)$

$\mathcal{N}(\mathcal{O})$: Null space of $\mathcal{O}$

$N(A) = \{x \mid Ax = 0\} \subset \mathbb{R}^n$

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P.220 Lemma 5.36

$$\mathcal{N}(\mathcal{O}) = \mathcal{N}(W_o(0,T))$$

Proof

① $x \in \mathcal{N}(\mathcal{O})$

$\Rightarrow \mathcal{O}x = 0$

$\Rightarrow \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x = 0$

$$\begin{cases} Cx = 0 \\ CAx = 0 \\ CA^{n-1}x = 0 \end{cases}$$

$\Rightarrow CA^n x \overset{?}{=} 0$

$C(-\alpha_1 A^{n-1} - \alpha_{n-1}A^{n-2} \cdots - \alpha_0 I)x = 0$ (C-H)

$\Rightarrow CA^n x = 0$, $CA^{n+1}x = 0$, $CA^{n+2}x = 0$, ...

$$Ce^{At}x = C\left(\sum_{k=0}^{\infty} \frac{t^k}{k!}A^k\right)x = 0$$

$$W_o(0,T)x = \left(\int_0^T e^{A'\tau}C'Ce^{A\tau}d\tau\right)x$$

$e^{A'\tau}C'(Ce^{A\tau}x)\,d\tau$, $= 0$

$\Rightarrow x \in \mathcal{N}(W_o(0,T))$

② $x \in \mathcal{N}(W_o(0,T))$

$x^T W_o(0,T) x = x^T 0 = 0$

$\Rightarrow \int_0^T x^T e^{A'\tau}C^T C e^{A\tau} x\, d\tau = 0$

$y^T y = \|y\|_2^2$

$\Rightarrow \int_0^T \|Ce^{A\tau}x\|_2^2\, d\tau = 0$

$\Rightarrow \|Ce^{A\tau}x\|_2^2 = 0$, $\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

$\Rightarrow Ce^{A\tau}x = 0$

$\tau = 0$: $Ce^{A0}x = 0 \Rightarrow Cx = 0$

$\frac{d}{d\tau}Ce^{A\tau}x = \frac{d}{d\tau}0$

$CAe^{A\tau}x = 0$

$\tau = 0$: $CAx = 0$

$CA^2x = 0$, ...

$\Rightarrow \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x = 0 \Rightarrow x \in \mathcal{N}(\mathcal{O})$

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P.220 Thm 5.39

x is unobservable

$\Leftrightarrow x \in N(\mathcal{O})$

$\Leftrightarrow x \in N(W_o(0,T))$

$\therefore$ unobservable subspace

$$\boxed{R_{\bar{o}} = N(\mathcal{O}) = N(W_o(0,T))}$$

proof "$\Rightarrow$" x is unobservable

$u(t)=0 \Rightarrow y(t)=0 \quad \forall t$

$\Rightarrow Ce^{At}x = 0 \quad \forall t \geq 0$

$t=0 \Rightarrow Cx = 0$

$\frac{d}{dt}\Big|_{t=0} \Rightarrow CAe^{At}x\Big|_{t=0} = 0$

$CAx = 0$

$\frac{d^2}{dt^2}\Big|_{t=0} \Rightarrow CA^2x = 0$

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} x = 0$$

$\mathcal{O}x = 0$

$\Rightarrow x \in N(\mathcal{O})$

$\Rightarrow x \in N(W_o(0,T))$

"$\Leftarrow$" $\mathcal{O}x = 0$

$\Rightarrow \begin{cases} Cx = 0 \\ CAx = 0 \\ CA^{n-1}x = 0 \end{cases}$

$\Rightarrow CA^n x = C(-\alpha_n A^{n-1} - \alpha_{n-1}A^{n-2} \cdots)x$

$= 0 + 0 + 0 \cdots + 0$

$= 0$

$\Rightarrow CA^{n+1}x = 0$

$Ce^{At}x = C\left(\sum \frac{t^k}{k!} A^k\right)x$

$= Cx + t\,CAx + \cdots$

$= 0 + 0 + \cdots$

$\underline{Ce^{At}x = 0}$

$\Rightarrow x$ is unobservable

P.221 Corollary 5.38

the system is (completely state) observable

(Def: $R_{\bar{o}} = \{0\}$)

$\Leftrightarrow \mathrm{rank}(\mathcal{O}) = n$

$\Leftrightarrow \mathrm{rank}(W_o(0,T)) = n$

$x_0 = W_o^{-1}(0,T)\left[\int_0^T e^{A'\tau}C'\,y(\tau)\,d\tau\right]$

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P.222 Def 5.40

① x is unconstructible

IF the zero-input response is zero $\forall t \leq 0$

i.e. $Ce^{At}x = 0, \ \forall t \leq 0$

② unconstructible subspace $R_{\bar{cn}}$

= the set of all unconstructible states

③ the system is (completely state) constructible

OR (A, C) is constructible

IF the only state that is unconstructible is $x = 0$ i.e. $R_{\bar{cn}} = \{0\}$

P.223 Thm 5.41

$\dot{x} = Ax + Bu$

$y = Cx + Du$

① x is unobservable $\Leftrightarrow$ x is unconstructible

② $R_{\bar{o}} = R_{\bar{cn}}$

③ the system is observable (A, C)

$\Leftrightarrow$ the system is constructible (A, C)

proof

unobservable $\Leftrightarrow Ce^{At}x = 0 \quad \forall t \geq 0$

unconstructible $\Leftrightarrow Ce^{At}x = 0 \quad \forall t \leq 0$

$\Leftrightarrow Cx = 0, \ CAx = 0, \ CA^2x = 0 \ldots$

$\mathcal{O} \qquad W_o(0,T)$

Constructibility Gramian

$W_{cn}(0,T) \triangleq \int_0^T e^{A'(\tau - T)} C'C e^{A(\tau - T)} d\tau$

$W_o(0,T) = e^{A'T}\, W_{cn}(0,T)\, e^{AT}$

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P.223-224 Thm 5.43

$\dot{x} = Ax + Bu$

$y = Cx + Du$

is observable (constructible)

① $\mathrm{rank}\, W_o(0,T) = n$ for some finite $T > 0$

$W_c(0,T)$

② the n columns of Ce^{At} are linearly indep.

$C(sI - A)^{-1}$

$e^{At}B, \quad (sI-A)^{-1}B$

③ $\mathrm{rank}(\mathcal{O}) = n \qquad \mathrm{rank}(\mathcal{C})$

④ $\mathrm{rank}\begin{bmatrix} s_i I - A \\ C \end{bmatrix} = n \qquad \mathrm{rank}\begin{bmatrix} s_i I - A & B \end{bmatrix}$

$\forall s_i \in \mathbb{C}$

or $s_i \in \mathrm{eig}(A)$

$W_c(0,T) = \int_0^T e^{A\tau} B B' e^{A'\tau} d\tau$

$W_o(0,T) = \int_0^T e^{A'\tau} C' C e^{A\tau} d\tau$

$\mathcal{C} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$

$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$

$(x \in \mathbb{R}^n)$

controllable: $x \in R(\mathcal{C}) = \mathbb{R}^n \quad \mathrm{rank}(\mathcal{C}) = n$

$x \in R(W_c(0,T)) = \mathbb{R}^n \quad \mathrm{rank}(W_c(0,T)) = n$

observable: only $x_0 = 0 \in N(\mathcal{O}) \quad \mathrm{rank}(\mathcal{O}) = n$

$x_0 = 0 \in N(W_o(0,T)) \quad \mathrm{rank}(W_o(0,T)) = n$