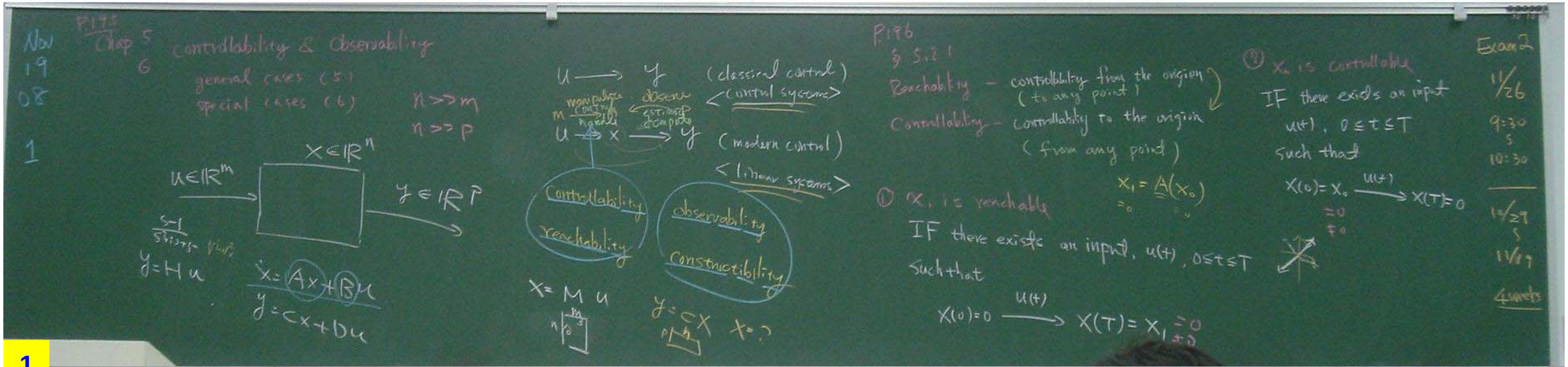
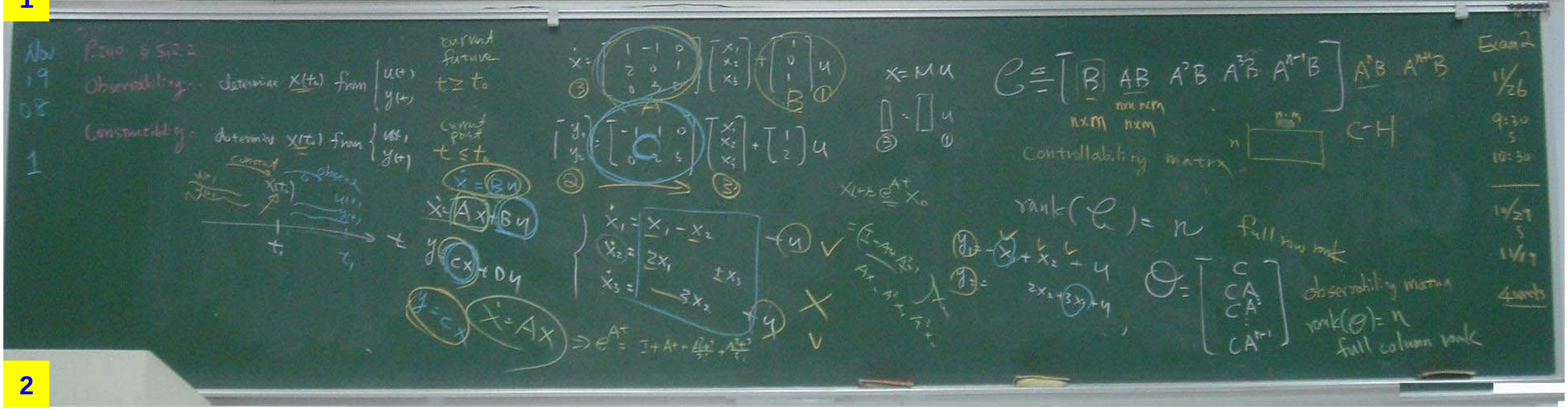
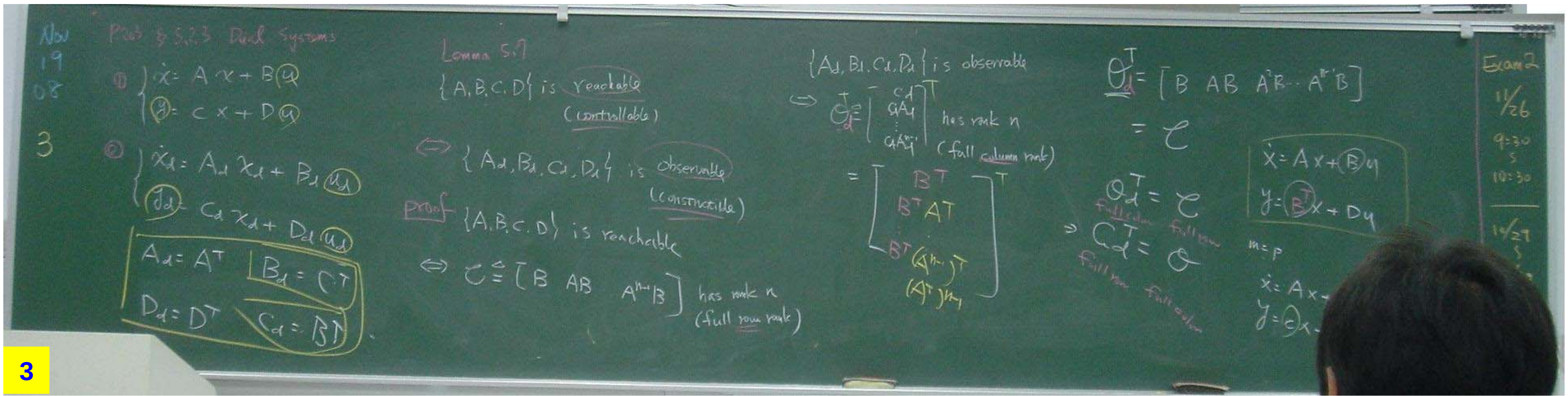

Nov 19 08
1
P.175 Chap 5 6
controllability & Observability
general cases (5)
special cases (6)
n >> m
n >> p
x ∈ ℝ^n
u ∈ ℝ^m
y ∈ ℝ^p
y = H u
ẋ = Ax + Bu
y = Cx + Du
u → y (classical control) <control systems>
manipulate control handle
observe estimate compute
u → x → y (modern control) <linear systems>
controllability
reachability
observability
constructibility
x = M u
y = CX
x = ?
P.196
§ 5.2.1
Reachability - controllability from the origin (to any point)
Controllability - controllability to the origin (from any point)
x₁ = A(x₀)
① x₁ is reachable
IF there exists an input, u(t), 0 ≤ t ≤ T
such that
x(0) = 0 → x(T) = x₁
② x₀ is controllable
IF there exists an input u(t), 0 ≤ t ≤ T
such that
x(0) = x₀ → x(T) = 0
Exam 2
11/26
9:30 ~ 10:30
11/29 ~ 11/19
4 weeks

1


Nov 19 08
1
P.200 § 5.2.2
Observability - determine x(t₀) from u(t), y(t)   t ≥ t₀
current future
Constructibility - determine x(t₀) from u(t), y(t)   t ≤ t₀
current past
ẋ = Bu
ẋ = Ax + Bu
y = Cx + Du
y = Cx
ẋ = Ax
⇒ e^{At} = I + At + A²t²/2! + A³t³/3!
x = M u
x(t) = e^{At} x₀
𝒞 = [B AB A²B A³B Aⁿ⁻¹B] AⁿB Aⁿ⁺¹B
n×m n×m
Controllability matrix
C-H
rank(𝒞) = n full row rank
𝒪 = [C CA CA² ⋮ CAⁿ⁻¹]
observability matrix
rank(𝒪) = n
full column rank
Exam 2
11/26
9:30 ~ 10:30
11/29 ~ 11/19
4 weeks

2


Nov 19 08
3
P.203 § 5.2.3 Dual Systems
① ẋ = Ax + Bu
y = Cx + Du
② ẋ_d = A_d x_d + B_d u_d
y_d = C_d x_d + D_d u_d
A_d = Aᵀ   B_d = Cᵀ
D_d = Dᵀ   C_d = Bᵀ
Lemma 5.7
{A, B, C, D} is reachable (controllable)
⇔ {A_d, B_d, C_d, D_d} is observable (constructible)
proof {A, B, C, D} is reachable
⇔ 𝒞 = [B AB Aⁿ⁻¹B] has rank n (full row rank)
{A_d, B_d, C_d, D_d} is observable
⇔ 𝒪_d = [C_d C_dA_d ⋮ C_dA_dⁿ⁻¹] has rank n (full column rank)
= [Bᵀ BᵀAᵀ ⋮ Bᵀ(Aⁿ⁻¹)ᵀ]ᵀ
𝒪_dᵀ = [B AB A²B ⋯ Aⁿ⁻¹B]
= 𝒞
𝒪_dᵀ = 𝒞
full column full row
⇒ 𝒞_dᵀ = 𝒪
full row full column
ẋ = Ax + Bu
y = Bᵀx + Du
m = p
Exam 2
11/26
9:30 ~ 10:30
11/29

3

Nov 19 08

4

P.204 §5.3 R & C

C.T. L.T.I. systems

$\dot{x} = Ax + Bu$

$x \in \mathbb{R}^n$

$u \in \mathbb{R}^m$ — piecewise continuous

$A \in \mathbb{R}^{n\times n}$

$B \in \mathbb{R}^{n\times m}$

$\Rightarrow x(t) = \Phi(t,t_0)x(t_0) + \int_{t_0}^{t} \Phi(t,\tau)Bu(\tau)d\tau$

S.T.M.

$\Phi(t,t_0) = e^{A(t-t_0)}$

$x(t_0) = x_0 \xrightarrow{u(t)} x(t_1) = x_1$

∵ T.I. $t_0 = 0,\ t_1 = T$

$x_1 = e^{AT}x_0 + \int_0^T e^{A(T-\tau)}B\boxed{u(\tau)}d\tau$

$\underbrace{x_1 - e^{AT}x_0}_{\text{Known}} = \int_0^T e^{A(T-\tau)}B\boxed{u(\tau)}d\tau$

$x = My$, $\mathbb{R}^n$ n:3, $\mathbb{R}^{m}$

$x = Mu$

P.205 Def 5.8

① x_1 is reachable

IF exists $u(t),\ 0 \le t \le T$ such that $x(0)=0 \xrightarrow{u(t)} x(T) = x_1$

x_s

② the reachable subspace R_r = the set of all reachable ...

③ $\dot{x} = Ax + Bu$, (A,B) is (completely state ...) IF every state is reachabl...

$R_r = \mathbb{R}^n$

Exam 2 11/26 9:30 ~ 10:30

10/29 ~ 11/19

5

P.206 Def 5.9

reachability Gramian of $\dot{x} = Ax + Bu$

$z^T z = \|z\|_2^2 \ge 0$

$(M^Ty)^T (M^T)^T (M)^T$, n×m, m×n

$z(y^TM)(M^Ty)^T$

$y^T W_r(0,T) y = y^T\int_0^T e^{A(T-\tau)}BB^Te^{A^T(T-\tau)}d\tau\, y$

n×n

$\mathbb{R}^n \to \mathbb{R}$

$N = MM^T$

$I + A + A^2$

$\mathcal{C} \triangleq [B\ AB\ \cdots\ A^{n-1}B]$, n×nm, $\mathbb{R}^{nm} \to \mathbb{R}^n$

row rank

① $W_r^T = W_r$ (symmetric)

② $W_r \ge 0$, n×n

$y^T W_r y \ge 0 \quad \forall y \in \mathbb{R}^n$

>, ≤, <

Range of $W_r(0,T)$: $\mathcal{R}(W_r(0,T)) \subseteq \mathbb{R}^n$

Range of $\mathcal{C}$: $\mathcal{R}(\mathcal{C}) \subseteq \mathbb{R}^n$

$\mathcal{R}(A)$: $y = Ax \subset \mathbb{R}^n$, n×n

Lemma 5.10

$\mathcal{R}(W_r(0,T)) = \mathcal{R}(\mathcal{C})$

Exam 2 11/26 9:30 ~ 10:30

10/29 ~ 11/19

6

P.207 Thm 5.12

$\dot{x} = Ax + Bu$, $x(0) = 0$, $\mathcal{C} \triangleq [B\ AB\ \cdots\ A^{n-1}B]$

ⓐ $\exists\, u(t)$: $x(0)=0 \xrightarrow{u(t)} x(T) = x_1$

⇔ ⓑ $x_1 \in \mathcal{R}(\mathcal{C})$

⇔ ⓒ $x_1 \in \mathcal{R}(W_r(0,T))$

ⓓ one of possible inputs!

$u(t) = B^T e^{A^T(T-t)}\eta_1$

$W_r(0,T)\eta_1 = x_1$ (n×n, n×1, n×1), $\eta_1 \in \mathbb{R}^n$

$x_1 - e^{AT}x_0 = \int_0^T e^{A(T-\tau)}B\boxed{u(\tau)}d\tau$

$= \int_0^T e^{A(T-\tau)}BB^Te^{A^T(T-\tau)}\eta_1\, d\tau$

$T>0$: $= \left(\int_0^T e^{A(T-\tau)}BB^Te^{A^T(T-\tau)}d\tau\right)\eta_1$, n×n $W_r(0,T)$

$\Rightarrow [B\ AB\ A^2B\ \cdots\ A^{n-1}B]$, nm

$x \ne 0$

$x - 0 = W_r(0,T)\eta_1$

Known

$y = Ax$

$y \in \mathcal{R}(A)$

P.208 Corollary 5.15

$\dot{x} = Ax + Bu$ is completely state reachable

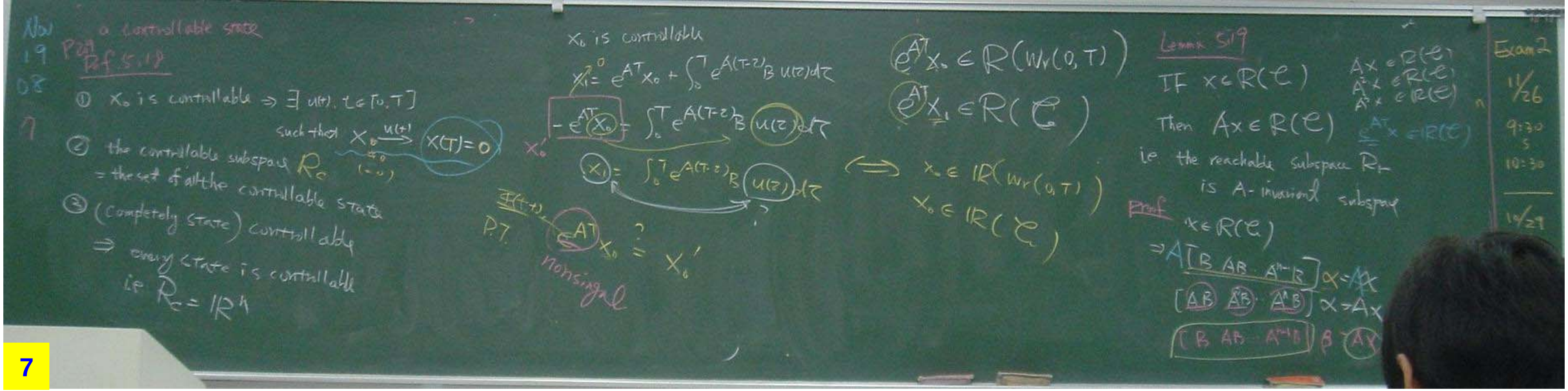
(A,B) is reachable

$\Rightarrow x_0 \xrightarrow{u(t)} x_1$

$u(t) = B^T e^{A^T(T-t)}\left[W_r^{-1}(0,T)\left[x_1 - e^{AT}x_0\right]\right]$, $t \in [0,T]$

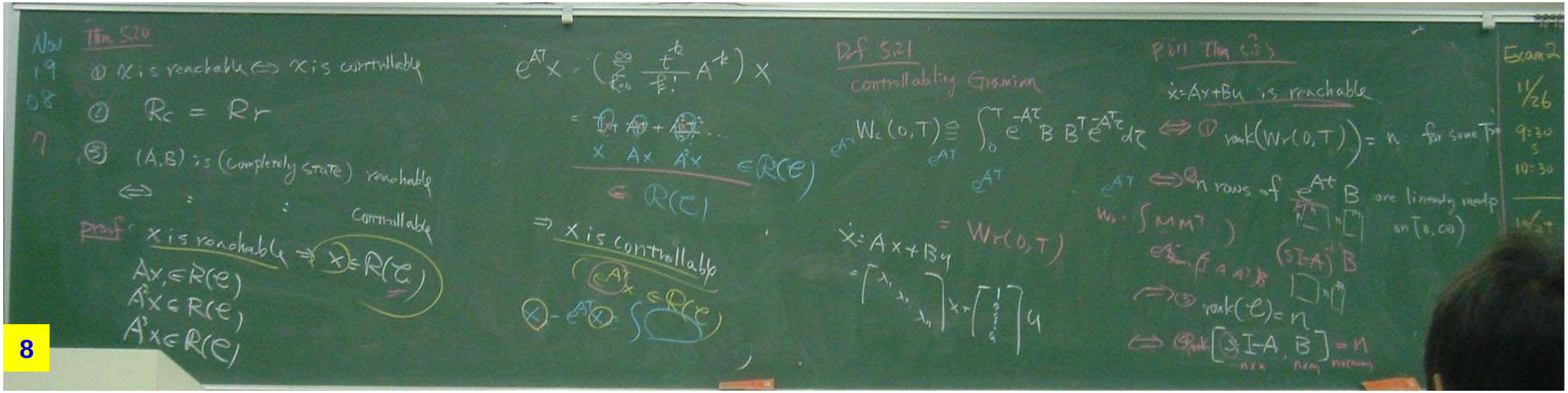
η_1

Exam 2 11/26 9:30 ~ 10:30



Nov 19 08
a controllable state
Def 5.18
① x_0 is controllable $\Rightarrow \exists\, u(t),\ t \in [0,T]$ such that $x_0 \xrightarrow{u(t)} x(T)=0$
② the controllable subspace R_c = the set of all the controllable state
③ (completely state) controllable $\Rightarrow$ every state is controllable i.e. $R_c = \mathbb{R}^n$
x_0 is controllable
$x_1 = e^{AT}x_0 + \int_0^T e^{A(T-\tau)}B\,u(\tau)d\tau$
$-e^{AT}x_0 = \int_0^T e^{A(T-\tau)}B\,u(\tau)d\tau$
$x_1 = \int_0^T e^{A(T-\tau)}B\,u(\tau)d\tau$
P.7
$e^{AT}x_0 = x_0'$
nonsingular
$e^{AT}x_0 \in R(W_r(0,T))$
$e^{AT}x_1 \in R(\mathcal{C})$
$\Longleftrightarrow x_0 \in \mathbb{R}(W_r(0,T))$
$x_0 \in \mathbb{R}(\mathcal{C})$
Lemma 5.19
IF $x \in R(\mathcal{C})$
Then $Ax \in R(\mathcal{C})$
i.e. the reachable subspace R_r is A-invariant subspace
proof $x \in R(\mathcal{C})$
$\Rightarrow A[B\ AB\ \cdots\ A^{n-1}B]\alpha = Ax$
$[AB\ A^2B\ \cdots\ A^nB]\alpha = Ax$
$[B\ AB\ \cdots\ A^{n-1}B]\beta = Ax$
$Ax \in R(\mathcal{C})$
$A^2x \in R(\mathcal{C})$
$A^3x \in R(\mathcal{C})$
$e^{AT}x \in R(\mathcal{C})$
Exam 2
11/26
9:30 ~ 10:30
10/29

7



Nov 19 08
Thm 5.20
① x is reachable $\Leftrightarrow$ x is controllable
② $R_c = R_r$
③ (A,B) is (completely state) reachable $\Leftrightarrow$ controllable
proof x is reachable $\Rightarrow x \in R(\mathcal{C})$
$Ax \in R(\mathcal{C})$
$A^2x \in R(\mathcal{C})$
$A^3x \in R(\mathcal{C})$
$e^{AT}x = \left(\sum_{k=0}^{\infty} \frac{t^k}{k!}A^k\right)x$
$= Ix + Ax + A^2x \cdots \in R(\mathcal{C})$
$\Rightarrow x$ is controllable
Def 5.21
controllability Gramian
$W_c(0,T) \triangleq \int_0^T e^{-A\tau}BB^Te^{-A^T\tau}d\tau$
$= W_r(0,T)$
$\dot{x} = Ax + Bu$
P.11 Thm 5.23
$\dot{x} = Ax + Bu$ is reachable
$\Leftrightarrow$ ① $\mathrm{rank}(W_r(0,T)) = n$ for some $T>0$
$\Leftrightarrow$ ② n rows of $e^{At}B$ are linearly indep on $[0,\infty)$
$(sI-A)^{-1}B$
$\Leftrightarrow$ ③ $\mathrm{rank}(\mathcal{C}) = n$
$\Leftrightarrow$ ④ $\mathrm{rank}[sI-A,\ B] = n$
Exam 2
11/26
9:30 ~ 10:30
10/29

8

9