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P.148 §4.4 Lyapunov Stability of Linear Systems

$\dot{x} = Ax, \; t \ge 0, \quad x(0) = x_0$

$\Rightarrow x_e = 0$ an equilibrium

$\Rightarrow \phi(t, x_0) = \Phi(t, 0)\, x_0$ (S.T.M.)

$= \Phi(t-0, 0)\, x_0$

$= \Phi(t)\, x_0$

$= e^{At} x_0$

$\Phi(t, t_0)$, $e^{A(t-t_0)}$

P.148 Thm 4.12

$x_e = 0$ of $\dot{x} = Ax$ is stable

$\Leftrightarrow$ the solutions of $\dot{x} = Ax$ is bounded

$\Leftrightarrow$ (i.e.) $\sup_{t \ge 0} \|\Phi(t)\| \triangleq k < \infty$

$\Phi(t)$ bounded; STM; matrix norm; finite

proof

"$\Rightarrow$" $x_e$ is stable

$\therefore$ for $\varepsilon = 1$, $\exists\, \delta = \delta(\varepsilon) = \delta(1) > 0$

$\forall \|x_0\| < \delta$ such that $\|\phi(t, x_0)\| < 1$, $\forall t \ge 0$

$\|\phi(t, x_0)\| = \|\Phi(t)\, x_0\, \frac{\delta}{\delta} \frac{\|x_0\|}{\|x_0\|}\|$

$= \|\Phi(t) \frac{x_0}{\|x_0\|} \delta\| \frac{\|x_0\|}{\delta} < \frac{\|x_0\|}{\delta}$

$\|\Phi(t) \frac{x_0}{\|x_0\|} \delta\| \le \|\Phi(t)\| \|\frac{x_0}{\|x_0\|}\delta\| < 1$

$\|Ax\| \le \|A\|\|x\|$

$\Rightarrow \|\Phi(t)\|\, \delta < 1$

$\Rightarrow \|\Phi(t)\| < \frac{1}{\delta}$

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"$\Leftarrow$" all $\phi(t, x_0) = \Phi(t) x_0$ bounded

Then $\{e_1, e_2, \dots, e_n\}$, $e_j = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j\text{th}$

$\|\phi(t, e_j)\| < \beta_j$

$x_0 = \sum_{j=1}^{n} \alpha_j e_j \Leftarrow x_0 = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$

$\|x_0\| = \sum_{j=1}^{n} |\alpha_j|$

$\|\phi(t, x_0)\| = \|\phi(t, \sum_{j=1}^{n} \alpha_j e_j)\|$

$= \|\Phi(t) \sum_{j=1}^{n} \alpha_j e_j\|$

$= \|\sum_{j=1}^{n} \alpha_j \Phi(t) e_j\|$

$= \|\sum_{j=1}^{n} \alpha_j \phi(t, e_j)\|$

$\le \sum_{j=1}^{n} |\alpha_j| \|\phi(t, e_j)\|$ ($< \beta_j$)

$\le \max_j \beta_j \sum_{j=1}^{n} |\alpha_j|$ ($= \|x_0\|$)

$= (\max_j \beta_j) \cdot \|x_0\|$ ($= k$)

$= k \|x_0\|$

For given $\varepsilon > 0$

choose $\delta = \frac{\varepsilon}{k}$

If $\|x_0\| < \delta = \frac{\varepsilon}{k}$

then $\|\phi(t, x_0)\| \le k \|x_0\|$

$\le k \frac{\varepsilon}{k}$

$= \varepsilon$

$\Rightarrow$ By definition, $x_e = 0$ is stable

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P.150 Thm 4.14

$x_e = 0$ of $\dot{x} = Ax$ is A.S.

$\Leftrightarrow x_e = 0$ of $\dot{x} = Ax$ is E.S.

proof

① E.S. $\Rightarrow$ A.S. By definition

② A.S. $\Rightarrow$ E.S.

If it is A.S.

$\exists\, \delta > 0$, and $T > 0$

such that if $\|x_0\| \le \delta$

then $\|\Phi(t+T) x_0\| < \frac{\delta}{2} = \frac{1}{2}\delta$

$\forall t \ge 0$.

$\Rightarrow \|\Phi(t+T)\| < \frac{1}{2}$

$\|\Phi(t+2T)\| = \|\Phi((t+2T)-(t+T))\, \Phi(t+T)\|$

$t \to t+T \to t+2T$

$\le \|\Phi((t+2T)-(t+T))\| \|\Phi(t+T)\|$

$= \|\Phi(T)\| \|\Phi(t+T)\|$

$\le \frac{1}{2} \cdot \frac{1}{2} = (\frac{1}{2})^2$

$\|\Phi(t+nT)\| \le 2^{-n}$

$\|\phi(t+nT, x_0)\|$

$= \|\Phi(t+nT) x_0\|$

$\le 2^{-n} \|x_0\|$

$= 2 \|x_0\| e^{-\frac{\ln 2}{T}(t+nT)}$

$\Rightarrow x_e = 0$ is E.S.

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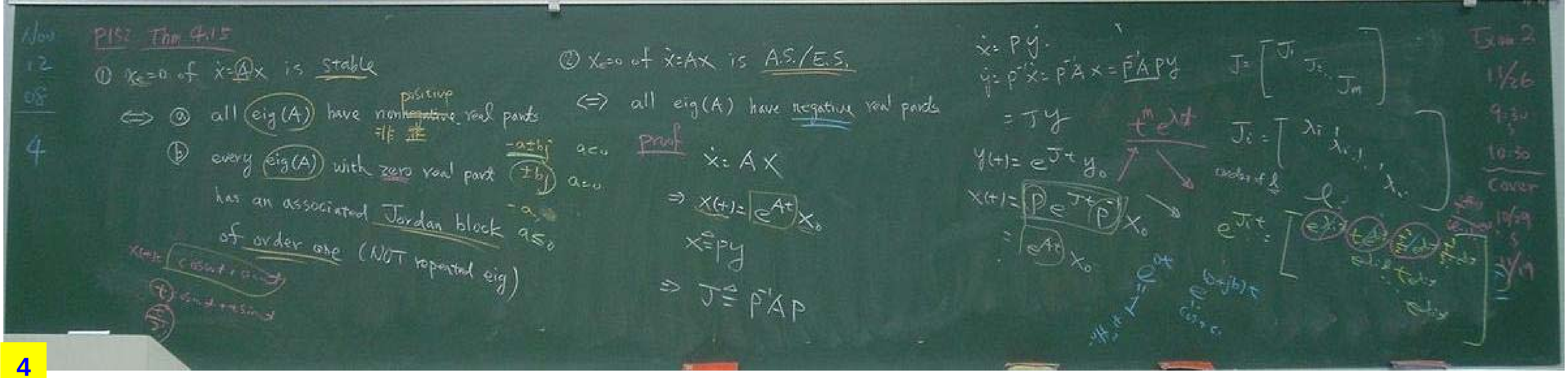
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P152 Thm 4.15
① x_e=0 of ẋ=Ax is stable
⟺ ⓐ all eig(A) have nonnegative positive real parts
ⓑ every eig(A) with zero real part has an associated Jordan block of order one (NOT repeated eig)
② x_e=0 of ẋ=Ax is A.S./E.S.
⟺ all eig(A) have negative real parts
proof ẋ = Ax
⇒ x(t) = e^{At} x_0
x = Py
⇒ J = P^{-1}AP
ẋ = Pẏ
ẏ = P^{-1}ẋ = P^{-1}Ax = P^{-1}APy = Jy
y(t) = e^{Jt} y_0
x(t) = P e^{Jt} P^{-1} x_0 = e^{At} x_0
J = [J_1 J_2 ... J_m]
J_i = [λ_i 1; λ_i 1; ...; λ_i]
e^{J_i t} =
t^m e^{λt}
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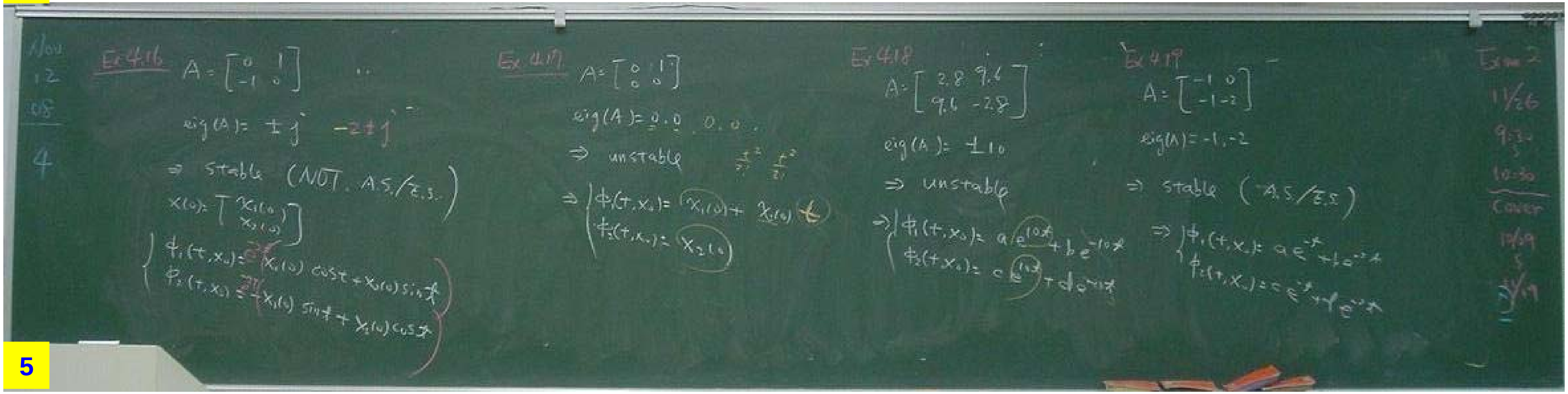
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Ex 4.16 A = [0 1; -1 0]
eig(A) = ±j
⇒ stable (NOT A.S./E.S.)
x(0) = [x_1(0); x_2(0)]
φ_1(t,x_0) = x_1(0) cos t + x_2(0) sin t
φ_2(t,x_0) = -x_1(0) sin t + x_2(0) cos t
Ex 4.17 A = [0 1; 0 0]
eig(A) = 0, 0
⇒ unstable
⇒ φ_1(t,x_0) = x_1(0) + x_2(0) t
φ_2(t,x_0) = x_2(0)
Ex 4.18 A = [2.8 9.6; 9.6 -2.8]
eig(A) = ±10
⇒ unstable
⇒ φ_1(t,x_0) = a e^{10t} + b e^{-10t}
φ_2(t,x_0) = c e^{10t} + d e^{-10t}
Ex 4.19 A = [-1 0; -1 -2]
eig(A) = -1, -2
⇒ stable (A.S./E.S.)
⇒ φ_1(t,x_0) = a e^{-t} + b e^{-2t}
φ_2(t,x_0) = c e^{-t} + d e^{-2t}
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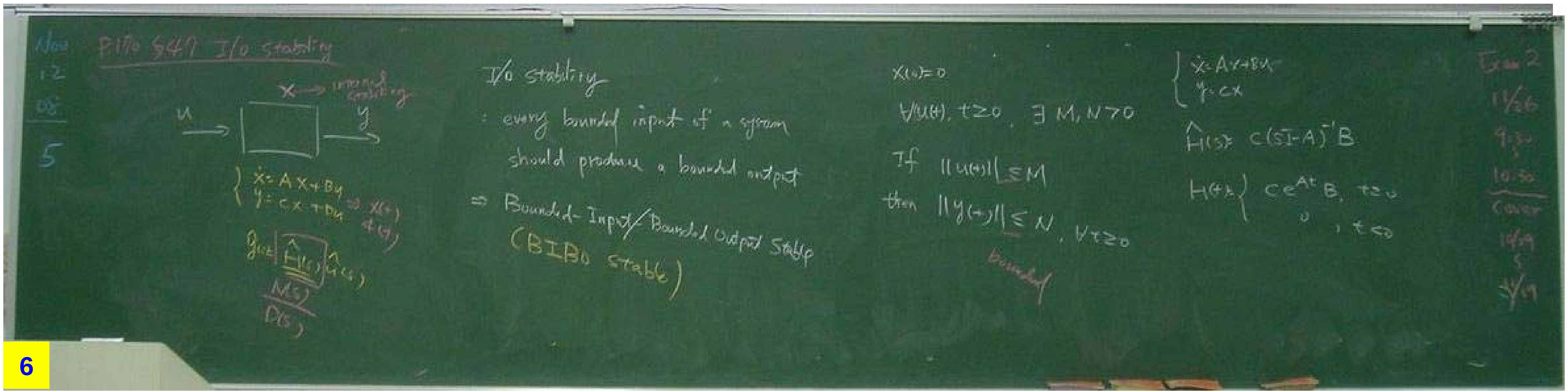
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P170 §4.7 I/O stability
x → internal stability
u → y
ẋ = Ax + Bu
y = Cx + Du
I/O stability
: every bounded input of a system should produce a bounded output
⇒ Bounded-Input/Bounded-Output Stable (BIBO stable)
x(0) = 0
∀u(t), t ≥ 0, ∃ M, N > 0
If ||u(t)|| ≤ M
then ||y(t)|| ≤ N, ∀t ≥ 0
bounded
ẋ = Ax + Bu
y = Cx
Ĥ(s) = C(sI-A)^{-1}B
H(t) = Ce^{At}B, t ≥ 0; 0, t < 0
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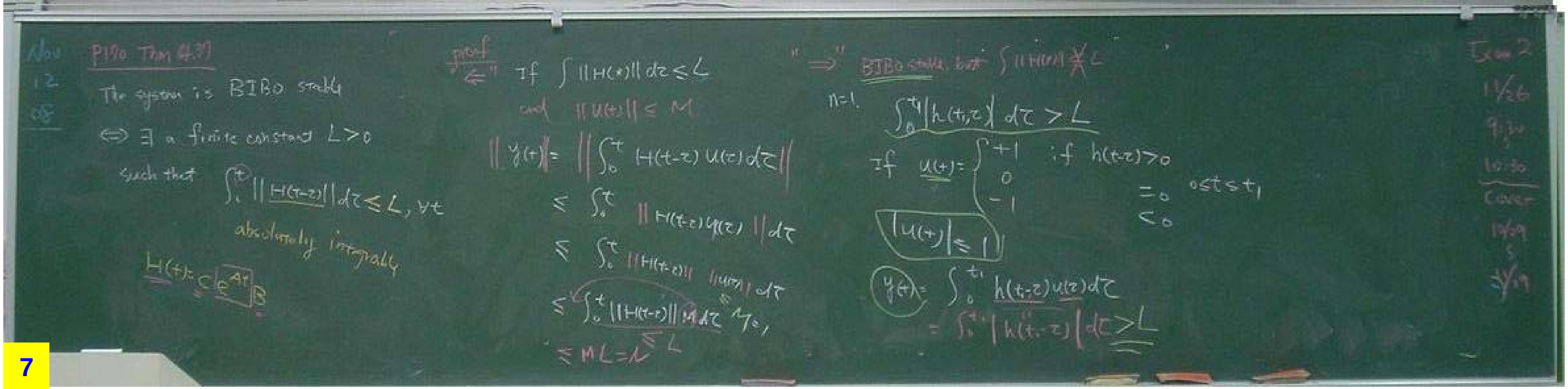
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P190 Thm 4.37
The system is BIBO stable
⟺ ∃ a finite constant L>0
such that ∫₀ᵗ ||H(t-τ)|| dτ ≤ L, ∀t
absolutely integrable
H(t) = C e^{At} B
proof "⟸" If ∫ ||H(τ)|| dτ ≤ L
and ||u(t)|| ≤ M
||y(t)|| = || ∫₀ᵗ H(t-τ) u(τ) dτ ||
≤ ∫₀ᵗ || H(t-τ) u(τ) || dτ
≤ ∫₀ᵗ ||H(t-τ)|| ||u(τ)|| dτ
≤ ∫₀ᵗ ||H(t-τ)|| M dτ
≤ ML = N ≤ L
"⟹" BIBO stable but ∫ ||H(τ)|| ≰ L
n=1
∫₀^{t₁} |h(t₁,τ)| dτ > L
If u(t) = +1 if h(t-τ) > 0; 0 if = 0; -1 if < 0; 0 ≤ t ≤ t₁
|u(t)| ≤ 1
y(t₁) = ∫₀^{t₁} h(t₁-τ) u(τ) dτ
= ∫₀^{t₁} |h(t₁-τ)| dτ > L
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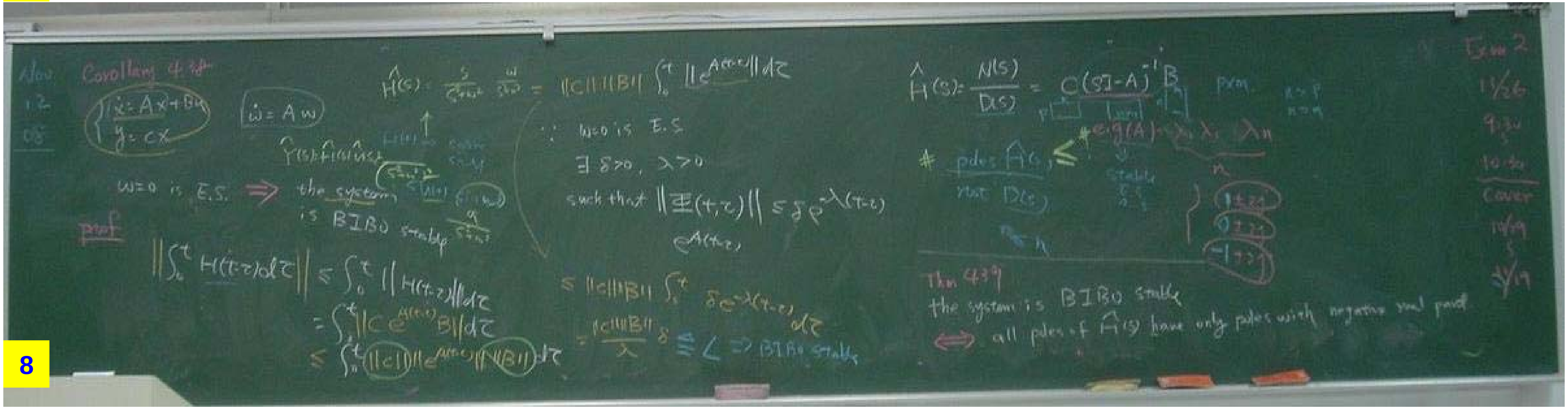
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Corollary 4.38
ẋ = Ax + Bu
y = Cx
ẇ = Aw
w=0 is E.S. ⟹ the system is BIBO stable
proof
|| ∫₀ᵗ H(t-τ) dτ || ≤ ∫₀ᵗ || H(t-τ) || dτ
= ∫₀ᵗ || C e^{A(t-τ)} B || dτ
≤ ∫₀ᵗ ||C|| ||e^{A(t-τ)}|| ||B|| dτ
Ĥ(s) = ||C|| ||B|| ∫₀ᵗ ||e^{A(t-τ)}|| dτ
∵ w=0 is E.S.
∃ δ>0, λ>0
such that ||Φ(t,τ)|| ≤ δ e^{-λ(t-τ)}
e^{A(t-τ)}
≤ ||C|| ||B|| ∫₀ᵗ δ e^{-λ(t-τ)} dτ
= ||C|| ||B|| δ / λ ≤ L ⟹ BIBO stable
Ĥ(s) = N(s)/D(s) = C(sI-A)⁻¹B
e.g.(A) λ₁ λ₂ λₙ
poles Ĥ(s) ≤
roots D(s)
Thm 4.39
the system is BIBO stable
⟺ all poles of Ĥ(s) have only poles with negative real part
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