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P.100 §3.4 State Equation of CT systems — I/O Description

§3.4.1 Response of Linear CT systems

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du \quad \text{State eqn}$$

$$\phi(t, t_0, x_0) = \Phi(t, t_0)\, x_0 + \int_{t_0}^{t} \Phi(t, \tau) B u(\tau)\, d\tau, \qquad \Phi(t,t_0) = e^{A(t-t_0)}$$

$\hat{x}(s), f(s), \hat{y}(s)$; $y = x * h$; $y(s) = \hat{f}(s)\hat{x}(s)$

$$y(t) = C\,\Phi(t, t_0)\, x_0 = y_{zi}(t) \quad \text{zero-input}$$
$$+ C\int_{t_0}^{t} \Phi(t,\tau) B u(\tau)\, d\tau + D u(t) \quad y_{zs}(t) \ \text{zero-state}$$

$$y_{zs}(t) = \int_{t_0}^{t} \left[ C\,\Phi(t,\tau) B + D\,\delta(t-\tau) \right] u(\tau)\, d\tau = \int_{t_0}^{t} H(t,\tau)\, u(\tau)\, d\tau$$

impulse response matrix

P.102 §3.4.2 Transfer function

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \quad x(t_0) = x_0$$

$$\Rightarrow \begin{cases} s\hat{x}(s) - x_0 = A\hat{x}(s) + B\hat{u}(s) \\ \hat{y}(s) = C\hat{x}(s) + D\hat{u}(s) \end{cases}$$

$$\Rightarrow \begin{cases} \hat{x}(s) = (sI-A)^{-1}x_0 + (sI-A)^{-1}B\hat{u}(s) \\ \hat{y}(s) = C(sI-A)^{-1}x_0 + C(sI-A)^{-1}B\hat{u}(s) + D\hat{u}(s) \end{cases}$$

$(sI-A)^{-1} \overset{\mathcal{L}}{\leftrightarrow} e^{At}$

$$y(t) = \mathcal{L}^{-1}\{\hat{y}(s)\} = C e^{A(t-t_0)} x_0 + C\int_{t_0}^{t} e^{A(t-\tau)} B u(\tau)\, d\tau + D u(t)$$

($y_{zi}(t)$, $y_{zs}(t)$)

$$\hat{y}_{zs}(s) = \left[ C(sI-A)^{-1}B + D \right] \hat{u}(s) = \hat{H}(s)\, \hat{u}(s)$$

transfer function

$H(t,\tau) \overset{\mathcal{L}}{\leftrightarrow} \hat{H}(s)$

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§3.4.3 Equivalence of state-space models

$$\begin{cases} \dot{x}_1 = A_1 x_1 + B_1 u \\ y = C_1 x_1 + D_1 u \end{cases}$$

$u \rightarrow \square \rightarrow y$, $x_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}^n$

$x_2 = P x_1$, $P \in \mathbb{R}^{n\times n}$ nonsingular

$x_1 = P^{-1} x_2$

$$\dot{x}_2 = P\dot{x}_1 = P(A_1 x_1 + B_1 u) = PA_1 x_1 + PB_1 u = \underbrace{PA_1P^{-1}}_{A_2} x_2 + \underbrace{PB_1}_{B_2} u$$

$$\boxed{\dot{x}_2 = A_2 x_2 + B_2 u}$$

$$y = \underbrace{C_1 P^{-1}}_{C_2} x_2 + \underbrace{D_1}_{D_2} u$$

$$\boxed{y = C_2 x_2 + D_2 u}$$

$$\begin{cases} A_2 = P A_1 P^{-1} \\ B_2 = P B_1 \\ C_2 = C_1 P^{-1} \\ D_2 = D_1 \end{cases}$$

$A_3 = Q^{-1} A_4 Q$ ($n\times n$)

$B_3 = Q^{-1} B_4$ ($n\times m$)

$C_3 = C_4 Q$ ($p\times n$)

$D_3 = D_4$

$Q x_3 = x_4$

$(ABC)^{-1}(T) = C^{-1}B^{-1}A^{-1}(T)$

$$\hat{H}_1(s) = C_1(sI-A_1)^{-1}B_1 + D_1$$
$$\hat{H}_2(s) = C_2(sI-A_2)^{-1}B_2 + D_2$$
$$= (C_1P^{-1})(sI - PA_1P^{-1})^{-1}PB_1 + D_1$$

$$(sI - PA_1P^{-1})^{-1} = (sPP^{-1} - PA_1P^{-1})^{-1} = \left(P(sI)P^{-1} - PA_1P^{-1}\right)^{-1} = \left(P(sI-A_1)P^{-1}\right)^{-1} = P(sI-A_1)^{-1}P^{-1}$$

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$$\hat{y}^1_{zs}(s) = \hat{H}_1(s)\,\hat{u}(s) = \left(C_1(sI-A_1)^{-1}B_1 + D_1\right)\hat{u}(s)$$

$$\hat{y}^2_{zs}(s) = \left(C_2(sI-A_2)^{-1}B_2 + D_2\right)\hat{u}(s) = \cdots = \hat{y}^1_{zs}(s)$$

$$y^1_{zs}(t) = C_1 e^{A_1(t-t_0)} x_0^1$$
$$y^2_{zs}(t) = C_2 e^{A_2(t-t_0)} x_0^2 = C_1 P^{-1} e^{(PA_1P^{-1})(t-t_0)} P x_0^1$$

$P e^{A(t-t_0)} P^{-1}$

$(PAP^{-1})^2 = PAP^{-1}PAP^{-1}$

$$= C_1 e^{A_1(t-t_0)} x_0^1$$

$$\begin{cases} \dot{x}_1 = A_1 x_1 + B_1 u \\ y = C_1 x_1 + D_1 u \end{cases}$$
$$\begin{cases} \dot{x}_2 = A_2 x_2 + B_2 u \\ y = C_2 x_2 + D_2 u \end{cases}$$
$$\begin{cases} \dot{x}_3 = A_3 x_3 + B_3 u \\ y = C_3 x_3 + D_3 u \end{cases}$$

$A_1 = P_1 A_3 P_1^{-1}$ ⟵ $P_2^{-1} A_2 P_2 = A_3$

$B_1 = P_1 B_3$

$C_1 = C_3 P_1^{-1}$

$D_1 = D_3$

$A_1 = P_1 (P_2^{-1} A_2 P_2) P_1^{-1} = (P_1 P_2^{-1}) A_2 (P_2 P_1^{-1})$, with $P_1P_2^{-1} = P$, $P_2P_1^{-1} = P^{-1}$

$P^{-1} = (P_1P_2^{-1})^{-1} = P_2 P_1^{-1}$

$A_3 = P_2 A_2 P_2^{-1}$

$B_3 = P_2 B_2$

$C_3 = C_2 P_2$

$D_3 = D_2$

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## Chap 4 Stability

$\dot{x}=f(x)$

- equilibrium ⇒ many

Definition

- stability of an equilibrium
  - stable
  - uniformly stable
  - asymptotically stable
  - exponentially stable
  - unstable

$y^{(n)}+a_1y^{(n-1)}+\dots$

$\dot{x}=Ax+Bu$, $y=Cx+Du$ ⇒ $\Phi(t,t_0,x_0)$ … Thm 555

→ A, B, C, D → eig(A)

$\hat{y}(s)=\hat{H}(s)\hat{u}(s)$ → $\hat{H}(s)$ → pole($\hat{H}(s)$)

### P.42 §4.2 the concept of an Equilibrium

$\dot{x}=f(x)$, ~~$x(t_0)=x_0$~~

solution ~~$\phi(t,t_0,x_0)$~~ $\forall t\ge t_0$

(Time-invariant: $\phi(t,x_0)$, $x(0)=x_0$)

Def 4.1 an equilibrium point $x_e$: $f(x_e)=0$ (state space)

$x_0\to x_e$

⇒ $f(x_e)=0_{n\times 1}$

⇒ $\dot{x}=0$

⇒ $\phi(t,x_e)=x_e$ $\forall t\ge t_0$

If $x_e\neq 0$, $f(x_e)=0$, $w\triangleq x-x_e$

$\dot{w}=\dot{x}=f(x)=f(w+x_e)=F(w)$

$f(x_e)=0$ ⇒ $F(0)=0$

$\dot{w}=F(w)$, $F(0)=0$

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### Ex 4.2 (Ex 1.4) Simple Pendulum

$\ddot{x}+k\sin(x)=0$ (nonlinear)

$\Rightarrow \begin{cases}x_1=x\\ x_2=\dot{x}\end{cases}\quad \begin{cases}\dot{x}_1=x_2 & =f_1(x_1,x_2)\\ \dot{x}_2=-k\sin x_1 & =f_2(x_1,x_2)\end{cases}$

$f(x)=0\Rightarrow\begin{cases}f_1(x_1,x_2)=0\\ f_2(x_1,x_2)=0\end{cases}$

$\Rightarrow\begin{cases}x_2=0\\ -k\sin x_1=0\end{cases}$

$\Rightarrow x_2=0$, $x_1=n\pi$, $n=0,\pm1,\pm2,\dots$

mathematically, $\infty$ equilibrium points

physically, 2

$x_e=\begin{pmatrix}n\pi\\ 0\end{pmatrix}$, $n=0,\pm1,\pm2$

### P.44 Ex 4.3 (linear system)

$\dot{x}=Ax$

⇒ $\boxed{Ax=0}$ ⇒ $x\in \text{Null}(A)$

⇒ $x$ ~~$\neq$~~ $0$

If $A$ is nonsingular ⇒ $x_e=0$

If $A$ is singular $x_e=0, a, bc$

### §4.3 Qualitative Characterization of an Equilibrium

$\dot{x}=f(x)$

$f(0)=0$ i.e. $x=0$ is an equilibrium

P.145 Def 4.6 Stable

$x=0$ is stable ($x=x_e$)

IF for every $\varepsilon>0$ there exists a $\delta(\varepsilon)>0$

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### Def 4.6 Stable $\quad\dot{x}=f(x)$

$x=0$ ($x=x_e$) is stable

IF for every finite $\varepsilon>0$

Then exists a finite $\delta(\varepsilon)>0$

whenever norm $\|x_0\|<\delta(\varepsilon)$

such that $\|\phi(t,x_0)\|<\varepsilon$, $\forall t\ge 0$ bounded

P.483 A.7

$\|\cdot\|$: norm

$x=\begin{bmatrix}x_1\\ x_2\end{bmatrix}$

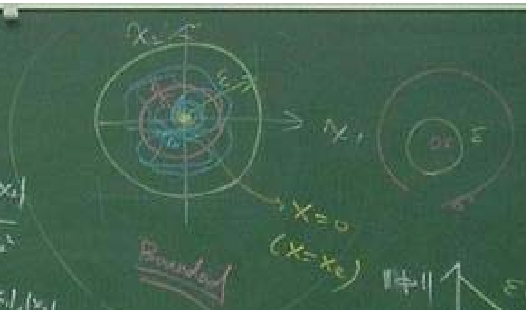
$\|x\|_1=|x_1|+|x_2|$

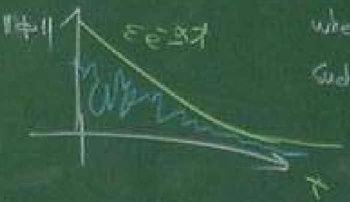
$\|x\|_2=\sqrt{x_1^2+x_2^2}$

$\|x\|_\infty=\max\{|x_1|,|x_2|\}$

$\|x\|_p=\left(|x_1|^p+|x_2|^p\right)^{\frac{1}{p}}$

equilvant




### P.145 Def 4.7 Asymptotically Stable

If ① $x=0$ is stable

② there exists an finite $\eta>0$

whenever $\|x_0\|<\eta$

such that $\lim_{t\to\infty}\phi(t,x_0)=0$

$\lim_{t\to\infty}\|\phi(t,x_0)\|=0$

### Def 4.8 Exponentially stable

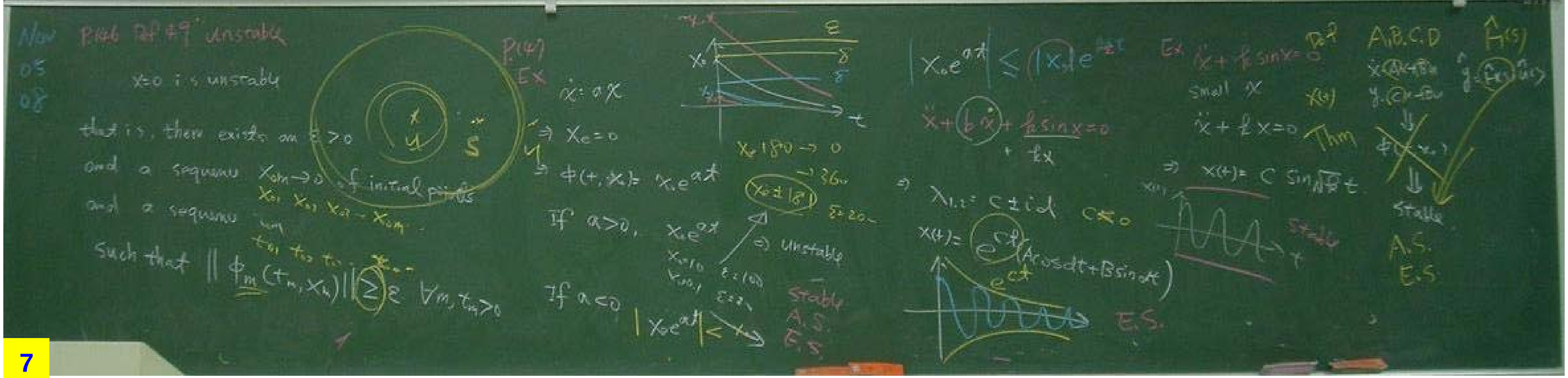
If exists an $\alpha>0$

and for every $\varepsilon>0$

there exists a $\delta(\varepsilon)>0$

whenever $\|x_0\|<\delta(\varepsilon)$

such that $\|\phi(t,x_0)\|\le\varepsilon e^{-\alpha t}$ $\forall t\ge 0$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P.146 Def 4.9' unstable
x=0 is unstable
that is, there exists an ε>0
and a sequence $x_{0m} \to 0$ of initial points
and a sequence $t_{0m}$
such that $\|\phi_m(t_m, x_0)\| \geq \varepsilon \ \forall m, t_m > 0$
P(4) Ex
$\dot{x} = ax$
$\Rightarrow x_e = 0$
$\Rightarrow \phi(t, x_0) = x_0 e^{at}$
If a>0, $x_0 e^{at}$
unstable
If a<0
$|x_0 e^{at}| < \varepsilon$
stable
A.S.
E.S.
$|x_0 e^{at}| \leq |x_0| e^{at}$
$\ddot{x} + k\dot{x} + \frac{g}{l}\sin x = 0$
$\lambda_{1,2} = c \pm id$
$x(t) = e^{ct}(A\cos dt + B\sin dt)$
E.S.
Ex
small x
$\ddot{x} + \frac{g}{l}x = 0$
$\Rightarrow x(t) = C\sin\sqrt{\frac{g}{l}}\,t$
Stable
Def
Thm
A,B,C,D
stable
A.S.
E.S.

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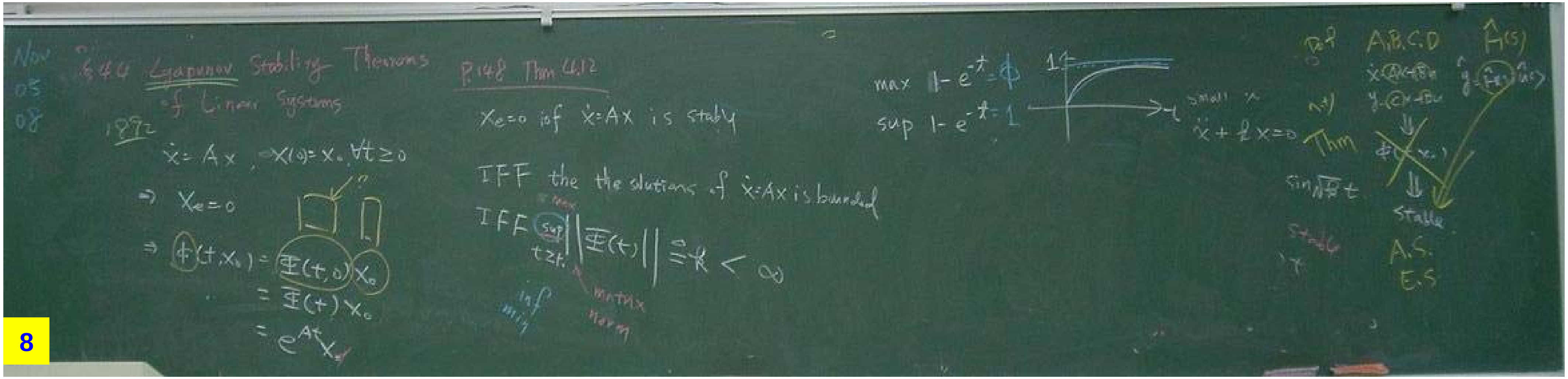
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§4.4 Lyapunov Stability Theorems of Linear Systems
1892
$\dot{x} = Ax$, $x(0) = x_0 \ \forall t \geq 0$
$\Rightarrow x_e = 0$
$\Rightarrow \phi(t, x_0) = \Phi(t,0)x_0$
$= \Phi(t)x_0$
$= e^{At}x_0$
P.148 Thm 4.12
$x_e = 0$ of $\dot{x} = Ax$ is stable
IFF the solutions of $\dot{x} = Ax$ is bounded
IFF $\sup_{t \geq t_0} \|\Phi(t)\| \triangleq k < \infty$
max
inf
min
matrix norm
max $1 - e^{-t} = \phi$
sup $1 - e^{-t} = 1$
Thm
stable
A.S.
E.S.

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