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P86 §3.3.2 How to determine e^{At}

$\dot{x} = Ax$
 sol: $\{\phi_1, \phi_2, \dots, \phi_n\}$
 F.M. $\Phi = [\phi_1 \ \phi_2 \ \dots \ \phi_n]$
 $\Phi^{-1} = [\phi_1^* \ \phi_2^* \ \dots \ \phi_n^*]$
 $\Phi^{-1} \Phi = I$

the S.F.M. $\dot{X} = AX$
 $X(t) = \Xi(t) \Phi$
 $\frac{d}{dt} \Xi = A \Xi$
 $\Xi(t, t_0) = I$
 $\Rightarrow \Phi^{-1} X(t) = \Xi(t, t_0) \Phi^{-1} X(t_0)$
 $\Rightarrow \text{STM}(t, t_0) = \Phi^{-1} \Xi(t, t_0) \Phi$

LTI
 $\text{STM}(t, t_0) = \text{STM}(t, t_0)$
 $= e^{A(t-t_0)}$
 $\Rightarrow e^{At} = I + \sum_{k=1}^{\infty} \frac{t^k}{k!} A^k$

P87
 ① Infinite Series Method
 $e^{At} = I + tA + \frac{t^2}{2!} A^2 + \frac{t^3}{3!} A^3 + \dots$
 $e^{At} = I + tJ + \frac{t^2}{2!} J^2 + \dots$
 ② Similarity Transformation Method
 $x = Py \Rightarrow \dot{y} = Jy$
 $y = P^{-1}x$

$y = P^{-1}x$
 $\dot{y} = P^{-1}Ax$
 $= P^{-1}APy$
 $= Jy$
 $J = P^{-1}AP$

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$\dot{y} = Jy$
 $\Rightarrow \Psi(t, t_0, y_0) = e^{J(t-t_0)} y_0$
 $J = P^{-1}AP \Rightarrow A = PJP^{-1}$
 $x = Py \Rightarrow \dot{x} = P\dot{y}$
 $e^{At} = I + tA + \frac{t^2}{2!} A^2 + \dots$
 $= I + t(PJP^{-1}) + \frac{t^2}{2!} (PJP^{-1})^2 + \dots$
 $= P (I + tJ + \frac{t^2}{2!} J^2 + \dots) P^{-1}$
 $= P e^{Jt} P^{-1}$

$J = P^{-1}AP$
 $A = PJP^{-1}$
 $e^{At} = P e^{Jt} P^{-1}$
 $e^{Jt} = P^{-1} e^{At} P$
 $\Psi(t, t_0, y_0) = (P^{-1} e^{A(t-t_0)} P) (P y_0)$
 $= (P^{-1} e^{A(t-t_0)} P) x_0$
 $\Psi(t, t_0, x_0) = e^{A(t-t_0)} x_0$

case 2a
 "A" has linearly indep eigenvectors v_1, v_2, \dots, v_n
 eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$
 $P = [v_1 \ v_2 \ \dots \ v_n]_{n \times n}$: nonsingular
 $\Rightarrow Av_i = \lambda_i v_i \quad i=1, \dots, n$
 $A[v_1 \ v_2 \ \dots \ v_n] = [v_1 \ v_2 \ \dots \ v_n] \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_n \end{bmatrix}$
 $\Rightarrow AP = PJ$
 $\Rightarrow J = P^{-1}AP$

$e^{At} \rightarrow e^{Jt}$
 $e^{At} = P e^{Jt} P^{-1}$
 $e^{Jt} = P^{-1} e^{At} P$
 $e^{Jt} = I + tJ + \frac{t^2}{2!} J^2 + \dots$
 $= I + t \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_n \end{bmatrix} + \frac{t^2}{2!} \begin{bmatrix} \lambda_1^2 & & \\ & \lambda_2^2 & \\ & & \lambda_n^2 \end{bmatrix} + \dots$
 $= \begin{bmatrix} e^{\lambda_1 t} & & \\ & e^{\lambda_2 t} & \\ & & e^{\lambda_n t} \end{bmatrix}$

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case 2b repeated eigenvalues
 $\lambda_i = a, b, b, c, c, c$
 $v_i = v_1, v_2, v_3, v_4, v_5, v_6$

$A \xrightarrow{J=AP} J = \begin{bmatrix} a & & & & & \\ & b & * & & & \\ & & b & & & \\ & c & & c & & \\ & & c & c & & \\ & & & & c & c \end{bmatrix}$
 Jordan Form

$Ax = \lambda x + b$
 $Ax = \lambda x + b_1$
 $Ax = \lambda x + b_2$
 $J = [J_1 \ J_2 \ \dots \ J_m]$
 $J_i = \begin{bmatrix} \lambda_i & & \\ & \lambda_i & \\ & & \lambda_i \end{bmatrix}$
 $J_i^k = \begin{bmatrix} \lambda_i^k & & \\ & \lambda_i^k & \\ & & \lambda_i^k \end{bmatrix}$

$J_0^k = [x_1 \ x_2 \ \dots \ x_n]^k$
 $\begin{bmatrix} c & 1 & 0 \\ 0 & c & 1 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} c & 1 & 0 \\ 0 & c & 1 \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} c^2 & 2c & 1 \\ c^2 & 2c & 1 \\ c^2 & 2c & 1 \end{bmatrix}$
 $J_i^k = \begin{bmatrix} \lambda_i^k & \binom{k-1}{1} \lambda_i^{k-1} & \dots \\ & \lambda_i^k & \\ & & \lambda_i^k \end{bmatrix}$

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$$C = \begin{bmatrix} c_1 & 0 \\ c_2 & 0 \\ c_3 & 1 \end{bmatrix}$$

$$e^{ct} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + t \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{t^2}{2!} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \dots$$

$$= \begin{bmatrix} 1+t+\frac{t^2}{2!}c_1^2+\frac{t^3}{3!}c_1^3 & 0+t+\frac{t^2}{2!}c_2+\frac{t^3}{3!}c_2^2+\frac{t^4}{4!}c_2^3 & 0+0+\frac{t^2}{2!}1+\frac{t^3}{3!}3c+\frac{t^4}{4!}6c^2+\frac{t^5}{5!}10c^3 \\ e^{ct} & e^{ct} & e^{ct} \end{bmatrix}$$

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PP1 Cayley-Hamilton Thm -

A → char poly

$$\alpha(\lambda) = \det(\lambda I - A) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 \in \mathbb{R}$$

$$\alpha(A) = A^n + a_{n-1}A^{n-1} + \dots + a_1A + a_0I \in \mathbb{R}^{n \times n}$$

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \dots + \frac{t^{n-1}}{(n-1)!}A^{n-1} + \frac{t^n}{n!}A^n + \dots$$

PP 9.93 The Laplace Transform Method

$$\dot{x} = Ax \quad x(0) = x_0 \Rightarrow x(t) = \mathcal{L}^{-1}\{\hat{x}(s)\}$$

$$\mathcal{L}\{\dot{x}\} = A \hat{x}(s) \Rightarrow \hat{x}(s) = (sI - A)^{-1} x_0$$

$$\hat{x}(s) = \mathcal{L}\{x(t)\} \Rightarrow x(t) = \mathcal{L}^{-1}\{(sI - A)^{-1} x_0\}$$

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PP4 Ex 3.18

$$\dot{x}_1 = -x_1 + x_2 \quad x_1(0) = -1$$

$$\dot{x}_2 = -2x_2 + u(t) \quad x_2(0) = 0$$

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$x(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \Rightarrow \hat{x} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \dots = \begin{bmatrix} 1-t & t \\ 0 & 1-2t \end{bmatrix}$$

$$\Phi(t, 0) = \begin{bmatrix} 1-t & t \\ 0 & 1-2t \end{bmatrix}$$

$$x(t) = \Phi(t, 0) x(0) + \int_0^t \Phi(t, \tau) g(\tau) d\tau$$

$$x(t) = \mathcal{L}^{-1}\{(sI - A)^{-1} x_0\} + \mathcal{L}^{-1}\{(sI - A)^{-1} \hat{g}(s)\}$$

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③ Similar Transform

$$P^{-1}AP = J = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$e^{At} = P^{-1} e^{Jt} P = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$$

③ Cayley-Hamilton Thm

$$\alpha(\lambda) = \det(\lambda I - A) = \det \begin{bmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 2 \end{bmatrix} = \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2)$$

$$e^{At} = \alpha(t)I + \beta(t)A$$

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④ Laplace Transform

$$sI - A = \begin{bmatrix} s+1 & -1 \\ 0 & s+2 \end{bmatrix}$$

$$(sI - A)^{-1} = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix}$$

$$e^{At} = \mathcal{L}^{-1} \left\{ \begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix} \right\}$$

$$= \begin{bmatrix} e^{-t} & e^{-t} - e^{-2t} \\ 0 & e^{-2t} \end{bmatrix} = e^{At}$$

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⑤ $u_1 \rightarrow y_1$
 $u_2 \rightarrow y_2$

$$y_1(t) = C \int_0^t e^{A(t-\tau)} u_1(\tau) d\tau$$

$$y_2(t) = C \int_0^t e^{A(t-\tau)} u_2(\tau) d\tau$$

⑥ $u = \alpha_1 u_1 + \alpha_2 u_2 \rightarrow y = \alpha_1 y_1 + \alpha_2 y_2$

⑦ $\dot{x} + k^2 \sin(x) = 0$

$$\dot{x}_1 = \frac{1}{k} \dot{x}$$

$$\dot{x}_2 = \frac{1}{k} \dot{x}$$

$$x = \frac{1}{k} X$$

$$\dot{X} + k^2 \sin\left(\frac{1}{k} X\right) = 0$$

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⑩

$$\begin{aligned} \dot{x}_1 &= kx \\ x_0 &= x \\ \dot{x}_2 &= kx_1 = k^2 x = f_1(x, x_1, u) \\ \dot{x}_0 &= \dot{x}_1 = -k^2 \sin\left(\frac{1}{k} x_1\right) = f_2(x, x_1, u) \\ \dot{x} &= \begin{bmatrix} k & 0 \\ 0 & -k \end{bmatrix} x + u \\ 0 &= k \cdot 0 \\ 0 &= -k \sin(k \cdot 0) \end{aligned}$$

⑪

$$A = \frac{\partial f}{\partial x} \Big|_{x=\phi=0, u=\psi=0} = \begin{bmatrix} 0 & k \\ -k & 0 \end{bmatrix}$$

$$B = \frac{\partial f}{\partial u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$z = x - \phi \quad \psi = u - 0$

$$\dot{z} = A z + B v$$

$$y = x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

⑫

$$A^2 = \begin{bmatrix} 0 & k \\ k & 0 \end{bmatrix} \begin{bmatrix} 0 & k \\ -k & 0 \end{bmatrix} = \begin{bmatrix} -k^2 & 0 \\ 0 & -k^2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & -k^3 \\ k^3 & 0 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} k^4 & 0 \\ 0 & k^4 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t \begin{bmatrix} 0 & k \\ k & 0 \end{bmatrix} + \frac{t^2}{2!} \begin{bmatrix} -k^2 & 0 \\ 0 & -k^2 \end{bmatrix} + \frac{t^3}{3!} \begin{bmatrix} 0 & -k^3 \\ k^3 & 0 \end{bmatrix} + \dots$$

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⑬

$$e^{At} = \begin{bmatrix} \cos kt & \sin kt \\ -\sin kt & \cos kt \end{bmatrix}$$

⑭

$$\begin{aligned} v_1(t) &= \begin{bmatrix} -k \sin kt \\ -k \cos kt \end{bmatrix} \\ v_2(t) &= \begin{bmatrix} k \cos kt \\ -k \sin kt \end{bmatrix} \end{aligned}$$

⑮

$$\det \begin{bmatrix} \cos kt & \sin kt \\ -\sin kt & \cos kt \end{bmatrix} = \cos^2 kt + \sin^2 kt = 1$$

\Rightarrow non-singular

$\Rightarrow v_1, v_2$ linearly indep

$\therefore \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$ is FM

⑯

$$\frac{d}{dt} (e^{At}) = \frac{d}{dt} (I + tA + \frac{t^2}{2!} A^2 + \dots) = A (I + tA + \frac{t^2}{2!} A^2 + \dots) = A e^{At}$$

$\Rightarrow e^{At}$ is a solution of $\dot{z} = Az$

$\det(e^{At}) \neq 0$

$\det(I + tA + \frac{t^2}{2!} A^2 + \dots) \neq 0$

⑰

$$\text{STM}(t, t_0) = FM(t) FM(t_0)^{-1}$$

$$M(t, t_0) = \begin{bmatrix} \cos k(t-t_0) & \sin k(t-t_0) \\ -\sin k(t-t_0) & \cos k(t-t_0) \end{bmatrix}$$

$M(t, t) = I$

⑱

$$\text{STM}(t, t_0) = e^{A(t-t_0)} (e^{-A(t-t_0)})^{-1} = e^{A(t-t_0)} e^{A(t-t_0)} = I$$

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