1

Oct 15 08

1

Thm 3.1

The set of solutions of $\boxed{\dot x = Ax}$ on J forms an n-dim vector space.

$\operatorname{span}\{\phi_1, \phi_2, \dots, \phi_n\} = V$, L.I. $v_1 + v_2 \in V$, $v_1 \in V$, $\alpha v_1 \in V$, $\alpha \in \mathbb{R}, \mathbb{C}$

Def 3.2

the (a) fundamental set of solutions of $\dot x = Ax$ on J

$\Rightarrow$ a set of linearly indep solutions $\{\phi_1(t)\ \phi_2(t) \cdots \phi_n(t)\}$

a fundamental matrix of $\dot x = Ax$

$\Rightarrow \Phi(t) = [\phi_1(t)\ \phi_2(t) \cdots \phi_n(t)]$ $n \times n$

Thm 3.3

A fundamental matrix of $\dot x = Ax$ satisfies $\dot X = AX$ on J, $X \in \mathbb{R}^{n\times n}$

Thm 3.4

If Φ is a solution of $\boxed{\dot X = AX}$ on J

then for $\tau \in J$, $\forall t \in J$

$$\det \Phi(t) = \det \Phi(\tau) \exp\left[\int_\tau^t \operatorname{tr} A \, ds\right]$$

Thm 3.5

a solution $\Phi(t)$ of $\boxed{\dot X = AX}$, $X \in \mathbb{R}^{n\times n}$, is a fundamental matrix of $\boxed{\dot x = Ax}$

iff (if and only if) $\det \Phi(t) \neq 0$, $\forall t \in J$

10/22 9:30 ~ 10:30 Exam 1

10:40 ~ 11:30

HW2 Matlab

2

Oct 15 08

2

Proof

$\Rightarrow$ If $\Phi(t) = [\phi_1(t)\ \phi_2(t) \cdots \phi_n(t)]$ is a fundamental matrix of $\dot x = Ax$

$\{\phi_1(t), \phi_2(t), \dots, \phi_n(t)\}$ L.I.

$\exists\ \phi(t)$ a nontrivial solution of $\dot x = Ax$

By uniqueness, $\exists\ \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}, \mathbb{C}$ such that $\phi(t) = \sum_{i=1}^{n} \alpha_i \phi_i(t)$

$$= [\phi_1\ \phi_2 \cdots \phi_n] \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}_{n\times 1}$$

$\boxed{\phi(t)}_{n\times 1} = \underbrace{\Phi(t)}_{n\times n}\ \underbrace{\alpha}_{n\times 1} \Rightarrow \det \Phi(t) \neq 0$

$y = Ax$

$\Rightarrow x = A^{-1} y$

$\Rightarrow$ unique solution

a MIMO system

"$\Leftarrow$" $\Phi(t)$ is a solution of $\dot X = AX \Rightarrow \phi_i(t)$ is a solution of $\dot x = Ax$

assume that $\det \Phi(t) \neq 0$ on J

to prove $\Phi(t)$ is a fundamental matrix of $\dot x = Ax$

$\Phi(t) = [\phi_1(t)\ \phi_2(t) \cdots \phi_n(t)]$ $n \times n$, L.I.

$\therefore \det \Phi(t) \neq 0$

$\Rightarrow \{\phi_1(t), \phi_2(t), \dots, \phi_n\}$ L.I.

$\Rightarrow \Phi(t)$ is F.M. of $\dot x = Ax$

P.81 Thm 3.6

If Φ is an F.M. of $\dot x = Ax$

If C is any nonsingular (invertible) constant matrix $C \in \mathbb{R}^{n\times n}$ or $\mathbb{C}^{n\times n}$

Then ① $\Phi(t) C$ is also an F.M. of $\dot x = Ax$

② Ψ is any F.M. of $\dot x = Ax$

$\Rightarrow \exists\ P \in \mathbb{R}^{n\times n}$ or $\mathbb{C}^{n\times n}$ such that $\Psi = \Phi P$

10/22 9:30 ~ 10:30 Exam 1

10:40 ~ 11:30

HW2 Matlab

3

Oct 15 08

3

proof

① $\Phi(t) = [\phi_1\ \phi_2 \cdots \phi_n]$

(a) $\phi_i(t)$ is a solution of $\dot x = Ax$

(b) $\{\phi_i\}$ L.I.

(a) $\frac{d}{dt}\boxed{\Phi(t) C} = \left(\frac{d}{dt}\Phi(t)\right) C = \left(A \Phi(t)\right) C = A\,\boxed{\Phi(t) C}$

$\Phi(t) C$ is a solution of $\dot X = AX$

columns of $\Phi(t) C$ is a solution of $\dot x = Ax$

(b) $\det(\Phi(t) C) = \det \Phi(t) \det C$, $\det \Phi(t) \neq 0$, $\det C \neq 0$

② $\frac{d}{dt}\left(\Phi(t)\Phi^{-1}(t)\right) = \frac{d}{dt} I$

$\left(\frac{d}{dt}\Phi(t)\right)\Phi^{-1}(t) + \Phi(t)\frac{d}{dt}\Phi^{-1}(t) = 0$

$\frac{d}{dt}\Phi^{-1}(t) = -\Phi^{-1}(t)\left(\frac{d}{dt}\Phi(t)\right)\Phi^{-1}(t)$

$\frac{d}{dt}\left(\Phi^{-1}(t)\Psi(t)\right)$

$= \left(\frac{d}{dt}\Phi^{-1}(t)\right)\Psi(t) + \Phi^{-1}(t)\frac{d}{dt}\Psi(t)$

$= -\Phi^{-1}(t)\Big(\underbrace{\frac{d}{dt}\Phi(t)}_{=A\Phi(t)}\Big)\Phi^{-1}(t)\,\Psi(t) + \Phi^{-1}(t)\underbrace{\frac{d}{dt}\Psi(t)}_{=A\Psi(t)}$

$= -\Phi^{-1} A \underbrace{\Phi \Phi^{-1}}_{I} \Psi + \Phi^{-1} A \Psi = 0_{n\times n}$

$\Phi^{-1}(t)\Psi(t) = \text{constant} \triangleq P$

$\det(\Psi(t)) = \det \Phi(t)\, P$, $\det(\Psi) = \det \Phi \det P$

$\Phi(t) = \Psi(t) P^{-1}$

$\det(AB) = \det A \det B$

10/22 9:30 ~ 10:30 Exam 1

10:40 ~ 11:30

HW2 Matlab

Oct 15 08

4

HW#1

A 矩陣維度不正確

B 變數引用錯誤

C 方程式列錯

D 使用近似公式時 未說明理由

E 計算錯誤

F 計算過程跳太多

Ex 3.7 $\ddot{x} + 2\dot{x} + 5x = \cdots$

$\dot{x}_1 = 5x_1 - 2x_2$

$\Rightarrow \dot{x}_2 = 4x_1 - x_2$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\phi_1(t) = \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix}$, $\phi_2(t) = \begin{bmatrix} e^{t} \\ 2e^{t} \end{bmatrix}$

$\Rightarrow$ linearly indep

$\phi_3(t) = \begin{bmatrix} 2e^{3t} - e^{t} \\ 2e^{3t} - 2e^{t} \end{bmatrix} = 2\phi_1(t) - \phi_2(t)$

$\alpha \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} + \beta \begin{bmatrix} e^{t} \\ 2e^{t} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\forall t \in J \Rightarrow \alpha = \beta = 0$

$\Phi(t) = \begin{bmatrix} e^{3t} & e^{t} \\ e^{3t} & 2e^{t} \end{bmatrix}$

F.M., $\Phi(0) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

$\Rightarrow \Psi(t)$? $\Psi(0) = I$

$\Psi(t) = \Phi(t) P$

$\Psi(0) = \Phi(0) P$

$I = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} P$

$P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$\Psi(t) = \Phi(t) P$

$= \begin{bmatrix} e^{3t} & e^{t} \\ e^{3t} & 2e^{t} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2e^{3t} - e^{t} & -e^{3t} + e^{t} \\ 2e^{3t} - 2e^{t} & -e^{3t} + 2e^{t} \end{bmatrix}$

10/22 9:30 ~ 10:30 Exam 1

10:40 ~ 11:30 講 HW2 Hwlab

4

Oct 15 08

5

P82 §3.2.2 State Transition Matrix (S.T.M.)

$\dot{x} = A(t)x$

$\Rightarrow$ solution $\phi(t, t_0, x_0) = \Phi(t, t_0) x_0$

$\phi(t_0, t_0, x_0) = x_0$

$$\begin{cases} \frac{d}{dt}\Phi(t, t_0) = A(t)\Phi(t, t_0) \\ \Phi(t_0, t_0) = I \end{cases}$$

P83 Def 3.8 S.T.M.

on F.M. $\Phi(t)$ of $\dot{x} = Ax$

$\Phi(t) = [\phi_1(t)\ \phi_2(t) \cdots \phi_n(t)]$

$\phi_1(t), \phi_2(t), \cdots, \phi_n(t)$: L.I. solutions

$\phi_1(t_0) = e_1$, $\phi_2(t_0) = e_2$, $\cdots$ $\phi_n(t_0) = e_n$

$e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i$

$\Rightarrow \Phi(t)$ is S.T.M. of $\dot{x} = Ax$

$\Phi(t_0) = I$

IF $\Psi(t)$ is any F.M. of $\dot{x} = Ax$

$\Rightarrow$ S.T.M.

$\Phi(t, t_0) = \Psi(t) \Psi^{-1}(t_0)$

F.M. F.M.$^{-1}(t_0, t_0)$

$\Phi(t_0, t_0) = \Psi(t_0)\Psi^{-1}(t_0)$

$= I$

$\Psi_1(t) = \Psi_2(t) P$ F.M.

$\Psi_1(t) \Psi_1^{-1}(t_0)$

$= (\Psi_2(t) P)(\Psi_2(t_0) P)^{-1}$

A B

$= \Psi_2(t) P (P^{-1})(\Psi_2(t_0))^{-1}$

B^{-1} A^{-1} I

$= \Psi_2(t) \Psi_2^{-1}(t_0)$

10/22 9:30 ~ 10:30 Exam 1

10:40 ~ 11:30 講 HW2 Hwlab

5

Oct 15 08

6

P83 Properties of the S.T.M.

Thm 3.9

$t_0 \in J$ $\Phi(t, t_0)$: S.T.M. of $\dot{x} = Ax$ $\forall t \in J$

$\phi(t_0) = x_0$

$\dot{\underline{x}} = A\underline{x}$

① $\Phi(t, t_0)$: the unique solution of $\begin{cases} \frac{d}{dt}\Phi(t, t_0) = A\Phi(t, t_0) \\ \boxed{\Phi(t_0, t_0) = I} \end{cases}$ v.F.M.

② $\Phi(t, t_0)$ is nonsingular, $\forall t \in J$

$\phi(t_0, t, x_1)$

$= \Phi(t_0, t) x_1$

③ $t, \sigma, \tau \in J$ ($\cancel{t \le \sigma \le \tau}$)

$\Phi(t, \tau) = \Phi(t, \sigma) \Phi(\sigma, \tau)$

(semigroup property)

④ $[\Phi(t, t_0)]^{-1} \overset{\text{def}}{\triangleq} \Phi^{-1}(t, t_0)$

$= \Phi(t_0, t)$

$\forall t, t_0 \in J$

$\approx \approx$ $\triangleq$ $:=$

⑤ The unique solution $\phi(t, t_0, x_0)$ of $\dot{x} = Ax$ with $\phi(t_0, t_0, x_0) = x_0$

is $\boxed{\phi(t, t_0, x_0) = \Phi(t, t_0) x_0}$

$\forall t \in J$

proof

① $\Psi(t)$: any F.M. of $\dot{x} = Ax$

$\Phi(t, t_0) = \Psi(t) \Psi^{-1}(t_0)$

S.T.M. F.M.

$\frac{d}{dt}\Phi(t, t_0) = \frac{d}{dt}(\Psi(t)) \Psi^{-1}(t_0)$

$= A\Psi(t) \Psi^{-1}(t_0)$

$= A \Phi(t, t_0)$

$\Phi(t_0, t_0) = \Psi(t_0) \Psi^{-1}(t_0)$

$= I$

10/22 9:30 ~ 10:30 Exam 1

10:40 ~ 11:30 講 HW2 Hwlab

6

7

Oct 15 08 — 7

② $\det(\Phi(t,t_0)) = \det(\Psi(t)\Psi^{-1}(t_0)) = \underbrace{\det\Psi(t)}_{\neq 0}\,\underbrace{\det\Psi^{-1}(t_0)}_{\neq 0}$ $\Rightarrow \neq 0$

③ $\Phi(t,\tau) = \boxed{\Psi(t)\Psi^{-1}(\sigma)}\,\boxed{\Psi(\sigma)\Psi^{-1}(\tau)} = \Phi(t,\sigma)\,\Phi(\sigma,\tau)$

④ $(\Phi(t,t_0))^{-1} = (\Psi(t)\Psi^{-1}(t_0))^{-1} = (\Psi^{-1}(t_0))^{-1}\Psi^{-1}(t) = \Psi(t_0)\Psi^{-1}(t) = \Phi(t_0,t)$

⑤ $\phi(t,t_0,x_0) = \Phi(t,t_0)x_0$

$\frac{d}{dt}\phi(t,t_0,x_0) = \frac{d}{dt}\Phi(t,t_0)x_0 = A\Phi(t,t_0)x_0 = A\phi(t,t_0,x_0)$, $\dot{x}=Ax$

$\phi(t_0,t_0,x_0) = \Phi(t_0,t_0)x_0 = Ix_0 = x_0$

$\phi(t_0,t_0,x_0) = x_0$

P85 §3.8 Matrix Exponential e^{At}

$\dot{x}=Ax$

S.T.M. $\Phi(t,t_0) = e^{A(t-t_0)}$

In section 1.8 (P.30)

$\phi(t,t_0,x_0) = \left(I + \sum_{k=1}^{\infty}\frac{A^k(t-t_0)^k}{k!}\right)x_0 = \Phi(t,t_0)x_0$

TI $= \Phi(t-t_0)x_0$

LTI $= e^{A(t-t_0)}x_0$

10/22 9:30 ~ 10:30 Exam 1; 10:40 ~ 11:30 討論 HW2 Matlab

8

Oct 15 08 — 8

P85 Def 3.12

$A \in \mathbb{R}^{n\times n}$ or $\mathbb{C}^{n\times n}$

$e^{At} \triangleq I + \sum_{k=1}^{\infty}\frac{t^k}{k!}A^k$, $-\infty < t < \infty$

matrix exponential

$A = \begin{bmatrix} a & b \\ c & d\end{bmatrix}$

exp(A) $= \begin{bmatrix} e^a & e^b \\ e^c & e^d\end{bmatrix}$

expm(A) $= \begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix} + \begin{bmatrix} a & b \\ c & d\end{bmatrix} + \begin{bmatrix} \frac{a^2}{2!} & \frac{b^2}{2!} \\ \frac{c^2}{2!} & \frac{d^2}{2!}\end{bmatrix}$

help exp

help expm

Thm 3.13 $J \subset \mathbb{R}$, $t \in J$

$A \in \mathbb{R}^{n\times n}$ or $\mathbb{C}^{n\times n}$

$\dot{x} = Ax$

① $\Phi(t) \triangleq e^{At}$, an F.M. of $\dot{x}=Ax$

② $\Phi(t,t_0) \triangleq e^{A(t-t_0)}$ $\forall t \in J$ $= \Phi(t-t_0)$, $\forall t, t_0 \in J$ — the S.T.M. of $\dot{x}=Ax$

③ $e^{At_1}e^{At_2} = e^{A(t_1+t_2)}$, $\forall t_1,t_2 \in J$

④ $Ae^{At} = e^{At}A$, $\forall t \in J$

⑤ $(e^{At})^{-1} = e^{-At}$ $\forall t \in J$

Proof $\dot{X} = AX$

① $\det e^{At} \neq 0$

$\frac{d}{dt}(e^{At}) = \frac{d}{dt}\left(I + \sum_{k=1}^{\infty}\frac{t^k}{k!}A^k\right)$

$= 0 + \sum_{k=1}^{\infty}\frac{k\,t^{k-1}}{k!}A^k$

$= A\sum_{k=1}^{\infty}\frac{t^{k-1}}{(k-1)!}A^{k-1}$, $m = k-1$

$= A\left(\sum_{m=0}^{\infty}\frac{t^m}{m!}A^m\right)$

$= A\left(I + \sum_{k=1}^{\infty}\frac{t^k}{k!}A^k\right) = e^{At}$

10/22 9:30 ~ 10:30 Exam 1; 10:40 ~ 11:30 討論 HW2 Matlab

9a

Oct 15 08 — 9

$e^{A(0)} = I + 0_{n\times n} = I$

$\det e^{A(0)} \neq 0$

$\det\Phi(t) = \det\Phi(\tau)\exp\int_\tau^t \mathrm{tr}A\,ds$

$\det(e^{At}) = \underbrace{\det(e^{A(0)})}_{\neq 0}\exp\int_0^t \mathrm{tr}A\,ds$

$\neq 0$ $\qquad \underbrace{\exp(\mathrm{tr}A)(t-0)}_{\neq 0}$

$\Rightarrow e^{At}$ is an F.M. of $\dot{x}=Ax$

② $\Phi(t,t_0) = e^{A(t-t_0)}$

$\Rightarrow$ F.M.

$\Phi(t_0,t_0) = e^{A(t_0-t_0)} = e^{A(0)} = I$

$\Rightarrow e^{A(t-t_0)}$ is the S.T.M. of $\dot{x}=Ax$

③ $\Phi(t,t_0) = \Psi(t)\Psi^{-1}(t_0)$

$e^{A(t-t_0)} = e^{At}(e^{At_0})^{-1}$

$e^{A(t_1-(-t_2))} = e^{At_1}(e^{-At_2})^{-1}$

$e^{A(t_1+t_2)} = e^{At_1}(\quad)$

$\Phi(t_2,-t_1) = \Phi(t_2,0)\Phi(0,-t_1)$

$e^{A(t_2+t_1)} = e^{A(t_2-0)}e^{A(0-(-t_1))} = e^{At_2}e^{At_1}$

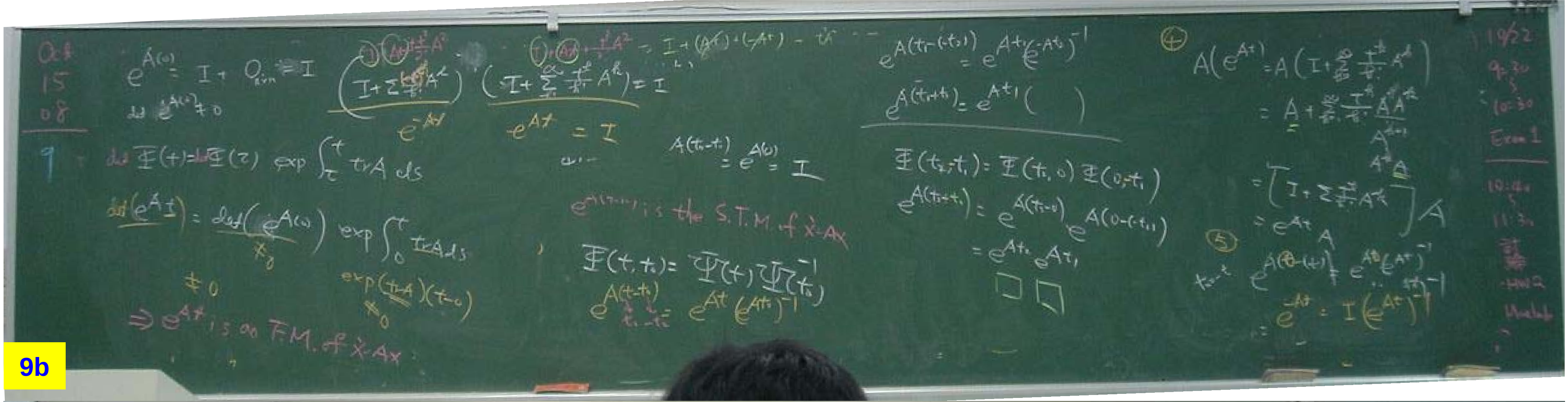
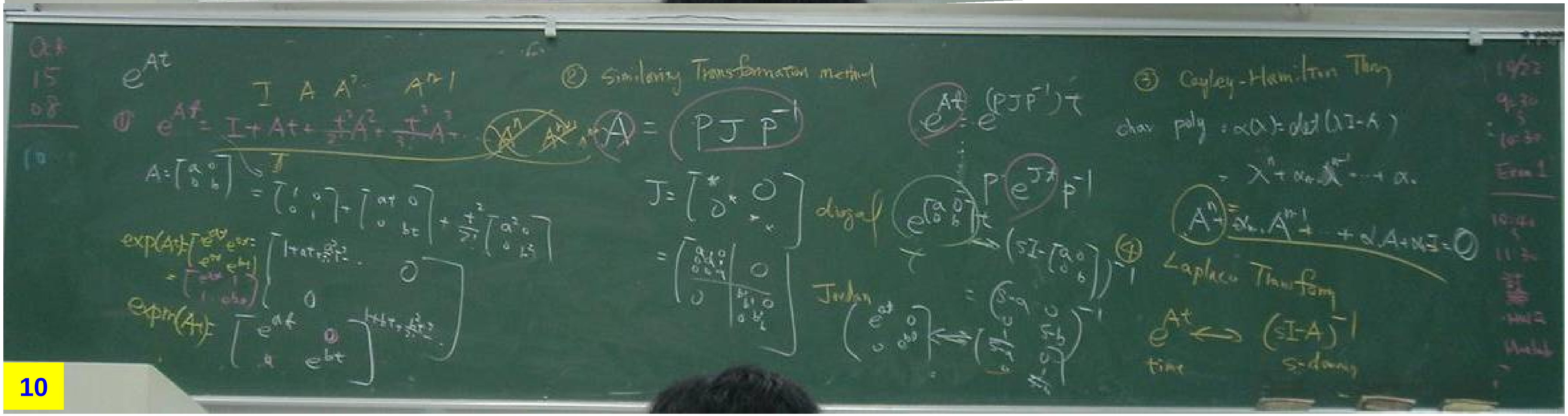
④ $A(e^{At}) = A\left(I + \sum_{k=1}^{\infty}\frac{t^k}{k!}A^k\right)$

$= A + \sum_{k=1}^{\infty}\frac{t^k}{k!}AA^k$

$= \left[I + \sum\frac{t^k}{k!}A^k\right]A$

$= e^{At}A$

10/22 9:30 ~ 10:30 Exam 1; 10:40 ~ 11:30 討論 HW2 Matlab


9b
Oct 15 08
10/22
9:30 ~ 10:30
Exam 1
$e^{A(t_1-t_0)}$ is the S.T.M. of $\dot{x}=Ax$
$\Rightarrow e^{At}$ is an F.M. of $\dot{x}=Ax$

10
Oct 15 08
10/22
9:30 ~ 10:30
Exam 1
② Similarity Transformation method
③ Cayley-Hamilton Thm
char poly
④ Laplace Transform
diagonal
Jordan
time
s-domain