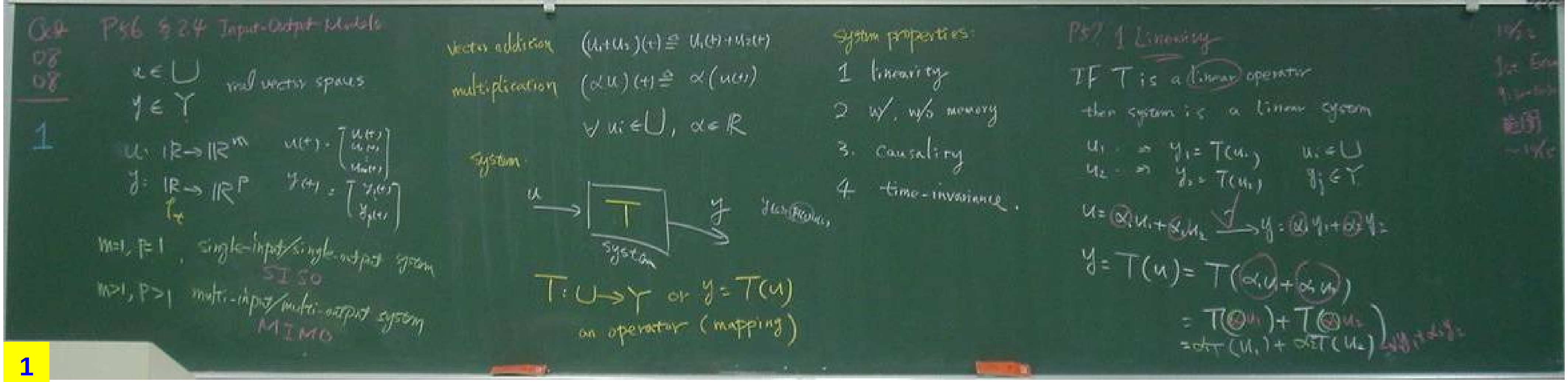

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P56 §2.4 Input-Output Models

$u \in U$ real vector spaces
$y \in Y$

$u: \mathbb{R} \to \mathbb{R}^m$ $u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}$

$y: \mathbb{R} \to \mathbb{R}^p$ $y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_p(t) \end{bmatrix}$

m=1, P=1 , single-input/single-output system
SISO

m>1, P>1 multi-input/multi-output system
MIMO

vector addition $(u_1+u_2)(t) \triangleq u_1(t)+u_2(t)$

multiplication $(\alpha u)(t) \triangleq \alpha(u(t))$

$\forall u_i \in U, \alpha \in R$

system

$T: U \to Y$ or $y = T(u)$
an operator (mapping)

system properties:
1 linearity
2 w/, w/o memory
3. causality
4 time-invariance.

P57 1 Linearity

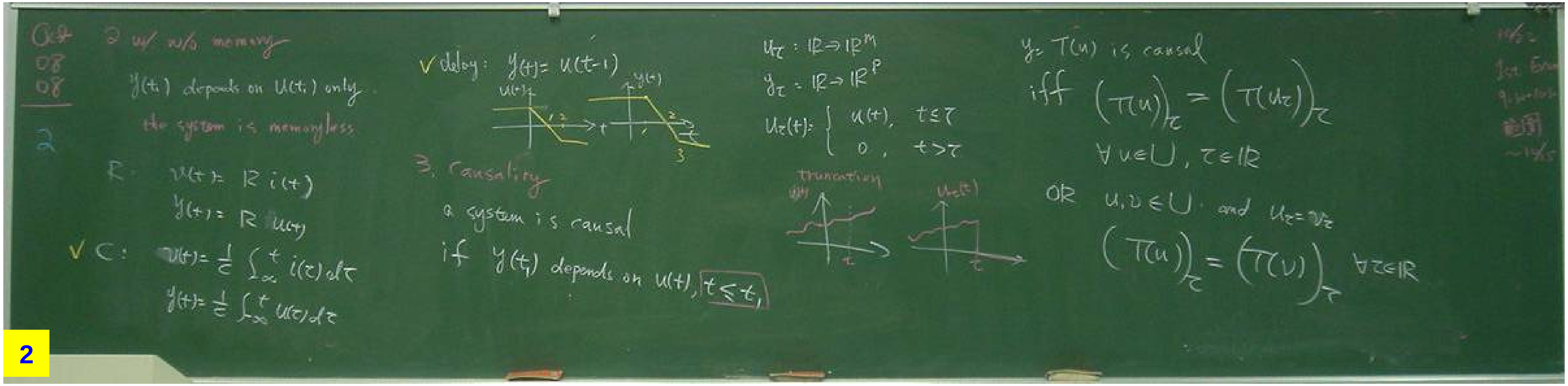
IF T is a linear operator
then system is a linear system

$u_1 \Rightarrow y_1 = T(u_1)$ $u_i \in U$
$u_2 \Rightarrow y_2 = T(u_2)$ $y_j \in Y$

$u = \alpha_1 u_1 + \alpha_2 u_2 \xrightarrow{\checkmark} y = \alpha_1 y_1 + \alpha_2 y_2$

$y = T(u) = T(\alpha_1 u_1 + \alpha_2 u_2)$
$= T(\alpha_1 u_1) + T(\alpha_2 u_2)$
$= \alpha_1 T(u_1) + \alpha_2 T(u_2) = \alpha_1 y_1 + \alpha_2 y_2$

1st Exam
範圍
~11/5



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2 w/ w/o memory

$y(t_1)$ depends on $u(t_1)$ only.
the system is memoryless

R: $v(t) = R\, i(t)$
$y(t) = R\, u(t)$

✓ C: $v(t) = \frac{1}{C} \int_{-\infty}^{t} i(\tau)\, d\tau$
$y(t) = \frac{1}{C} \int_{-\infty}^{t} u(\tau)\, d\tau$

✓ delay: $y(t) = u(t-1)$

3. Causality

a system is causal
if $y(t_1)$ depends on $u(t)$, $\boxed{t \le t_1}$

$u_\tau : \mathbb{R} \to \mathbb{R}^m$
$y_\tau : \mathbb{R} \to \mathbb{R}^p$

$u_\tau(t) = \begin{cases} u(t), & t \le \tau \\ 0, & t > \tau \end{cases}$

truncation

$y = T(u)$ is causal
iff $(T(u))_\tau = (T(u_\tau))_\tau$
$\forall u \in U, \tau \in \mathbb{R}$

OR $u, v \in U$ and $u_\tau = v_\tau$
$(T(u))_\tau = (T(v))_\tau$ $\forall \tau \in \mathbb{R}$

1st Exam
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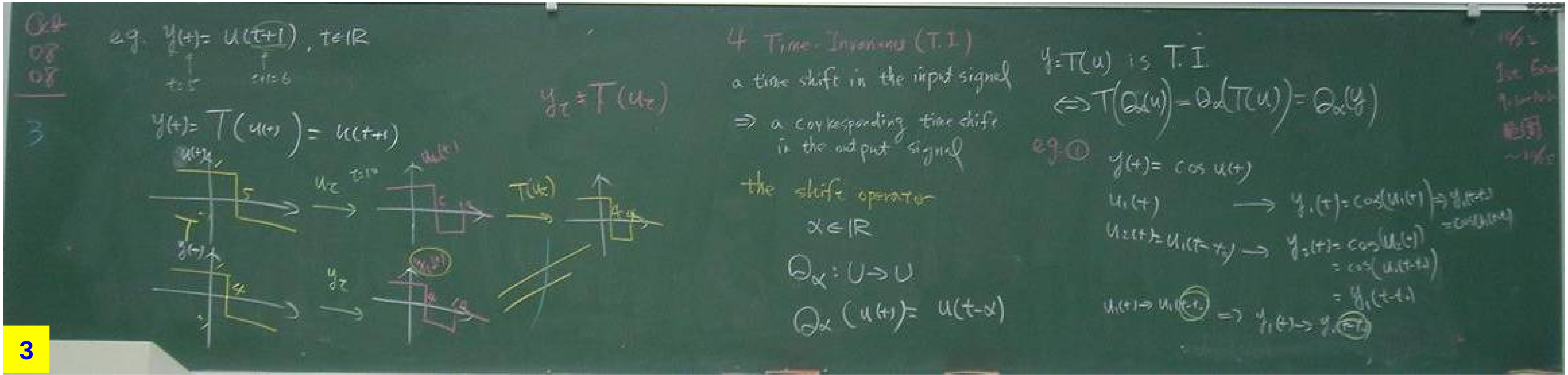

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e.g. $y(t) = u(t+1)$, $t \in \mathbb{R}$
t=5 t+1=6

$y(t) = T(u(t)) = u(t+1)$

$y_\tau \ne T(u_\tau)$

4 Time-Invariance (T.I)

a time shift in the input signal
⇒ a corresponding time shift in the output signal

the shift operator
$\alpha \in \mathbb{R}$
$Q_\alpha : U \to U$
$Q_\alpha(u(t)) = u(t-\alpha)$

$y = T(u)$ is T.I
$\Leftrightarrow T(Q_\alpha(u)) = Q_\alpha(T(u)) = Q_\alpha(y)$

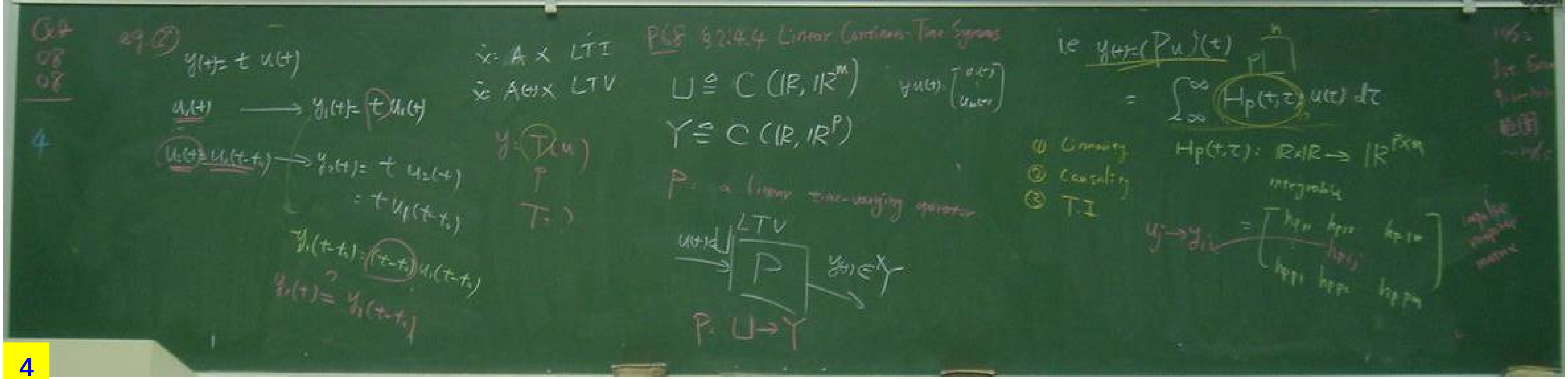
e.g.① $y(t) = \cos u(t)$

$u_1(t) \longrightarrow y_1(t) = \cos(u_1(t)) \Rightarrow y_1(t-t_0) = \cos(u_1(t-t_0))$

$u_2(t) = u_1(t-t_0) \longrightarrow y_2(t) = \cos(u_2(t))$
$= \cos(u_1(t-t_0))$
$= y_1(t-t_0)$

$u_1(t) \Rightarrow u_1(t-t_0) \Rightarrow y_1(t) \to y_1(t-t_0)$

1st Exam
範圍
~11/5



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eg.②

$y(t) = t\,u(t)$

$u_1(t) \longrightarrow y_1(t) = t\,u_1(t)$

$u_2(t) = u_1(t-t_0) \longrightarrow y_2(t) = t\,u_2(t) = t\,u_1(t-t_0)$

$y_1(t-t_0) = (t-t_0)\,u_1(t-t_0)$

$y_2(t) \overset{?}{=} y_1(t-t_0)$

$\dot{x} = Ax$ LTI

$\dot{x} = A(t)x$ LTV

$y = T(u)$

P

T:)

P68 §2.4.4 Linear Continuous-Time Systems

$U \triangleq C(\mathbb{R}, \mathbb{R}^m)$ $\quad u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix}$

$Y \triangleq C(\mathbb{R}, \mathbb{R}^p)$

P: a linear time-varying operator

LTV

$u(t) \in U \rightarrow \boxed{P} \rightarrow y(t) \in Y$

$P: U \rightarrow Y$

ie $y(t) = (Pu)(t)$

$= \int_{-\infty}^{\infty} H_p(t,\tau)\, u(\tau)\, d\tau$

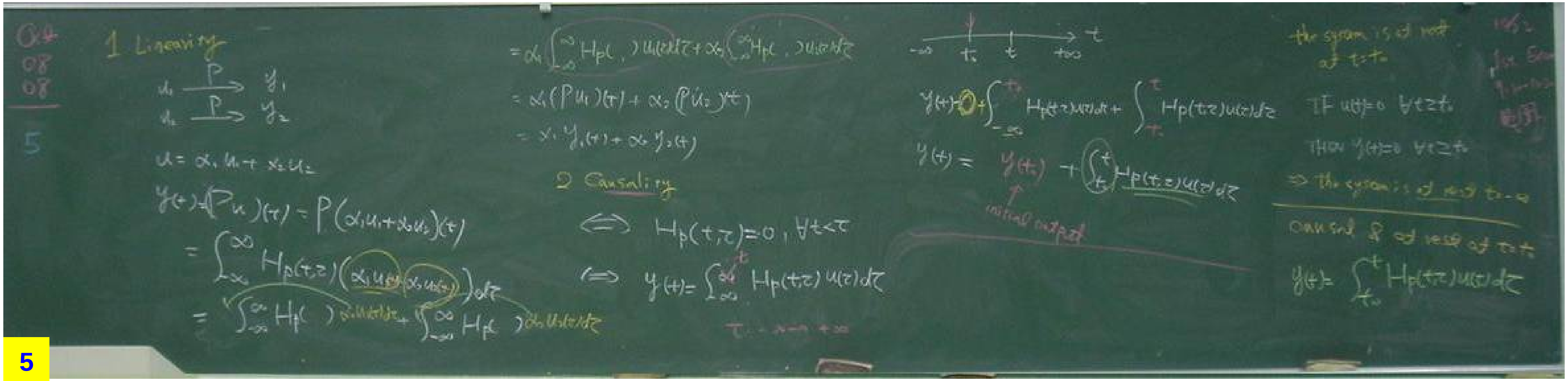
① Linearity
② Causality
③ T.I

$H_p(t,\tau): \mathbb{R}\times\mathbb{R} \rightarrow \mathbb{R}^{p\times m}$

integrable

$u_j \rightarrow y_i = \begin{bmatrix} h_{p11} & h_{p12} & & h_{p1m} \\ & & h_{pij} & \\ h_{pp1} & h_{pp2} & & h_{ppm} \end{bmatrix}$

impulse response matrix



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1 Linearity

$u_1 \xrightarrow{P} y_1$

$u_2 \xrightarrow{P} y_2$

$u = \alpha_1 u_1 + \alpha_2 u_2$

$y(t) = (Pu)(t) = P(\alpha_1 u_1 + \alpha_2 u_2)(t)$

$= \int_{-\infty}^{\infty} H_p(t,\tau)(\alpha_1 u_1(\tau) + \alpha_2 u_2(\tau))\,d\tau$

$= \int_{-\infty}^{\infty} H_p(\ )\,\alpha_1 u_1(\tau)\,d\tau + \int_{-\infty}^{\infty} H_p(\ )\,\alpha_2 u_2(\tau)\,d\tau$

$= \alpha_1 \int_{-\infty}^{\infty} H_p(\ ,\ )\,u_1(\tau)\,d\tau + \alpha_2 \int_{-\infty}^{\infty} H_p(\ ,\ )\,u_2(\tau)\,d\tau$

$= \alpha_1 (Pu_1)(t) + \alpha_2 (Pu_2)(t)$

$= \alpha_1 y_1(t) + \alpha_2 y_2(t)$

2 Causality

$\Longleftrightarrow H_p(t,\tau) = 0,\ \forall t < \tau$

$\Longleftrightarrow y(t) = \int_{-\infty}^{\cancel{\infty}\ t} H_p(t,\tau)\,u(\tau)\,d\tau$

$\tau: -\infty \rightarrow +\infty$

$-\infty \quad t_0 \quad t \quad +\infty \rightarrow t$

$y(t) = 0 + \int_{-\infty}^{t_0} H_p(t,\tau)u(\tau)d\tau + \int_{t_0}^{t} H_p(t,\tau)u(\tau)d\tau$

$y(t) = y(t_0) + \int_{t_0}^{t} H_p(t,\tau)u(\tau)d\tau$

initial output

the system is at rest at $t = t_0$

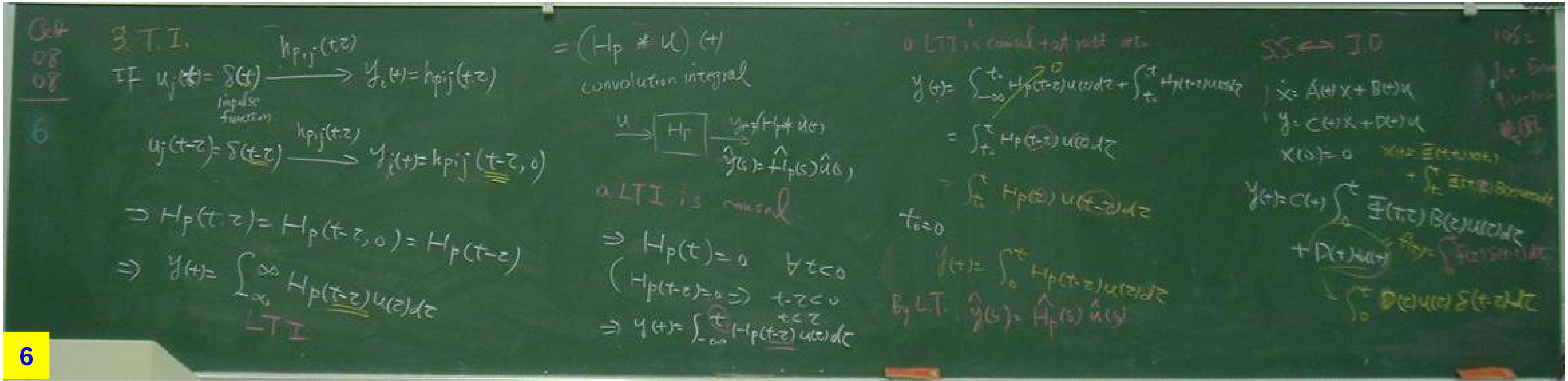
IF $u(t) = 0\ \forall t \geq t_0$

THEN $y(t) = 0\ \forall t \geq t_0$

$\Rightarrow$ the system is at rest $t_0 - \infty$

causal & at rest at $t = t_0$

$y(t) = \int_{t_0}^{t} H_p(t,\tau)u(\tau)d\tau$



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3 T.I.

IF $u_j(t) = \delta(t) \xrightarrow{h_{pij}(t,\tau)} y_i(t) = h_{pij}(t,\tau)$

impulse function

$u_j(t-\tau) = \delta(t-\tau) \xrightarrow{h_{pij}(t,\tau)} y_i(t) = h_{pij}(t-\tau, 0)$

$\Rightarrow H_p(t,\tau) = H_p(t-\tau, 0) = H_p(t-\tau)$

$\Rightarrow y(t) = \int_{-\infty}^{\infty} H_p(t-\tau)\,u(\tau)\,d\tau$

LTI

$= (H_p * u)(t)$

convolution integral

$u \rightarrow \boxed{H_p} \rightarrow y = (H_p * u)(t)$

$\hat{y}(s) = \hat{H}_p(s)\,\hat{u}(s)$

a LTI is causal

$\Rightarrow H_p(t) = 0 \quad \forall t < 0$

$(H_p(t-\tau) = 0 \Rightarrow t - \tau < 0$

$t < \tau$

$\Rightarrow y(t) = \int_{-\infty}^{t} H_p(t-\tau)\,u(\tau)\,d\tau$

a LTI is causal + at rest at t_0

$y(t) = \int_{-\infty}^{t_0} H_p(t-\tau)u(\tau)d\tau + \int_{t_0}^{t} H_p(t-\tau)u(\tau)d\tau$

$= \int_{t_0}^{t} H_p(t-\tau)\,u(\tau)\,d\tau$

$= \int_{t_0}^{t} H_p(\tau)\,u(t-\tau)\,d\tau$

$t_0 = 0$

$y(t) = \int_{0}^{t} H_p(t-\tau)u(\tau)d\tau$

By L.T. $\hat{y}(s) = \hat{H}_p(s)\,\hat{u}(s)$

SS $\Longleftrightarrow$ I.O

$\dot{x} = A(t)x + B(t)u$

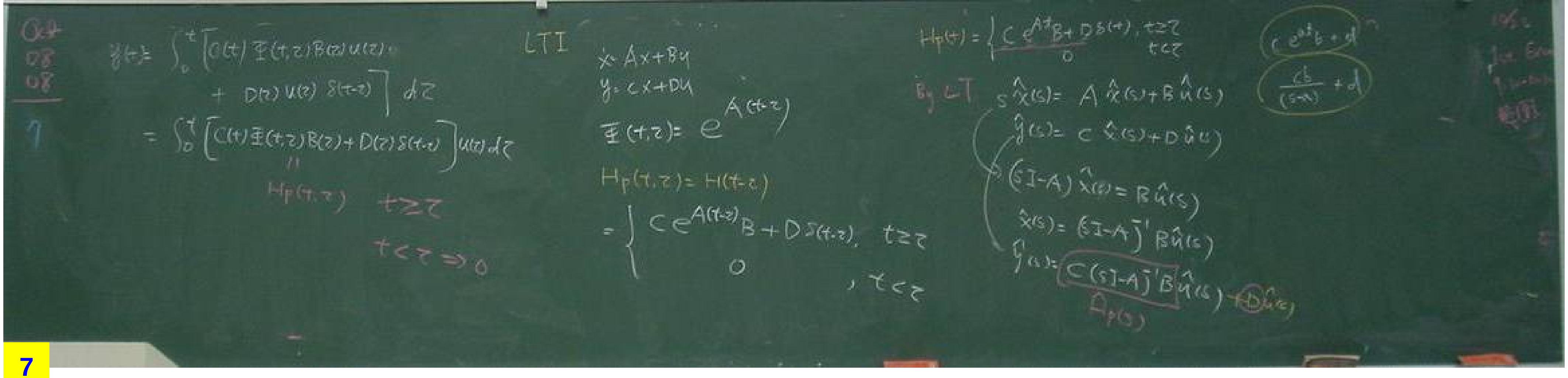
$y = C(t)x + D(t)u$

$x(0) = 0$ $\quad x(t) = \Phi(t,t_0)x(t_0) + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau$

$y(t) = C(t)\int_{0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau + D(t)u(t)$

$f(t) = \int f(\tau)\delta(t-\tau)d\tau$

$\int_{0}^{t} D(\tau)u(\tau)\delta(t-\tau)d\tau$



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$y(t) = \int_0^t \Big[C(t)\Phi(t,\tau)B(\tau)u(\tau) + D(\tau)u(\tau)\delta(t-\tau) \Big] d\tau$

$= \int_0^t \Big[\underbrace{C(t)\Phi(t,\tau)B(\tau) + D(\tau)\delta(t-\tau)}_{H_p(t,\tau)} \Big] u(\tau)\, d\tau$

$t \geq \tau$

$t < \tau \Rightarrow 0$

LTI

$\dot{x} = Ax + Bu$

$y = Cx + Du$

$\Phi(t,\tau) = e^{A(t-\tau)}$

$H_p(t,\tau) = H(t-\tau)$

$= \begin{cases} Ce^{A(t-\tau)}B + D\delta(t-\tau), & t \geq \tau \\ 0, & t < \tau \end{cases}$

$H_p(t) = \begin{cases} Ce^{At}B + D\delta(t), & t \geq \tau \\ 0 & t < \tau \end{cases}$

By LT

$s\hat{x}(s) = A\hat{x}(s) + B\hat{u}(s)$

$\hat{y}(s) = C\hat{x}(s) + D\hat{u}(s)$

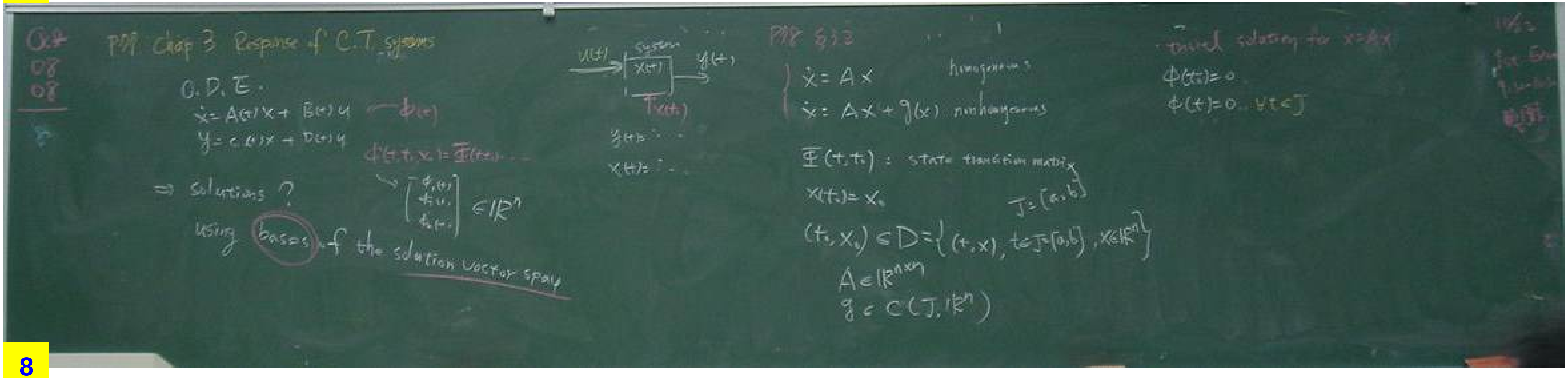
$(sI-A)\hat{x}(s) = B\hat{u}(s)$

$\hat{x}(s) = (sI-A)^{-1}B\hat{u}(s)$

$\hat{y}(s) = \underbrace{\boxed{C(sI-A)^{-1}B}}_{\hat{H}_p(s)}\hat{u}(s) + D\hat{u}(s)$

$ce^{at}b + d$

$\dfrac{cb}{(s-a)} + d$



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P.98 Chap 3 Response of C.T. systems

O.D.E.

$\dot{x} = A(t)x + B(t)u \leftarrow \phi(t)$

$y = C(t)x + D(t)u$

$\phi(t, t_0, x_0) = \Phi(t,t_0) \ldots$

$\begin{bmatrix} \phi_1(t) \\ \phi_2(t) \\ \phi_n(t) \end{bmatrix} \in \mathbb{R}^n$

$\Rightarrow$ solutions ?

using bases of the solution vector space

$u(t) \rightarrow \boxed{x(t)} \rightarrow y(t)$

system

$T(x(t))$

$y(t) = \ldots$

$x(t) = \ldots$

P.98 §3.2

$\begin{cases} \dot{x} = Ax & \text{homogeneous} \\ \dot{x} = Ax + g(x) & \text{nonhomogeneous} \end{cases}$

$\Phi(t,t_0)$: state transition matrix

$x(t_0) = x_0$

$J = (a, b)$

$(t_0, x_0) \in D = \{ (t,x),\ t \in J = (a,b),\ x \in \mathbb{R}^n \}$

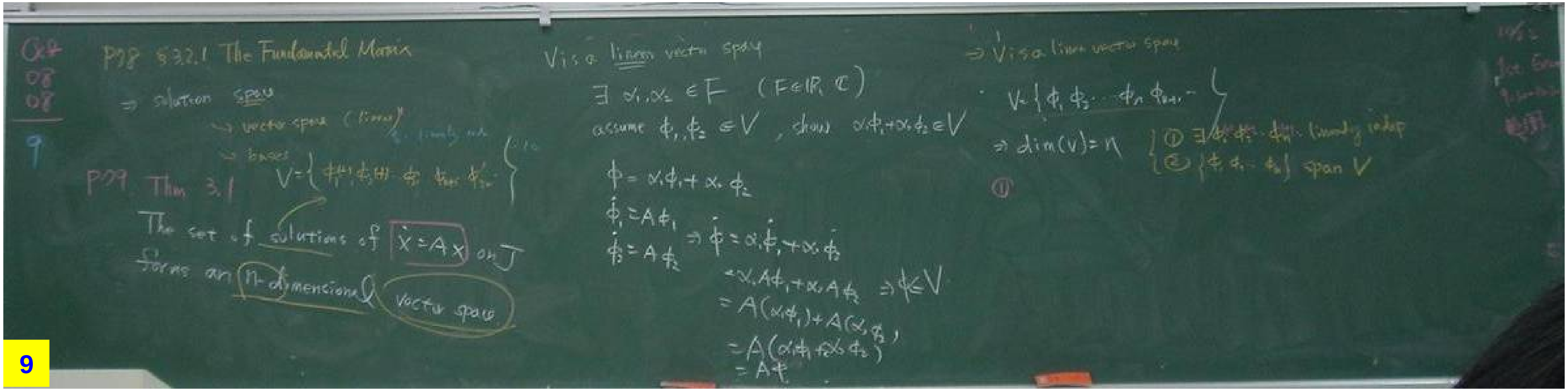
$A \in \mathbb{R}^{n \times n}$

$g \in C(J, \mathbb{R}^n)$

trivial solution for $\dot{x} = Ax$

$\phi(t_0) = 0$

$\phi(t) = 0 \quad \forall t \in J$



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P.98 §3.2.1 The Fundamental Matrix

$\Rightarrow$ solution space

$\rightarrow$ vector space (linear)

linearly ind.

$\rightarrow$ bases

P.99 Thm 3.1 $V = \{ \phi_1(t), \phi_2(t) \ldots \phi_n \ \phi_{n+1} \ \phi_{n+2} \ldots \}$

The set of solutions of $\boxed{\dot{x} = Ax}$ on J forms an n-dimensional vector space

V is a linear vector space

$\exists\ \alpha_1, \alpha_2 \in F \quad (F \in \mathbb{R}, \mathbb{C})$

assume $\phi_1, \phi_2 \in V$, show $\alpha_1\phi_1 + \alpha_2\phi_2 \in V$

$\phi = \alpha_1\phi_1 + \alpha_2\phi_2$

$\dot{\phi}_1 = A\phi_1$

$\dot{\phi}_2 = A\phi_2 \Rightarrow \dot{\phi} = \alpha_1\dot{\phi}_1 + \alpha_2\dot{\phi}_2$

$= \alpha_1 A\phi_1 + \alpha_2 A\phi_2 \Rightarrow \phi \in V$

$= A(\alpha_1\phi_1) + A(\alpha_2\phi_2)$

$= A(\alpha_1\phi_1 + \alpha_2\phi_2)$

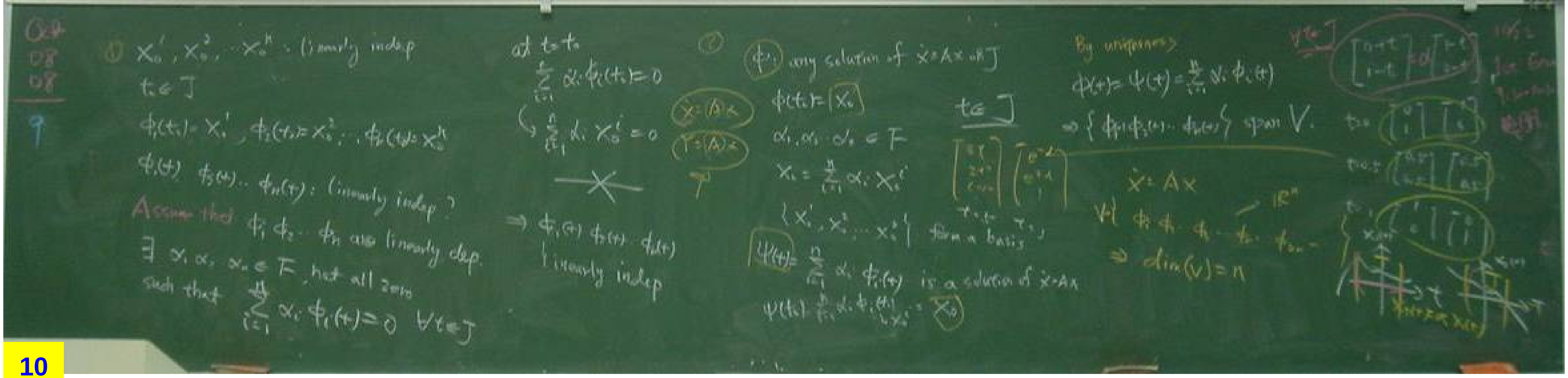
$= A\phi$

$\Rightarrow$ V is a linear vector space

$V = \{ \phi_1, \phi_2 \ldots \phi_n\ \phi_{n+1} \ldots \}$

$\Rightarrow \dim(V) = n \quad \begin{cases} ① \ \exists\ \phi_1, \phi_2 \ldots \phi_n \text{ linearly indep} \\ ② \ \{\phi_1, \phi_2 \ldots \phi_n\} \text{ span } V \end{cases}$

①



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① $X_0^1, X_0^2, \dots X_0^n$: linearly indep

$t_0 \in J$

$\phi_1(t_0) = X_0^1$, $\phi_2(t_0) = X_0^2$, … , $\phi_n(t_0) = X_0^n$

$\phi_1(t)$ $\phi_2(t)$ … $\phi_n(t)$: linearly indep ?

Assume that $\phi_1\ \phi_2 \dots \phi_n$ are linearly dep.

$\exists\ \alpha_1\ \alpha_2 \dots \alpha_n \in F$, not all zero

such that $\sum_{i=1}^{n} \alpha_i \phi_i(t) = 0 \quad \forall t \in J$

at $t = t_0$

$\sum_{i=1}^{n} \alpha_i \phi_i(t_0) = 0$

$\Rightarrow \sum_{i=1}^{n} \alpha_i X_0^i = 0$

$\Rightarrow \phi_1(t)\ \phi_2(t) \dots \phi_n(t)$ linearly indep

$\dot{x} = A x$

$\dot{Y} = A Y$

② $\phi(t)$ any solution of $\dot{x} = A x$ on J

$\phi(t_0) = \boxed{X_0}$

$\alpha_1, \alpha_2 \dots \alpha_n \in F$

$X_0 = \sum_{i=1}^{n} \alpha_i X_0^i$

$\{X_0^1, X_0^2, \dots X_0^n\}$ form a basis

$\boxed{\psi(t)} = \sum_{i=1}^{n} \alpha_i \phi_i(t)$ is a solution of $\dot{x} = Ax$

$\psi(t_0) = \sum_{i=1}^{n} \alpha_i \underbrace{\phi_i(t_0)}_{X_0^i} = X_0$

$t \in J$

By uniqueness

$\phi(t) = \psi(t) = \sum_{i=1}^{n} \alpha_i \phi_i(t)$

$\Rightarrow \{\phi_1(t), \phi_2(t), \dots \phi_n(t)\}$ span V.

$\dot{x} = A x$

$V \{ \phi_1\ \phi_2\ \phi_3 \dots \}$ — $\mathbb{R}^n$

$\Rightarrow \dim(V) = n$

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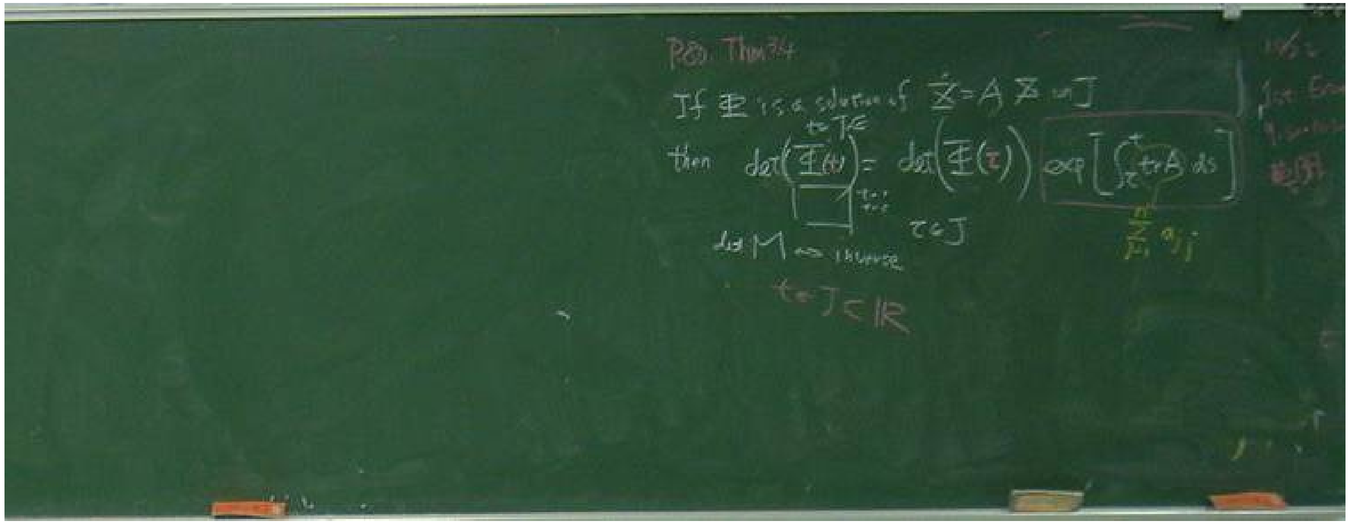

P80. Thm 3.4

If Φ is a solution of $\dot{X} = A X$ on J

then $\det(\Phi(t)) = \det(\Phi(\tau)) \boxed{\exp\left[\int_{\tau}^{t} \operatorname{tr} A \, ds\right]}$

$t \in J$ $\quad \tau \in J$

$\sum_{j=1}^{n} a_{jj}$

det M ⟺ inverse

$t \in J \subset \mathbb{R}$

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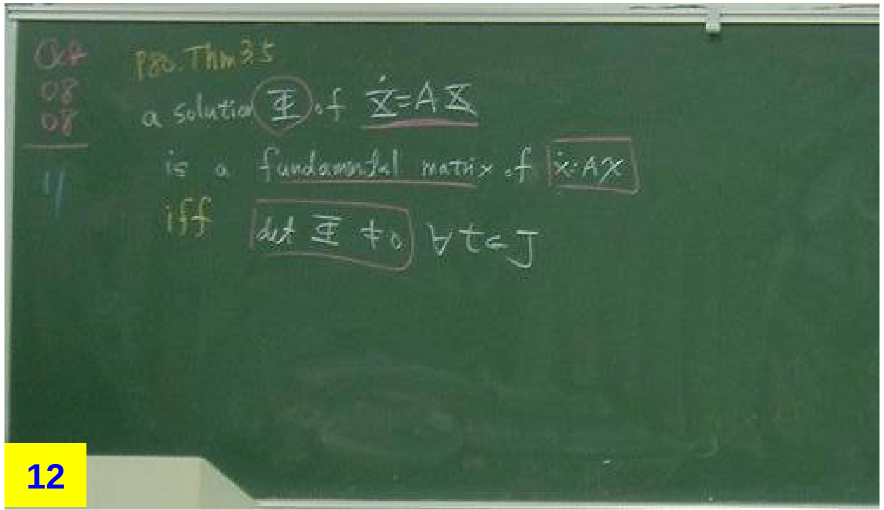

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P80. Thm 3.5

a solution Φ of $\dot{X} = A X$

is a fundamental matrix of $\boxed{\dot{x} = A x}$

iff $\boxed{\det \Phi \neq 0}$ $\forall t \in J$

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