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P2

Chap 1 System Models, Diff Eqn, IVP

Physical Process (Dynamical systems)

many ↓ laws of physics — Chemistry, Biology — Electronics, Circuit

# of variables

Models

many ↓ equations: Differential, Difference

Analysis — stability, Controllability, Observability

many ↓

20% Design

Transistor

→ low-frequency

→ high-freq

→ Semi conduct impurity

Time-varying / Time-invariant

deterministic / Stochastic

P4 classification of systems

lumped-parameter or finite-dimensional

distributed- : or infinite-dim $u(t) =$ random(x, t)

$R(s)$

continuous-time / discrete-time

$v(t)$ $t \in \mathbb{R}$

$v(k)$ $k \in \mathbb{Z}$

linear / nonlinear

$v = R \sin i$

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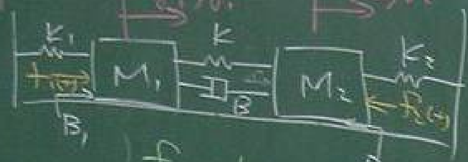
§ 1.4 Examples

Ex 1.1 $y_1, \dot{y}_1$ $y_2, \dot{y}_2$



$f_K = K \Delta y$

$f_B = B \Delta \dot{y}$

By Newton's 2nd Law

$$\begin{cases} M_1 \ddot{y}_1 + (B+B_1)\dot{y}_1 + (K+K_1)y_1 - B\dot{y}_2 - Ky_2 = f_1(t) \\ M_2 \ddot{y}_2 + (B+B_2)\dot{y}_2 + (K+K_2)y_2 - B\dot{y}_1 - Ky_1 = -f_2(t) \end{cases}$$

$x_1 = y_1$, $x_2 = \dot{y}_1$, $x_3 = y_2$, $x_4 = \dot{y}_2$

$\dot{x}_1 = \dot{y}_1 = x_2$

$\dot{x}_2 = \ddot{y}_1 = \frac{1}{M_1}(x_1, x_2, x_3, x_4)$

$\dot{x}_3 = \dot{y}_2 = x_4$

$\dot{x}_4 = \ddot{y}_2 = \frac{1}{M_2}(x_1, x_2, x_3, x_4)$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{(K+K_1)}{M_1} & -\frac{(B+B_1)}{M_1} & \frac{K}{M_1} & \frac{B}{M_1} \\ 0 & 0 & 0 & 1 \\ \frac{K}{M_2} & \frac{B}{M_2} & -\frac{(K+K_2)}{M_2} & -\frac{(B+B_2)}{M_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{1}{M_1} & 0 \\ 0 & 0 \\ 0 & -\frac{1}{M_2} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

State $\dot{x} = [A] x + [B] u$ — State equation, input

output $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + [D] \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ — output equation

State-Space Model

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Ex 1.2 R_1 v_1 R_2 v_2

node voltage $v_1\ v_2\ v_3$

Kirchhoff's law

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{C_1}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) & \frac{1}{R_2 C_1} & 0 \\ -\frac{1}{C_1}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) & -\frac{R_2}{L} - \frac{1}{R_2 C_1} & \frac{R_2}{L} \\ \frac{1}{R_2 C_2} & -\frac{1}{R_2 C_2} & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1 C_1} \\ \frac{1}{R_1 C_1} \\ 0 \end{bmatrix} V$$

$$\dot{x} = Ax + Bu$$

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Ex 1.4

$$\ddot{x} + f(x)\dot{x} + g(x) = 0$$

$f : \mathbb{R} \to \mathbb{R}$ $f \in C^1(\mathbb{R}, \mathbb{R})$

$g : \mathbb{R} \to \mathbb{R}$ $g \in C^1(\mathbb{R}, \mathbb{R})$

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IF $f(x) \geq 0,\ \forall x \in \mathbb{R}$

$x\,g(x) > 0,\ \forall x \in \mathbb{R}$

Lienard Eqn

RLC circuit w/ nonlinear circuit component

IF $f(x) = -\varepsilon(1-x^2)$

$g(x) = x$

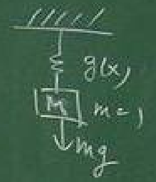
$\Rightarrow \ddot{x} - \varepsilon(1-x^2)\dot{x} + x = 0$

van der Pol Eqn

electronic oscillator

IF $f(x) = 0$

$\Rightarrow \ddot{x} + g(x) = 0$

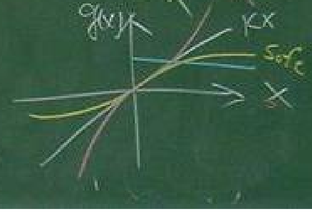


$k>0,\ a^2>0$

$g(x) = k(1+a^2x^2)x$ hard spring

$g(x) = k(1-a^2x^2)x$ soft spring

$g(x) = kx$ linear spring



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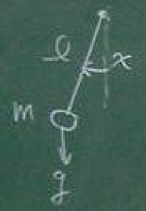
$g(x) = k\sin x \quad k>0$

$\ddot{x} + k\sin x = 0 \qquad k = \omega^2$



$k = \frac{g}{l}$

$M(l\ddot{x}) = -(Mg\sin x)$

$x_1 = x$

$x_2 = \dot{x}$

$\dot{x}_1 = x_2$

$\dot{x}_2 = -f(x)x_2 - g(x)$

$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} x_2 \\ -f(x_1)x_2 - g(x_1)\end{bmatrix}$

$= \begin{bmatrix} \ \ & \ \ \end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$

$\dot{x} = \begin{bmatrix} x_2 \\ -kx_1\end{bmatrix}$

$\dot{x} = \begin{bmatrix} 0 & 1 \\ -k & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$

P5 Finite-dimensional systems

$$\begin{cases}\dot{x}_1 = f_1(t, x_1, x_2, \cdots x_n, u_1, u_2, \cdots u_m) \\ \dot{x}_2 = f_2(t, x_1, x_2, \cdots x_n, u_1, u_2, \cdots u_m) \\ \vdots \\ \dot{x}_n = f_n(t, x_1, \cdots x_n, u_1, \cdots u_m)\end{cases}$$

State equation (diff eqn)

$$\begin{cases} y_1 = g_1(t, x_1, \cdots x_n, u_1, \cdots u_m) \\ \vdots \\ y_p = g_p(t, x_1, \cdots x_n, u_1, \cdots u_m)\end{cases}$$

output equation (algebraic eqn)

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$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$

$u = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} \in \mathbb{R}^m$

$y = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix} \in \mathbb{R}^p$

$f = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$

$g = \begin{bmatrix} g_1 \\ \vdots \\ g_p \end{bmatrix}$

in vector form

$\dot{x} = f(t, x, u)$

$y = g(t, x, u)$

$f: \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$

$g: \underset{t}{\mathbb{R}} \times \underset{x}{\mathbb{R}^n} \times \underset{u}{\mathbb{R}^m} \to \mathbb{R}^p$

linear time-varying state/output eqn

$\dot{x} = \underset{n\times n}{A(t)}\,x + B(t)\,u$

$y = C(t)\,x + D(t)\,u$

$A: \underset{t}{\mathbb{R}} \to \mathbb{R}^{n\times n}$

$B: \mathbb{R} \to \mathbb{R}^{n\times m}$

$C: \mathbb{R} \to \mathbb{R}^{p\times n}$

$D: \mathbb{R} \to \mathbb{R}^{p\times m}$

linear time-invariant state/output eqn

$\dot{x} = Ax + Bu$

$y = Cx + Du$

$A \in \mathbb{R}^{n\times n}$

$B \in \mathbb{R}^{n\times m}$

$C \in \mathbb{R}^{p\times n}$

$D \in \mathbb{R}^{p\times m}$

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P8 §1.3 IVP

$D \subset \mathbb{R}^{n+1}$ an open, nonempty, connected subset of $\mathbb{R}^{n+1}$

(t, x)-space
1 n

$f_i \in C(D, \mathbb{R})$

$f_i(t, x) \in \mathbb{R}$

$\dot{x}_i = f_i(t, x_1, \dots, x_n)$, $i = 1, \dots, n$

$(t, x) \in D$

a system of n 1st-order ODEs

Solution

$\phi_1(t), \phi_2(t), \dots \phi_n(t)$: continuously differentiable

$\phi_j : J \to \mathbb{R}$

$J = (a, b)$

$\dot{\phi}_i(t) = f_i(t, \phi_1(t), \phi_2(t), \dots, \phi_n(t))$

$\Rightarrow \phi_1(t), \dots \phi_n(t)$ is a solution of $\dot{x}_i = f_i(t, x)$

Initial-Value Problem

$\dot{x}_i = f_i(t, x)$, $x_i(t_0) = x_{i0}$ $i = 1, \dots, n$

$\Rightarrow \{\phi_1, \dots, \phi_n\} \Rightarrow \begin{cases} \dot{\phi}_i = f_i(t, \phi) \\ \phi_i(t_0) = x_{i0} \end{cases}$ $i = 1, \dots, n$

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$\dot{x} = f(t, x)$, $x(t_0) = x_0$

a solution of IVP

$\phi(t) = x_0 + \int_{t_0}^{t} f(s, \phi(s))\, ds$

$\begin{cases} \phi(t_0) = x_0 \quad \checkmark \\ \dot{\phi} = f(t, \phi(t)) \quad \checkmark \end{cases}$

P10 §1.3.2 classification

$\dot{x} = f(t, x)$

① $f(t, x) \equiv f(x)$

$\dot{x} = f(x)$

time-invariant

autonomous system

② periodic system

If $f(t+T, x) = f(t, x)$

$(t+T, x) \in D$

$\dot{x} = f(t+T, x) = f(t, x)$

③ $f(t, x) = A(t)x$

$\dot{x} = A(t)x$

a linear homogenous system

④ $f(t, x) = A(t)x + g(t)$

$\dot{x} = A(t)x + g(t)$

a linear nonhomogeneous system

$\ddot{y} + a\dot{y} \cdots = \cancel{0}$

⑤ $f(t, x) = Ax$ $A \in \mathbb{R}^{n \times n}$

$\boxed{\dot{x} = Ax}$

a linear, autonomous, homogenous system

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P11 nth-order ODEs

$y^{(n)} = h(t, y, y^{(1)}, \dots, y^{(n-1)})$
1 n

$h \in C(D, \mathbb{R})$

IVP

$\begin{cases} y(t_0) = x_{10} \\ y^{(1)}(t_0) = x_{20} \\ y^{(n-1)}(t_0) = x_{n0} \end{cases}$

a solution

$\phi \in C^n(J, \mathbb{R})$

$J = (a, b)$

$(t, \phi(t), \phi^{(1)}(t), \dots, \phi^{(n-1)}(t)) \in D$

$\forall t \in J$

$\Rightarrow \phi^{(n)}(t) = h(t, \phi(t), \phi^{(1)}(t), \dots, \phi^{(n-1)}(t))$

$\forall t \in J$

$\phi(t_0) = x_{10}$, $\dot{\phi}(t_0) = x_{20}$, $\phi^{(n-1)}(t_0) = x_{n0}$

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a linear nonhomogenous ODE of order n

$y^{(n)} + a_{n-1}(t) y^{(n-1)} + \dots + a_0(t) y = g(t)$

$a\, b\, y^{(2)} \cdot y^{(3)}$ $\sin(y^{(1)}(t))$

$a_i \in C(J, \mathbb{R})$

$g \in C(J, \mathbb{R})$

$g(t) = 0 \Rightarrow$ homogeneous

$a_i(t) \equiv a_i \Rightarrow$ autonomous

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n-th-order ODE $\Leftrightarrow$ n 1st-order ODEs P20

$y^{(n)} = h(t, y, y', y^{(n-1)})$

$\dot{x}_1 = f_1$

$\dot{x}_2 = \cdots$

n func

$x_1 = y$

$x_2 = y^{(1)}$

$x_{n-1} = y^{(n-2)}$

$x_n = y^{(n-1)}$

$\dot{x}_1 = y^{(1)} = x_2$

$\dot{x}_2 = y^{(2)} = x_3$

$\dot{x}_{n-1} = y^{(n-1)} = x_n$

$\dot{x}_n = y^{(n)} = h(t, x_1, x_2, \ldots x_n)$

$x_1 = y^{(n-1)}$ $x_1 = y$

$x_2 = y^{(n-2)}$ $x_2 = \dot{y} + 2y$

$\dot{x} = A(t)x + g(t)$

$\dot{x} = A(t)x$

$\dot{x} = Ax + g(t)$

$\boxed{\dot{x} = Ax}$

$\dot{x} = f(t, x) \Longrightarrow \dot{x} = Ax$

Linearization

$\dot{x} = f(t, x, u) \Longrightarrow \dot{x} = Ax + Bu$

$\ddot{x} + k \sin x = 0$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -k \sin x_1 \end{bmatrix} \overset{?}{\approx} \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B \end{bmatrix} \begin{bmatrix} u \end{bmatrix}$$