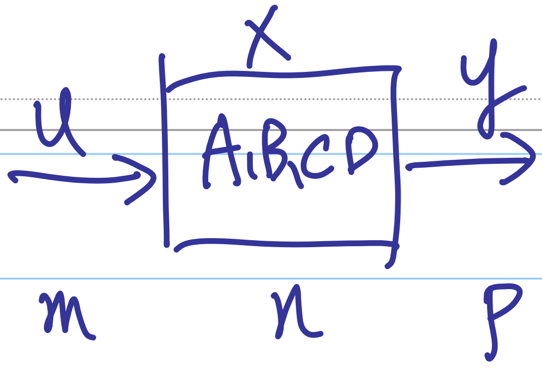
P318 § 9.3 Linear State Observer



$$\left.\begin{matrix} u(t),\ y(t) \\ A,B,C,D \end{matrix}\right\} \to \hat{x}(t) \to x(t)$$

as $t \to \infty$

as $t \geq T$

full-order $\quad \hat{x}(t) \in \mathbb{R}^{n=10}$

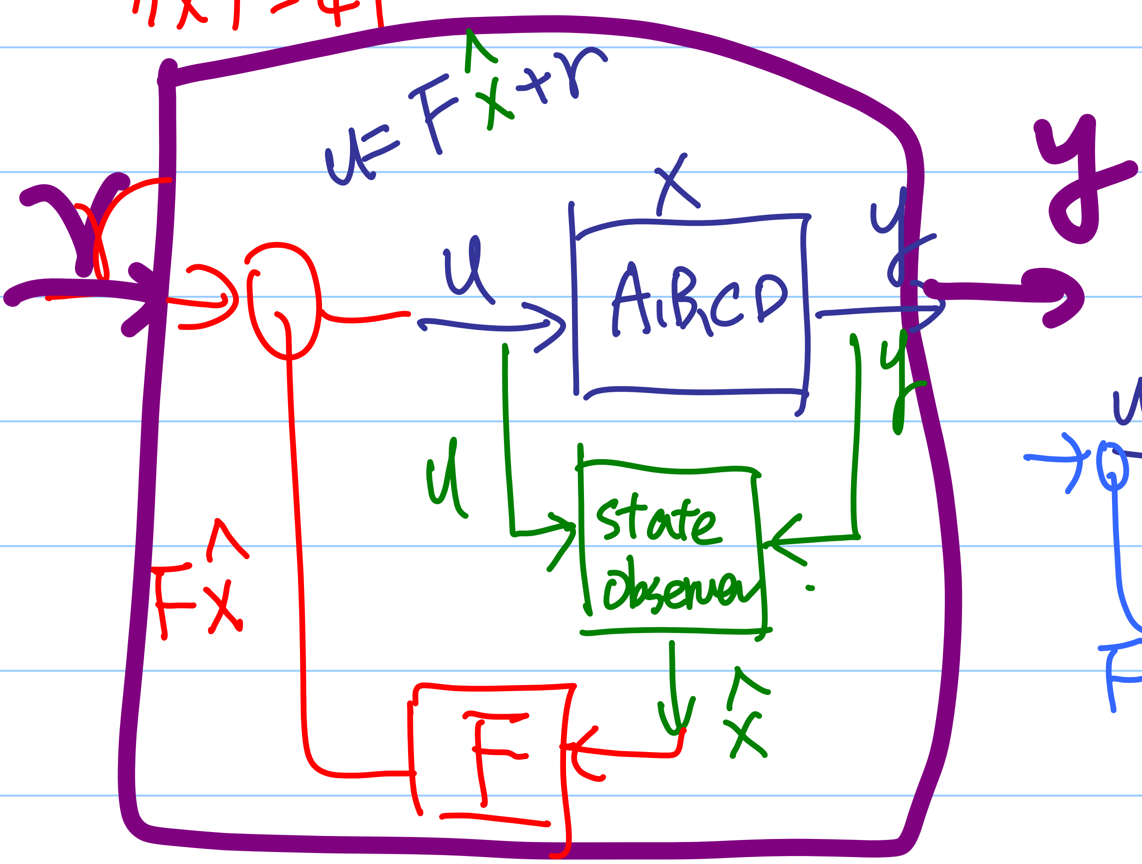
reduced-order $\quad \hat{x}(t) \in \mathbb{R}^{n-p}$ $\quad 10-3 = 7$

$y = Cx + Du$

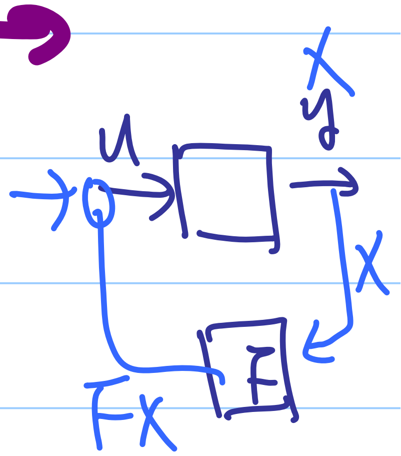
$\quad p \quad n$

$A: 10 \times 10 = 100$

$7 \times 7 = 49$

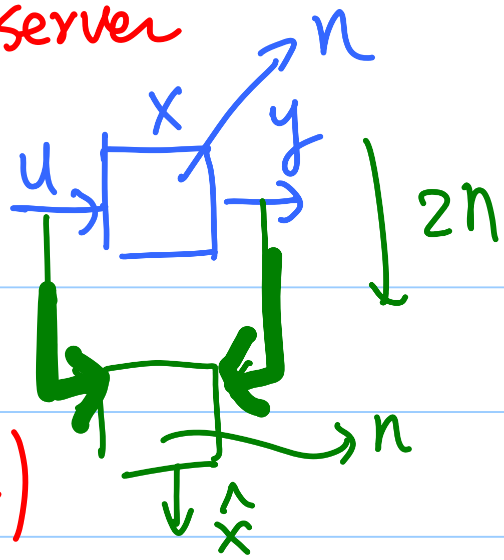


$C = I_n$



P318 § 9.3.1 : Full-order Observer

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$



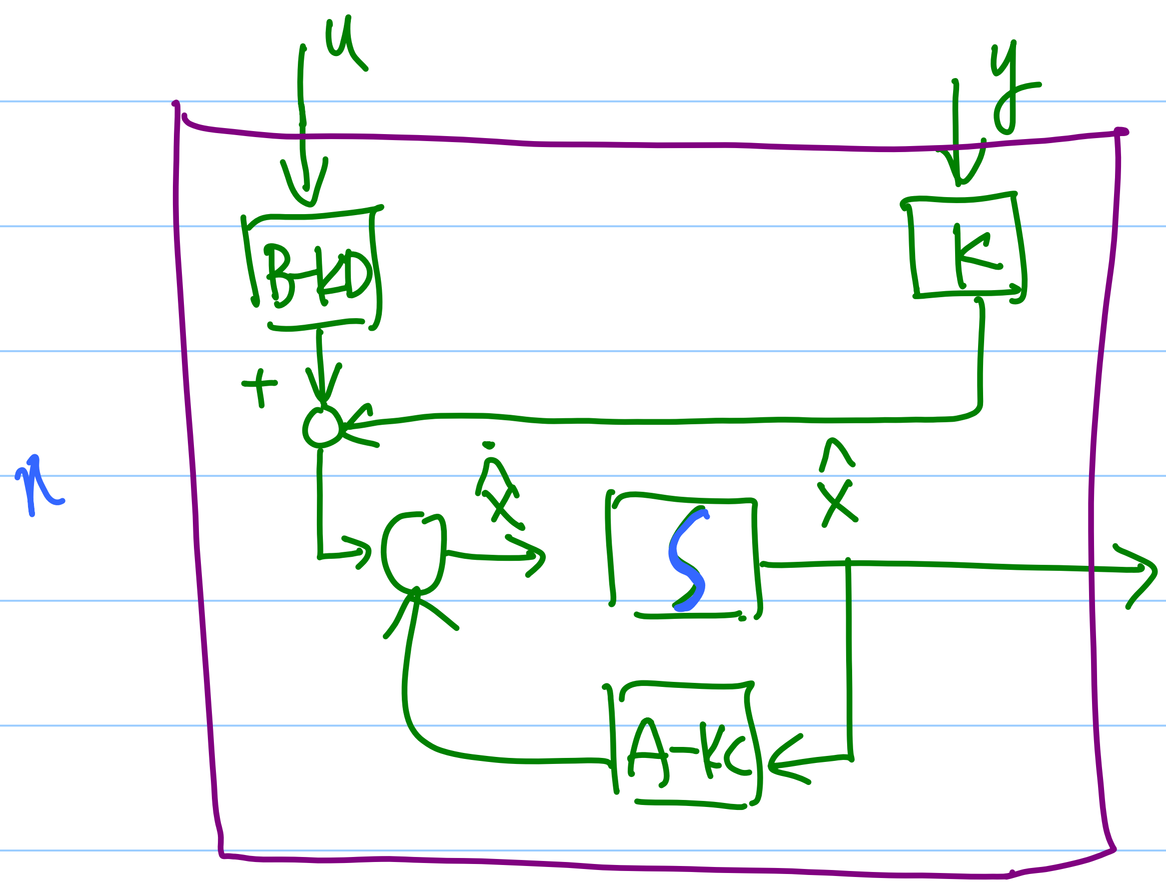
$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu + K(y - \hat{y})$$

$n \times p$ $\quad p \quad p$

$$\hat{y}(t) = C\hat{x}(t) + Du$$

$$\dot{\hat{x}} = A\hat{x} + Bu + Ky - K(C\hat{x} + Du)$$

$$= (A - KC)\hat{x} + [B - KD \quad K]\begin{bmatrix} u \\ y \end{bmatrix}$$



Analysis:

$e = x - \hat{x}$

$\dot{e} = \dot{x} - \dot{\hat{x}}$

$= (Ax + \cancel{Bu}) - (A\hat{x} + \cancel{Bu} + K(y - \hat{y}))$

$\hat{y} = C\hat{x} + \cancel{D}u$

$y = Cx + \cancel{D}u$

$A(x - \hat{x})$ $\qquad$ $KC(x - \hat{x})$

$= (A - KC)(x - \hat{x})$

$\dot{e} = (A - KC)\,e$

$e(t) = \exp(A - KC)\, e(0)$

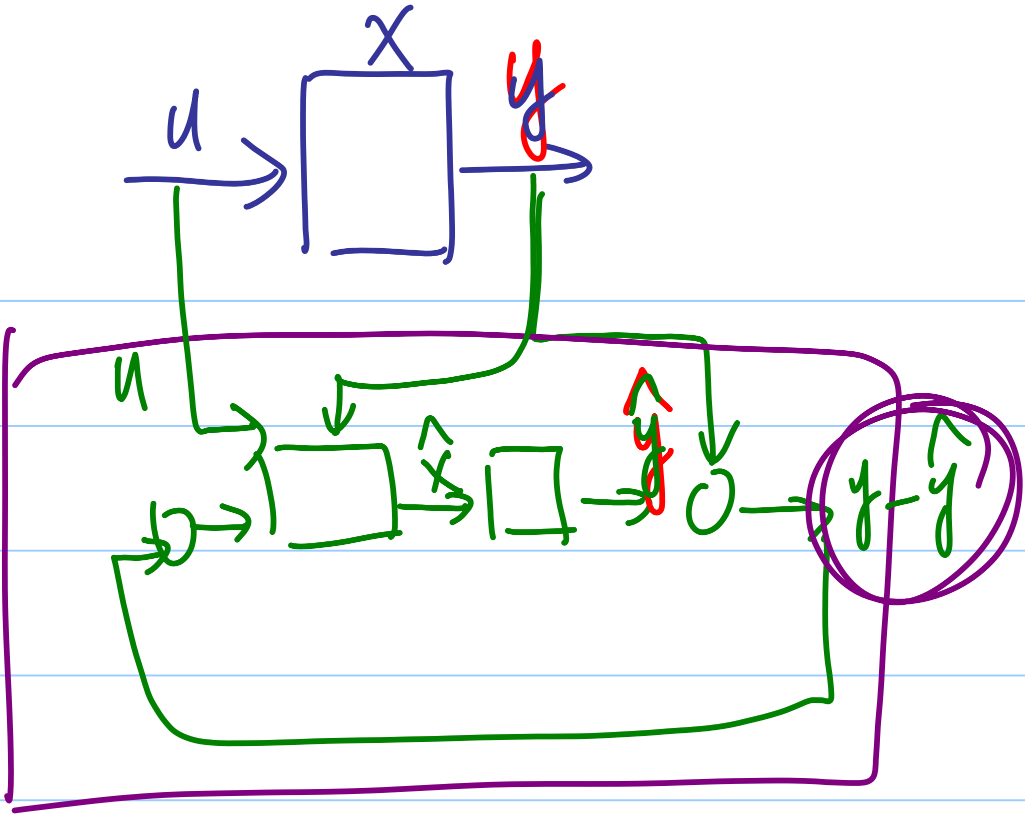
If Re{eig $(A - KC)$} < 0

Then $e(t) \to 0$ as $t \to \infty$, no matter what the value of $e(0) = x(0) - \hat{x}(0)$ is

$t > T$ $\|e(t)\| \le \delta$

$\Rightarrow$ asymptotic state observer
estimator

$\Rightarrow$ Luenberger observer



P319 Lemma 9.1.8

There exists $K \in \mathbb{R}^{n \times p}$ such that $\text{eig}(A-KC)$ are assigned to arbitrary real or complex conjugate locations

$\iff (A, C)$ is observable

Proof By Thm 9.2

$\text{eig}(A+BF) \iff (A, B)$ controllable

$A^T \quad C^T(-K^T) \qquad (A^T, C^T)$ controllable

$A^T - C^T K^T$

$\text{eig}(A-KC) \iff (A, C)$ observable

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

P381 Ex 9.19

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$\det(sI - A) = \cdots = 0 s^3 \Rightarrow \operatorname{eig}(A) = 0, 0, 0$$

$$\alpha_d(s) = s^3 + d_2 s^2 + d_1 s + d_0$$

$$A_D = A^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

$$B_D = C^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ controller form

$$\mathcal{C}_D = \begin{bmatrix} B_D & A_D B_D & A_D^2 B_D \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \mathcal{C}_D^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow q$$

$$P = \begin{bmatrix} q \\ qA_D \\ qA_D^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow A_{D/c} = P A_D P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$B_{D/c} = P B_D = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$F_{D/c} = \begin{bmatrix} -d_0 - a_0 & -d_1 - a_1 & -d_2 - a_2 \end{bmatrix}$$

$$-d_0 - 0 \qquad -d_1 - 2 \qquad -d_2 - (-1)$$

$$K = -F^T = \alpha_d(A_D)\,\Theta^{-1} e_n$$

$$F = -e_n^T \, \mathcal{C}_D^{-1} \, \alpha_d(A_D)$$

P383. §9.3.2. Reduced-order Observers

$$\dot{x} = Ax + Bu$$
$$y = Cx$$
$$= \begin{bmatrix} I_p & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{matrix} p \\ n-p \end{matrix}$$

$\Rightarrow y = x_1$ p : measured states

$x_2 \in \mathbb{R}^{n-p}$: to be estimated

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$

$$p \quad \dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1 u \rightarrow y$$

$$n-p \quad \boxed{\dot{x}_2 = A_{22}x_2 + \underbrace{\begin{bmatrix} A_{21} & B_2 \end{bmatrix}}_{\tilde{B}} \underbrace{\begin{bmatrix} x_1 \\ u \end{bmatrix}}_{\tilde{u}}}$$

$\in \mathbb{R}^{n-p}$ known

$$\tilde{y} \triangleq \dot{x}_1 - A_{11}x_1 + B_1 u = A_{12}x_2$$

$$\underset{\text{known}}{y} = C \underset{\text{unknown}}{x}$$

$$\dot{\hat{x}}_2 = A_{22}\hat{x}_2 + \tilde{B}\tilde{u} + \tilde{K}(\tilde{y} - \hat{y})$$

$$\hat{y} = A_{12}\hat{x}_2$$

$$= (A_{22} - \tilde{K}A_{12})\hat{x}_2 + (A_{21}x_1 + B_2 u) + \tilde{K}(\dot{x}_1 - A_{11}x_1 - B_1 u)$$

$$e = x_2 - \hat{x}_2$$

$$\dot{e} = \dot{x}_2 - \dot{\hat{x}}_2$$

$$= (A_{22}x_2 - \tilde{B}\tilde{u}) - \left((A_{22}\hat{x}_2 + \tilde{B}\tilde{u}) + \tilde{K}(\tilde{y} - \hat{y})\right)$$

$$= (A_{22} - \tilde{K}A_{12})(x_2 - \hat{x}_2)$$

$$(A - KC)(x - \hat{x})$$

$$\dot{e} = (A_{22} - \tilde{K}A_{12})\, e$$

$\Rightarrow$ IF (A_{22}, A_{12}) is observable

THEN $\operatorname{eig}(A_{22} - \tilde{K}A_{12})$ can be arbitrarily assigned

$$\overset{A}{\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}}, \quad \overset{C}{\begin{bmatrix} \underline{I_p} & 0 \end{bmatrix}} \text{ is observable}$$

$\Leftrightarrow$ (A_{22}, A_{12}) is observable

Another estimated state

$$\hat{x}_2 = w + \tilde{k} x_1$$

$$w = \hat{x}_2 - \tilde{k} x_1$$

$$\dot{w} = \dot{\hat{x}}_2 - \boxed{\tilde{k} \dot{x}_1}$$

$$= (A_{22} - \tilde{k} A_{12}) \hat{x}_2 + (A_{21} x_1 + B_2 u) \cancel{+ \tilde{k} (\dot{x}_1 - A_{11} x_1 - B_1 u)}$$

$$- \tilde{k} \cancel{\dot{x}_1}$$

$$= (A_{22} - \tilde{k} A_{12}) (\underbrace{\hat{x}_2 - \tilde{k} x_1}_{= w}) + (A_{22} - \tilde{k} A_{12}) \tilde{k} x_1$$

$$+ (A_{21} x_1 + B_2 u) + \tilde{k} (-A_{11} x_1 - B_1 u)$$

$$\dot{w} = \boxed{(A_{22} - \tilde{k} A_{12})} w + [(A_{22} - \tilde{k} A_{12}) \tilde{k} + A_{21} - \tilde{k} A_{11}] x_1$$

$$+ [B_2 - \tilde{k} B_1] u$$

(A_{22}, A_{12})

$$+ [\ast \quad \ast] \begin{bmatrix} x_1 \\ u \end{bmatrix}$$

$$[\ast \quad \ast] \begin{bmatrix} y \\ u \end{bmatrix}$$

F.O. $\hat{x}(t)$

R.O. $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} f(y) \\ \hat{x}_2 \end{bmatrix}$ computed / measured / estimated

P.385. Ex 9.21

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & -2 \\ 1 & -2 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u$$

$$y = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C} x \qquad y = x_2$$

$$n - p = 2 - 1 = 1$$

$$P = \begin{bmatrix} C \\ \hat{C} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \leftarrow \text{any row} \qquad P^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\bar{x} = Px$$

$$\Rightarrow \bar{A} = PAP^{-1} = \begin{bmatrix} -2 & 1 \\ -2 & 0 \end{bmatrix} \qquad y = \bar{x}_1$$

$$\bar{B} = PB = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\bar{C} = CP^{-1} = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad \bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} \qquad x_2$$

$$\dot{w} = (-\tilde{k}) w + \left[-\tilde{k}^2 + (-2) - \tilde{k}(-2) \right] y + (-\tilde{k}) u$$

$$\tilde{k} = -10$$

$$\dot{w} = 10 w - 122 y + 10 u$$

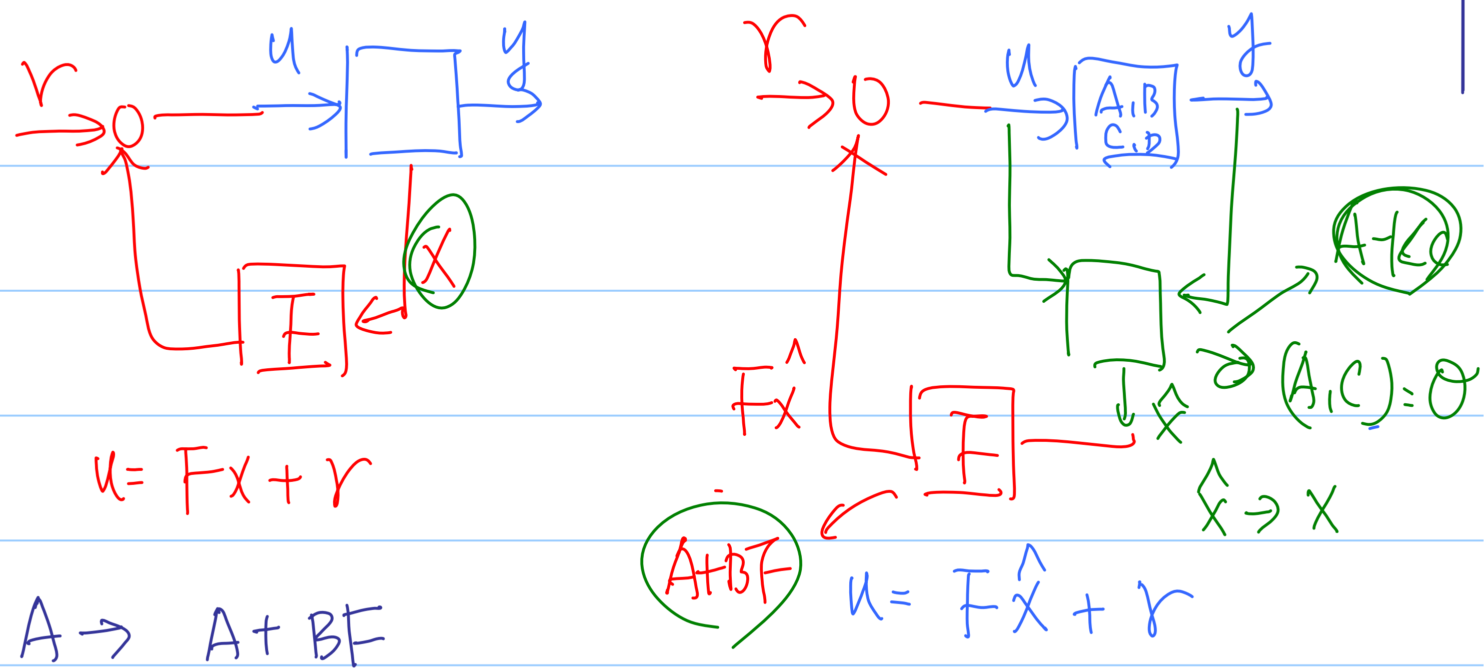
$$w + \tilde{k} y = w - 10 y = \hat{\bar{x}}_2$$

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} \Rightarrow \begin{bmatrix} y \\ w - 10y \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = P^{-1} \bar{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ w - 10y \end{bmatrix} = \begin{bmatrix} w - 10y \\ y \end{bmatrix}$$

x_2

P392 §9.4 Observer-Based Dynamic Controller



$u = Fx + r$

$A \to A + BF$

$\hat{x} \to x$

$u = F\hat{x} + r$

P393 §9.4.1 State-space Analysis

$$n \quad \dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$n \quad \dot{\hat{x}} = A\hat{x} + Bu + K(y - \hat{y})$$
$$= (A - KC)\hat{x} + [B - KD \quad K]\begin{bmatrix} u \\ y \end{bmatrix}$$

$$u = F\hat{x} + r$$

$$\dot{x} = Ax + B(F\hat{x} + r)$$
$$= Ax + BF\hat{x} + Br$$

$(A+BF)x$

$y = Cx + Du$

$$\dot{\hat{x}} = (A - KC)\hat{x} + (B - KD)u + Ky$$
$$+ (B - K\cancel{D})(F\hat{x} + r) + K(Cx + Du)$$

$$= (A-KC)\hat{X} + BF\hat{X} + Br + KCX \qquad +KCX + K\cancel{Du}$$

$$(F\hat{X}+r)$$

$$= (A-KC+BF)\hat{X} + Br + KCX$$

eig( )

$$\begin{array}{c} n \\ n \end{array}\begin{bmatrix} \dot{X} \\ \dot{\hat{X}} \end{bmatrix} = \begin{bmatrix} A & BF \\ KC & A-KC+BF \end{bmatrix}\begin{bmatrix} X \\ \hat{X} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} r$$

$$y = \begin{bmatrix} C & DF \end{bmatrix}\begin{bmatrix} X \\ \hat{X} \end{bmatrix} + [D]r$$

2n-th-order system

$$P = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}$$

$$P\begin{bmatrix} X \\ \hat{X} \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}\begin{bmatrix} X \\ \hat{X} \end{bmatrix} = \begin{bmatrix} X \\ X-\hat{X} \end{bmatrix} = \begin{bmatrix} X \\ e \end{bmatrix}$$

$$\begin{bmatrix} \dot{X} \\ \dot{e} \end{bmatrix} = P\begin{bmatrix} \dot{X} \\ \dot{\hat{X}} \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}\left(\begin{bmatrix} A & BF \\ KC & A-KC+BF \end{bmatrix}\begin{bmatrix} X \\ \hat{X} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} r\right)$$

$$= \underline{\begin{bmatrix} A & BF \\ A-KC & -A+KC \end{bmatrix}}\begin{bmatrix} X \\ \hat{X} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r$$

$$\underline{P^{-1}}\begin{bmatrix} X \\ e \end{bmatrix}$$

$$\begin{bmatrix} \dot{X} \\ \dot{e} \end{bmatrix} = \underline{\begin{bmatrix} A+BF & -BF \\ 0 & A-KC \end{bmatrix}}\begin{bmatrix} X \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r$$

$$y = \begin{bmatrix} C+DF & -DF \end{bmatrix}\begin{bmatrix} X \\ e \end{bmatrix} + Dr$$

$$\det\left(sI_{2n} - \begin{bmatrix} A+BF & -BF \\ 0 & A-KC \end{bmatrix}\right)$$

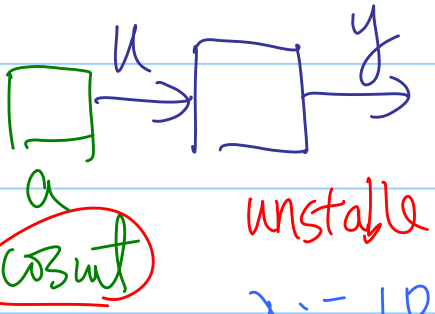
$$= \det\begin{pmatrix} sI_n-(A+BF) & +BF \\ 0 & sI_n-(A-KC) \end{pmatrix}$$

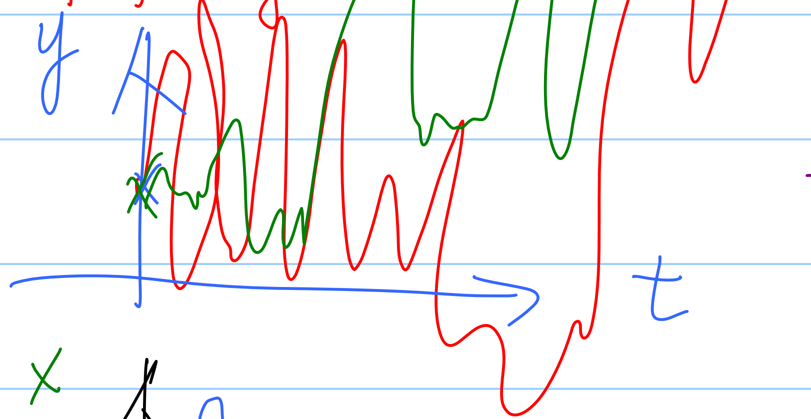
$$= \boxed{\det(sI_n-(A+BF))}\,\boxed{\det(sI_n-(A-KC))}$$

controller $\qquad\qquad$ observer

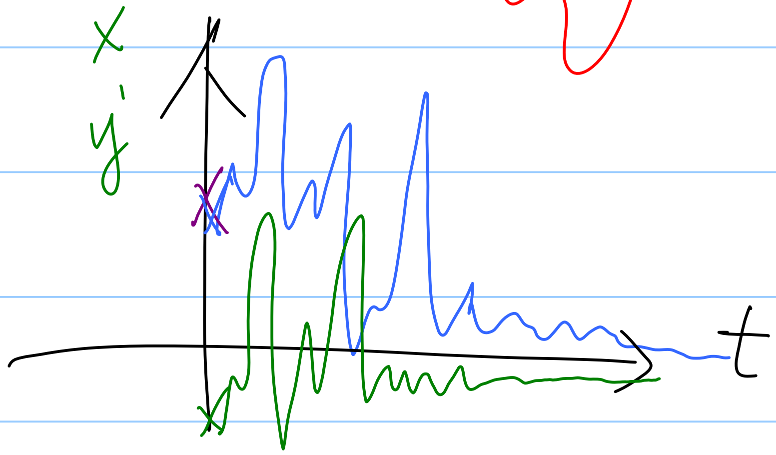
$\alpha_d^c(s) \qquad\qquad \alpha_d^o(s)$

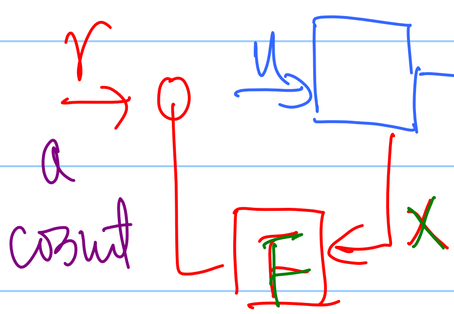
separation property





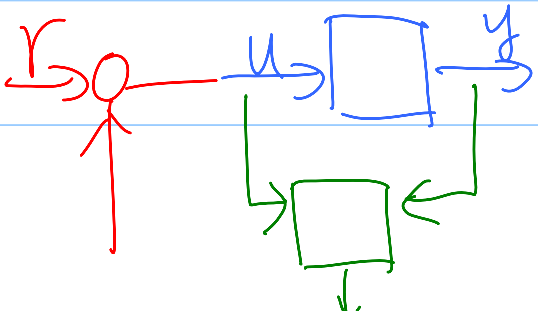
$x_i=10$

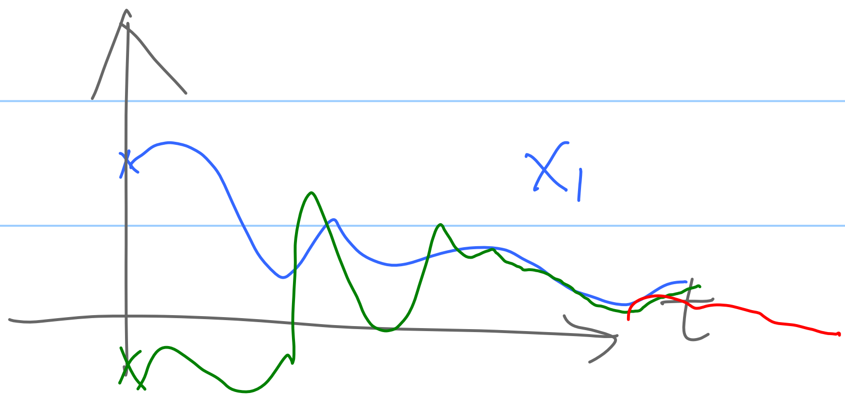


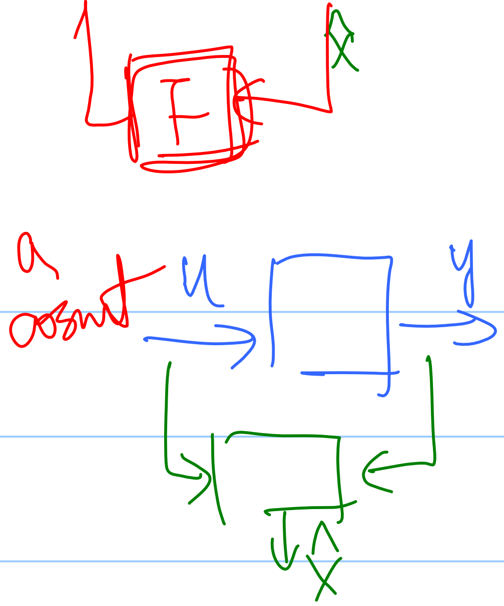


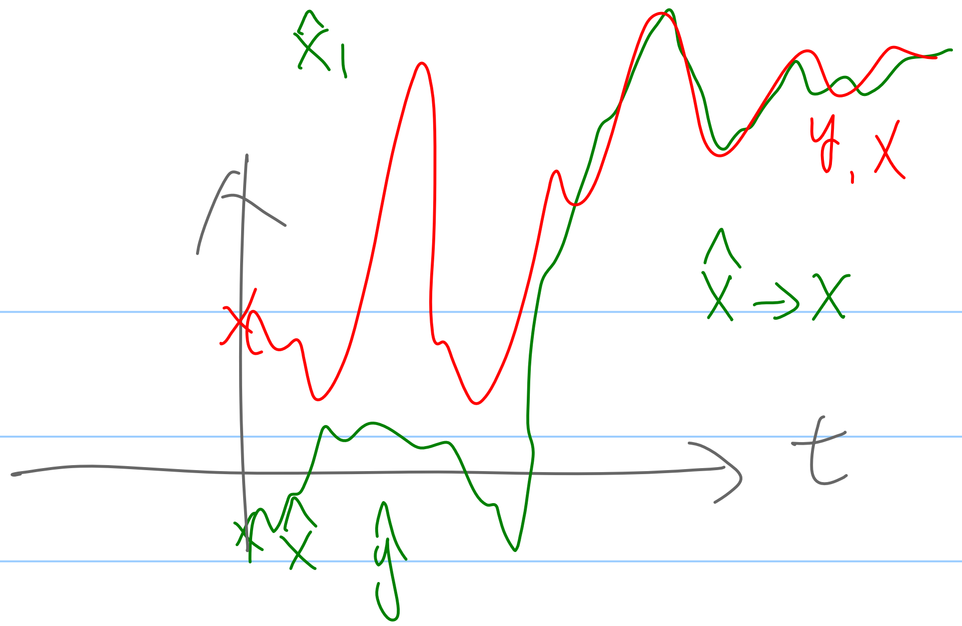
$x(t)\to 0$

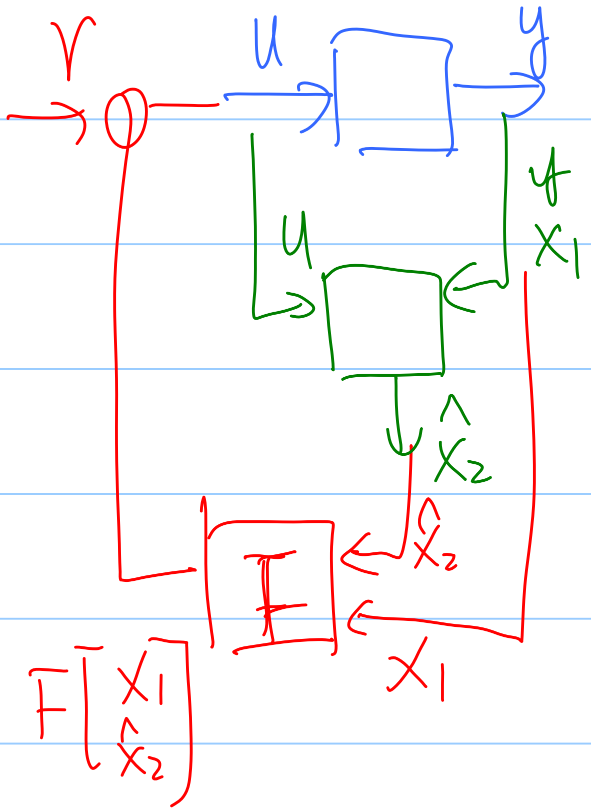
$x_i=10 \to x_i=-5$

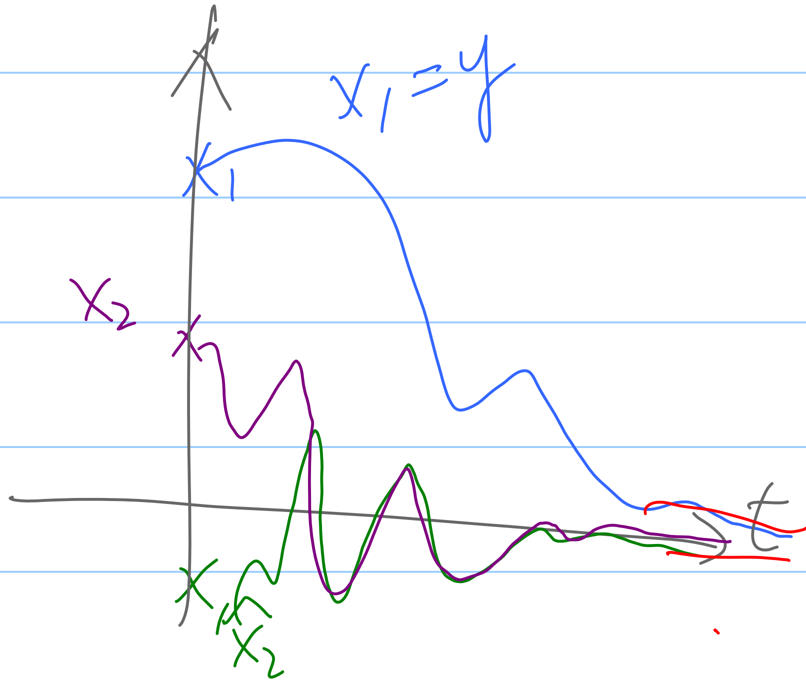












$$y = \underbrace{I_p}_{C} x \qquad I_p = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$y = \begin{bmatrix} a & b & a & 0 \\ c & d & d & 0 \end{bmatrix} \tilde{x} \Rightarrow y = \bar{C}\tilde{x}$$

$$x \xrightarrow{P} \tilde{x}$$

$$P = \begin{bmatrix} a & b & a & 0 \\ c & d & 0 & 0 \\ x & x & x & x \\ x & x & x & x \end{bmatrix} = \begin{bmatrix} C \\ * \end{bmatrix}$$

$$= \begin{bmatrix} I_p & 0 \end{bmatrix} \tilde{x}$$

$$\bar{C} = CP^{-1} \Rightarrow P\bar{C} = C$$

$$[I_p \; 0] =$$

$$P [I_p \; 0] = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} a & b & 0 \\ c & d & 0 \end{bmatrix}$$

$$\bar{C}_b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P \Rightarrow \bar{C} = CP^{-1} \Rightarrow P\bar{C} = C$$

$$P = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ x & x & x \end{bmatrix} \quad \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ x & 1 & 1 \end{bmatrix} [I_p \; 0] = \begin{bmatrix} \cancel{0} & 0 \\ a & b \\ c & d \end{bmatrix}$$