Problem 3-1:

[Ref: HW3_張峻豪 r07921012 張峻豪_DCS_HW3_107 0412_zTransform & sampling]

(a)

When the different equation of discrete time system is given, the pulse-transfer function can be derived in z-Transform format. In advance, substitute the input in z-Transform and take the inverse z-Transform. Finally, The output sequence will be derived.

The difference equation:

$\mathrm{y}[\mathrm{k}+2]-1.5\mathrm{y}[\mathrm{k}+1]+0.5\mathrm{y}[\mathrm{k}]=\mathrm{u}[\mathrm{k}+1]$	(1)

For initial condition, let k=-1:

$\mathrm{y}[1]-1.5\mathrm{y}[0]+0.5\mathrm{y}[-1]=\mathrm{u}[0]$	(2)

Substitute y[0], y[-1] into (2), derive y[1]:

y[1] = 1.25	

Consider the initial condition, the zTransform will be:

$\mathrm{y}[\mathrm{k}+2]\rightarrow z^2(Y(z)-y[0]-z^{-1}y[1])$ $\mathrm{y}[\mathrm{k}+1]\rightarrow \mathrm{z}(\mathrm{Y}(\mathrm{z})-\mathrm{y}[0])$ $\mathrm{y}[\mathrm{k}]\rightarrow \mathrm{Y}(\mathrm{z})$ $\mathrm{u}[\mathrm{k}+1]\rightarrow \mathrm{z}(\mathrm{U}(\mathrm{z})-\mathrm{u}[0])$	(3)

where

y[1] = 1.25, y[0] = 0.5, y[-1] = 1, u[0] = 1	

Substitute (3) into (1) and simplify, then we can derive the equation:

$\mathrm{Y}(\mathrm{z})=\mathrm{z}\dfrac{1}{(z-0.5)(z-1)}U(z)+0.5z\dfrac{1}{z-0.5}$	(4)
$\mathrm{Y}(\mathrm{z})=\left[\dfrac{-2z}{z-0.5}+\dfrac{2z}{z-1}\right]U(z)+0.5\dfrac{\mathrm{z}}{z-0.5}$	(5)

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The pulse-transfer function is:

$G(z)=\frac{Y(z)}{U(z)}=\frac{z}{(z-0.5)(z-1)}$	(7)

And, its poles and zeros are: poles: z=0.5, 1, zeros: z=0.

(b)

For the following input function:

$U(z)=\left[\frac{z}{z-1}\right]$	(8)

Finally, the output sequence becomes:

$\mathrm{Y(z)}=\left[\frac{-2z}{z-0.5}+\frac{2z}{z-1}\right]\left[\frac{z}{z-1}\right]+0.5\frac{z}{z-0.5}$	(9)
$=\left[\frac{z^2}{(z-0.5)(z-1)^2}\right]+0.5(\frac{z}{z-0.5})$	(10)
$=\frac{1}{z-0.5}+0.5*\frac{z}{z-0.5}+(-2)*\frac{1}{z-1}+(2)*\frac{z}{(z-1)^2}$	(11)

Finally, the output sequence is derived by inverse zTransform and simplification:

$\mathrm{y[k]}=(0.5)^{(k-1)}+(0.5)^{(k+1)}+(-2)(1)^{(k-1)}+2*k$	(12)
$=(0.5)^{(k-1)}1[k-1]+(0.5)^{(k+1)}1[k]-2*1[k-1]+2k*1[k]$	(13)

Moreover, (11) can be validated by initial conditions:

$\mathrm{y}[-1]=(0.5)^{(-2)}+(0.5)^{(0)}+(-2)(1)^{(-2)}+2*(-1)=1$	(14)
$\mathrm{y}[0]=(0.5)^{(-1)}+(0.5)^{(1)}+(-2)(1)^{(-1)}+2*(0)=0.5$	(15)
$\mathrm{y}[1]=(0.5)^{(0)}+(0.5)^{(2)}+(-2)(1)^{(0)}+2*1=1.25$	(16)

(c) poles: z=0.5 is for $(0.5)^{(k)}$,

poles: z=1 is for $(1)^{(k)} = 1$.

The effect of z can be considered as the weighting among different kernel (or basis) functions or between one function and its delayed function.

For example, (z-b)/(z-a) = z/(z-a) - b/(z-a) = z/

$G(z) = \frac{Y(z)}{U(z)} = \frac{(\mathrm{z}-\mathrm{c})}{(z-a)(z-b)}$	
$= \frac{\mathrm{A}}{(z-a)} + \frac{\mathrm{B}}{(z-b)}$	
$= Az^{-1}\frac{\mathrm{z}}{(z-a)} + Bz^{-1}\frac{\mathrm{z}}{(z-b)}$	
➔ $\mathrm{A}(a)^{(k-1)} + \mathrm{B}(b)^{(k-1)}$	
OR	
$G(z) = \frac{Y(z)}{U(z)} = \frac{(\mathrm{z}-\mathrm{c})}{(z-a)}$	
$= \frac{\mathrm{z}}{(z-a)} - \frac{\mathrm{c}}{(z-a)}$	
$= \frac{\mathrm{z}}{(z-a)} - \mathrm{c}z^{-1}\frac{\mathrm{z}}{(z-a)}$	
➔ $(a)^{(k)} - \mathrm{c}(a)^{(k-1)}$	

These features can be illustrated by simulation results in next example.

Problem 3-2:

[Ref: HW3_張峻豪 r07921012 張峻豪_DCS_HW3_107 0412_zTransform & sampling]

In discrete time system, the poles and zeros in z-Transform influence the properties of system, such as rising time, settling time, and overshoot etc. In this problem, the relationship between the percentage overshoot and the locations of poles and zeros are discussed.

The system is described in z-Transform as:

$$\frac{z+b}{(1+b)(z^2-1.1z+a)} \tag{2-1}$$

Where

$$\mathrm{a} \in [0.3, 0.5], \mathrm{b} \in [-0.75, 0.75]$$

After executing the [2: Lian Response_01_pole.m 2019] given by teacher, the Figure 2-1 and Figure 2-2 are derived.

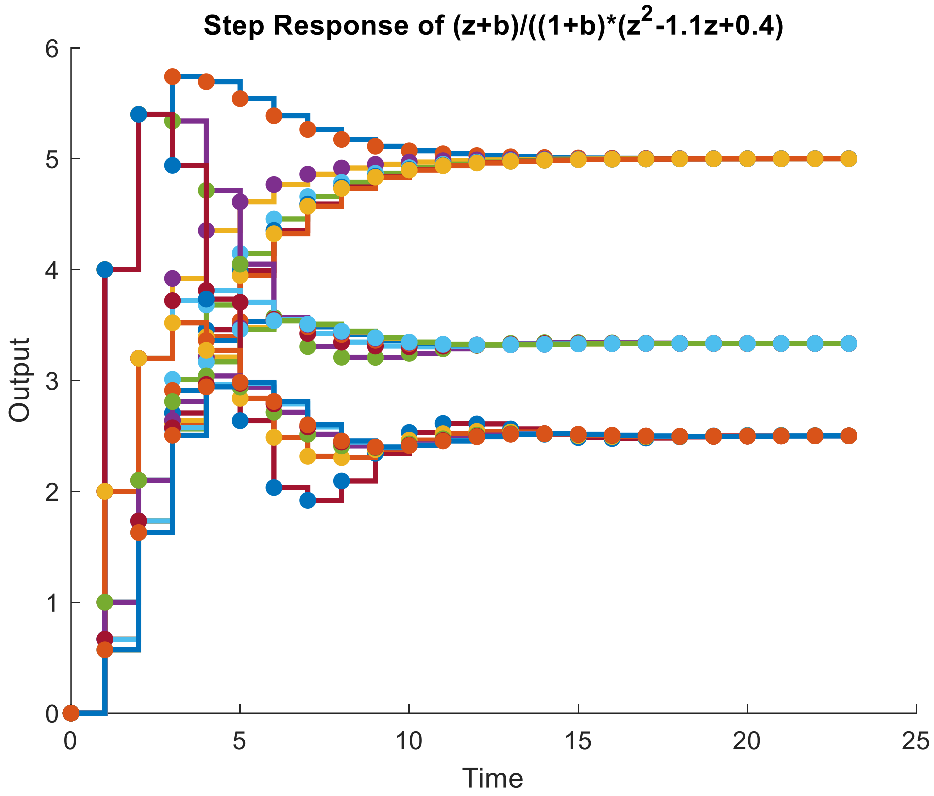


Figure 2-1. Step response for different pairs of a and b.

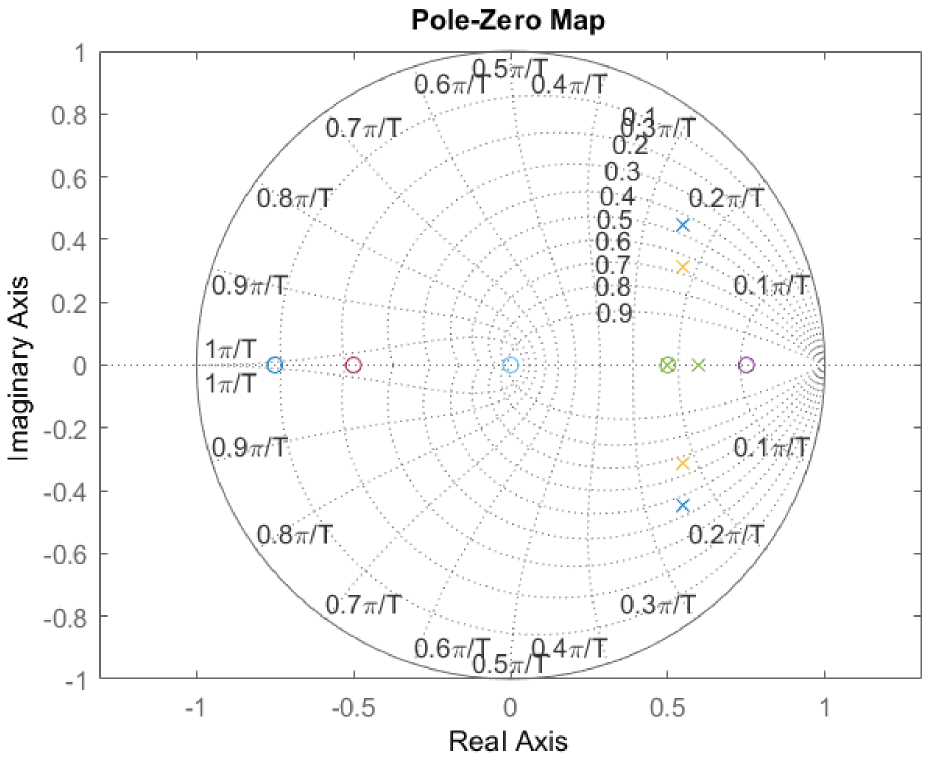
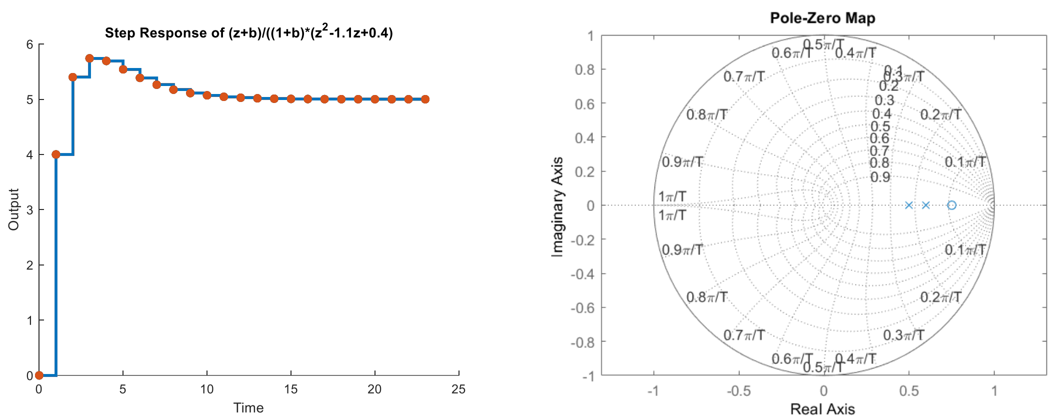


Figure 2-2. Poles and zeros for different pairs of a and b. The overshoot is derived for each pairs in MATLAB.

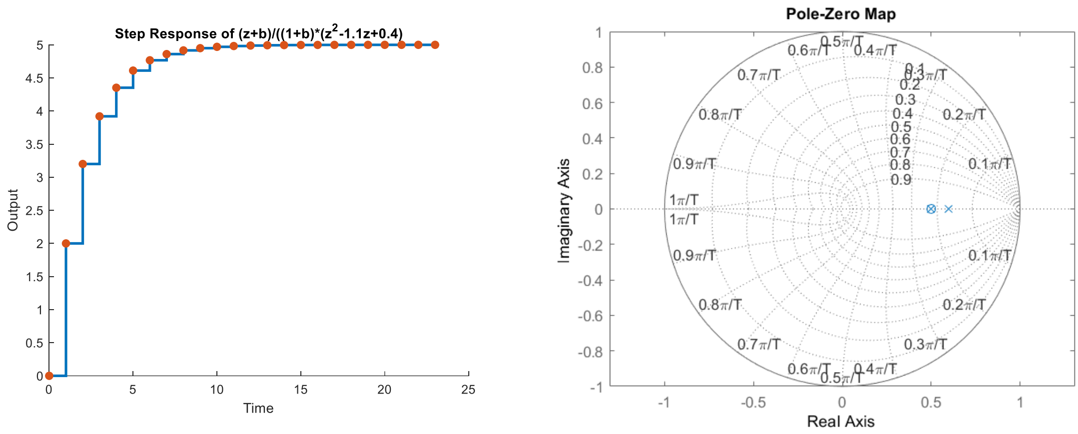
However, Figure 2-2 doesn't show all of pairs of a and b, the following will show all the results.

When $a = 0.3$, $b \in [-0.75, -0.5, 0, 0.5, 0.75]$:



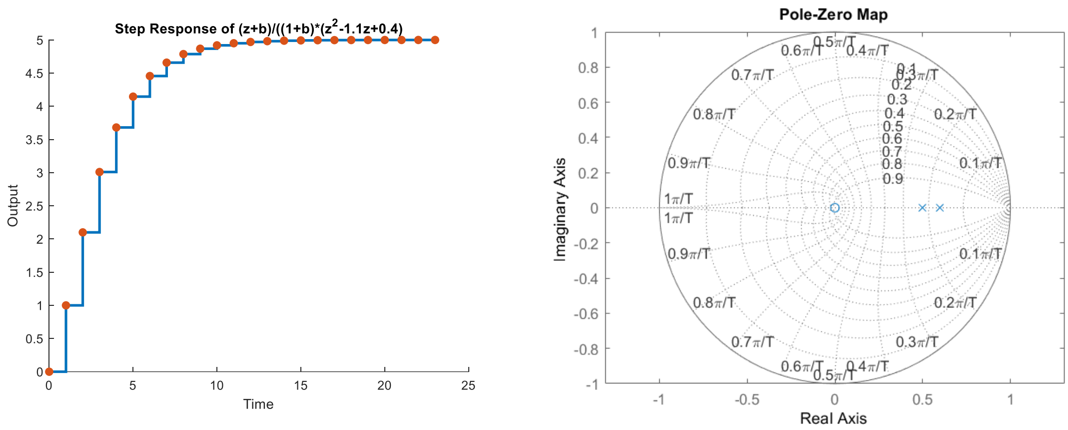
(a) (b)

Figure 2-3. When a = 0.3 and b = -0.75, (a) notes that step response, and (b) notes the poles and zeros in z-Transform.



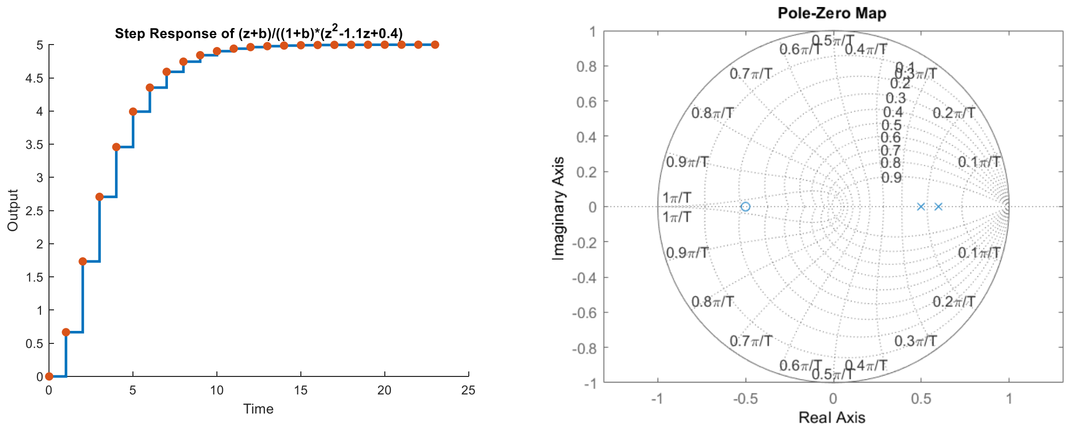
(a) (b)

Figure 2-4 When a = 0.3 and b = -0.5, (a) notes that step response, and (b) notes the poles and zeros in z-Transform.



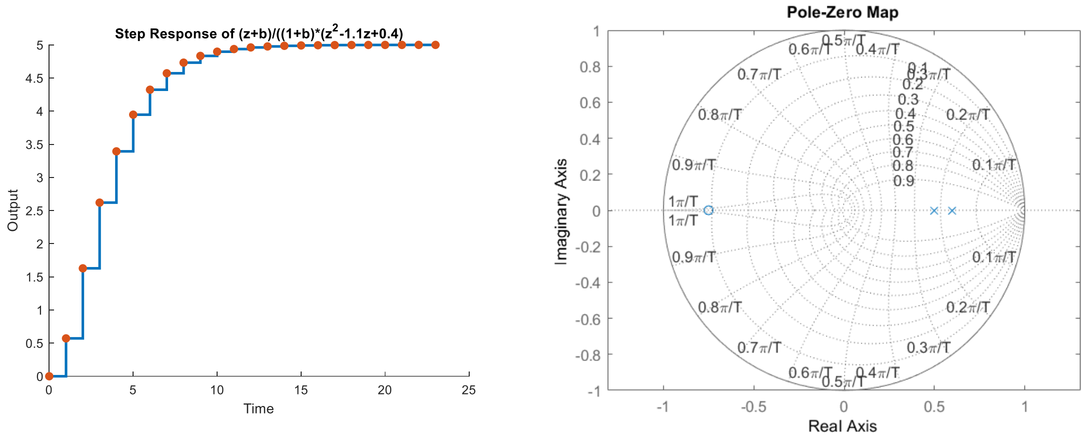
(a) (b)

Figure 2-5. When a = 0.3 and b = 0, (a) notes that step response, and (b) notes the poles and zeros in z-Transform.



(a) (b)

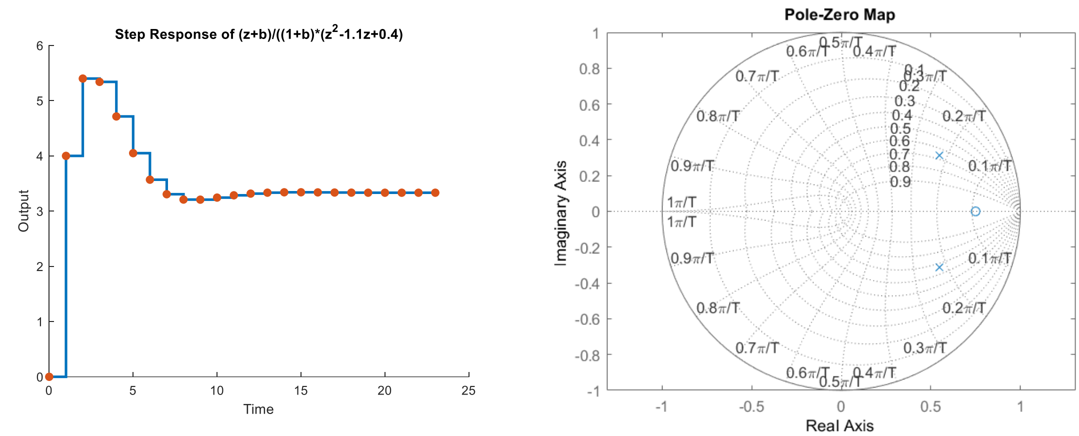
Figure 2-6 When a = 0.3 and b = 0.5, (a) notes that step response, and (b) notes the poles and zeros in z-Transform.



(a) (b)

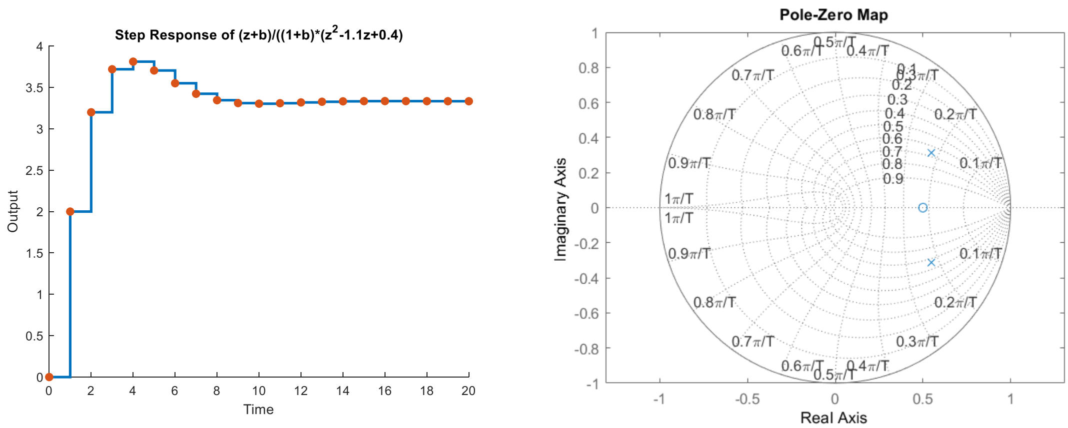
Figure 2-7. When a = 0.3 and b = 0.75, (a) notes that step response, and (b) notes the poles and zeros in z-Transform.

When $a = 0.4,\ b \in [-0.75, -0.5, 0, 0.5, 0.75]$:



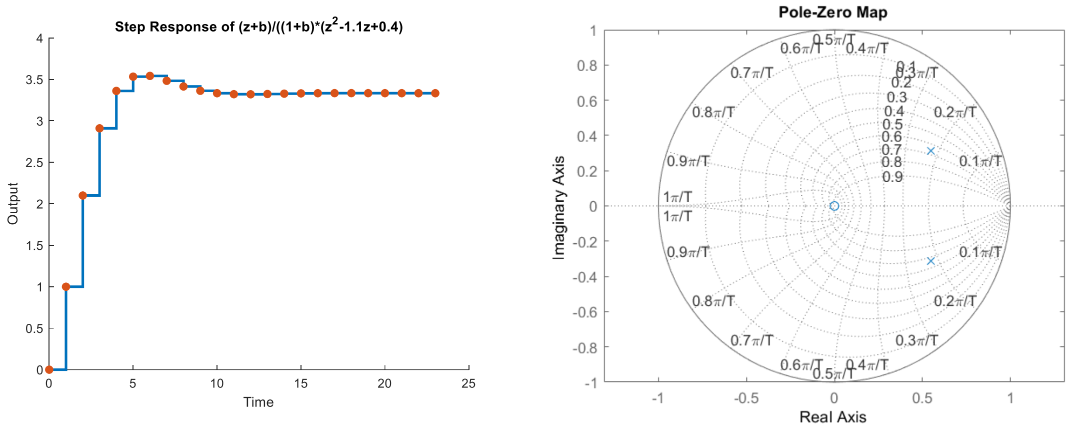
(a) (b)

Figure 2-8. When a = 0.4 and b = -0.75, (a) notes that step response, and (b) notes the poles and zeros in z-Transform.



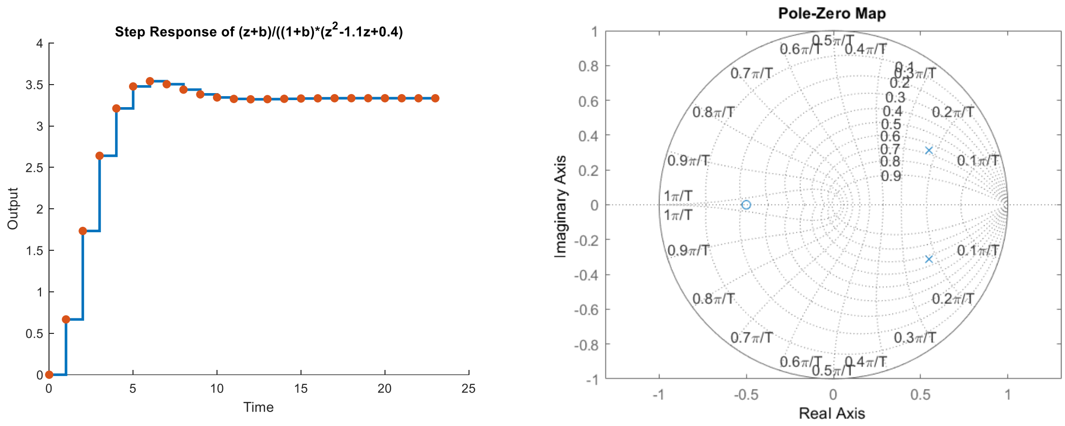
(a) (b)

Figure 2-9. When a = 0.4 and b = -0.5, (a) notes that step response, and (b) notes the poles and zeros in z-Transform.



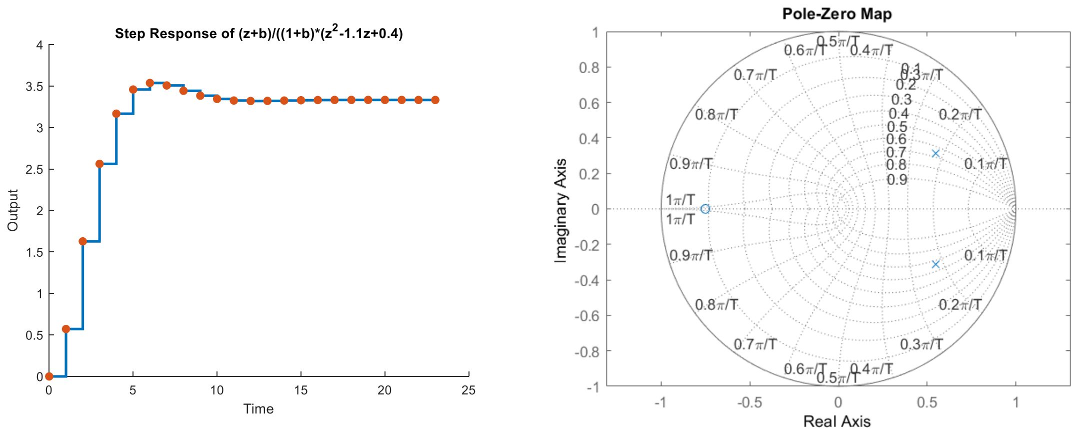
(a) (b)

Figure 2-10. When a = 0.4 and b = 0, (a) notes that step response, and (b) notes the poles and zeros in z-Transform.



(a) (b)

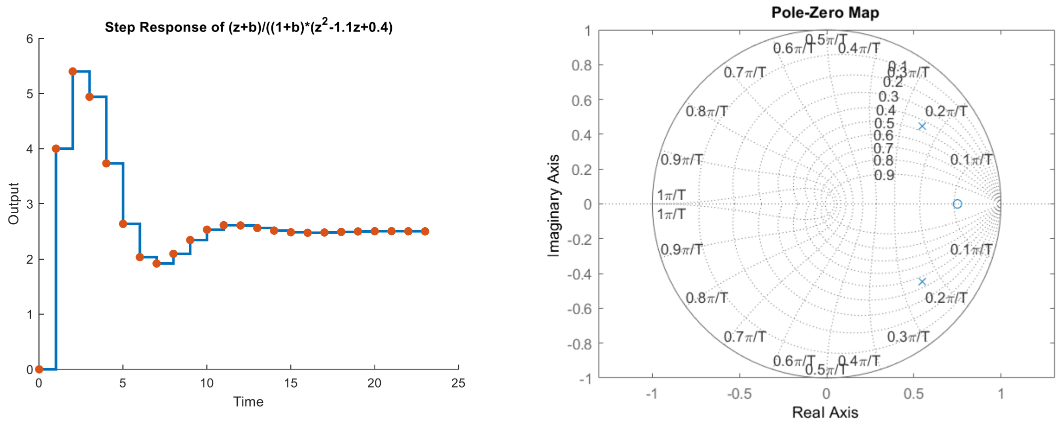
Figure 2-11. When a = 0.4 and b = 0.5, (a) notes that step response, and (b) notes the poles and zeros in z-Transform.



(a) (b)

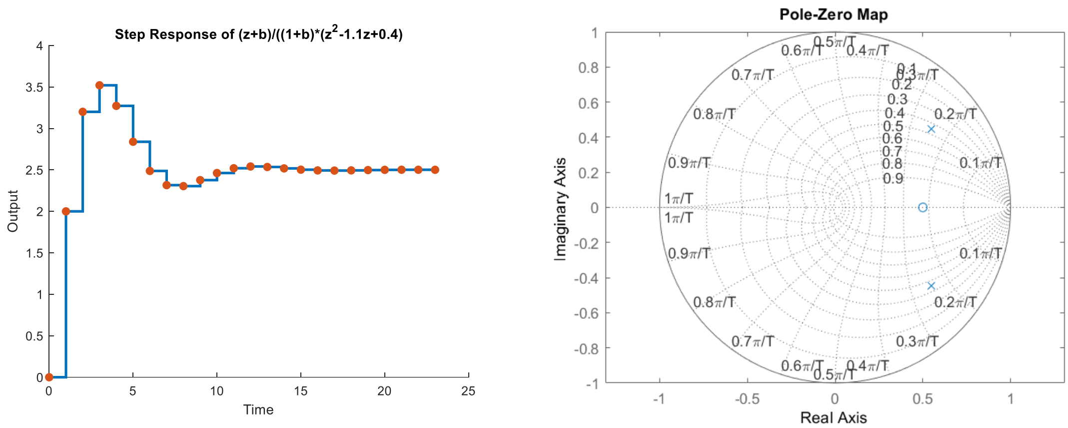
Figure 2-12. When a = 0.4 and b = 0.75, (a) notes that step response, and (b) notes the poles and zeros in z-Transform.

When $a = 0.5,\ b \in [-0.75, -0.5, 0, 0.5, 0.75]$:



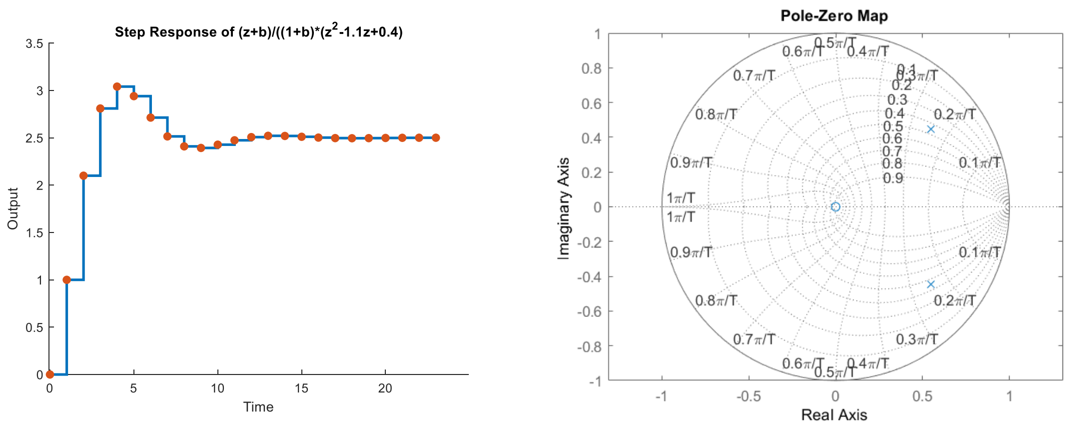
(a) (b)

Figure 2-13. When a = 0.5 and b = -0.75, (a) notes that step response, and (b) notes the poles and zeros in z-Transform.



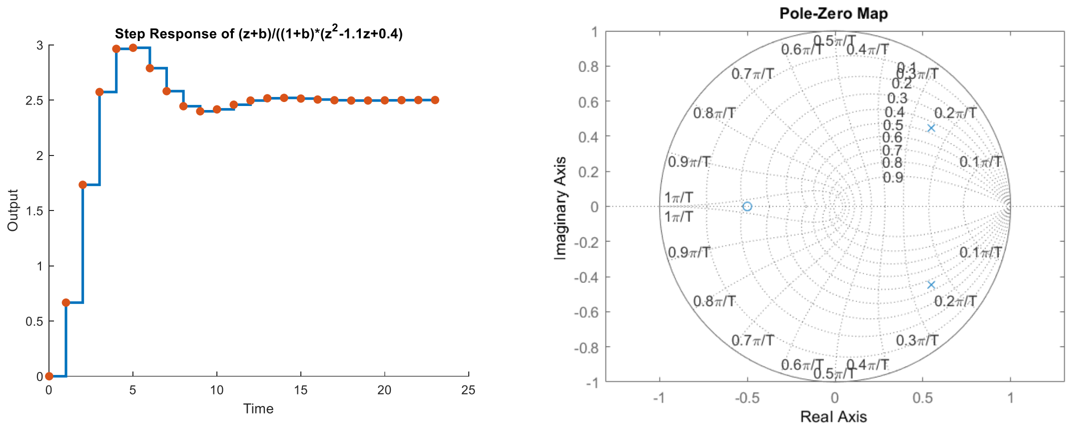
(a) (b)

Figure 2-14. When a = 0.5 and b = -0.5, (a) notes that step response, and (b) notes the poles and zeros in z-Transform.



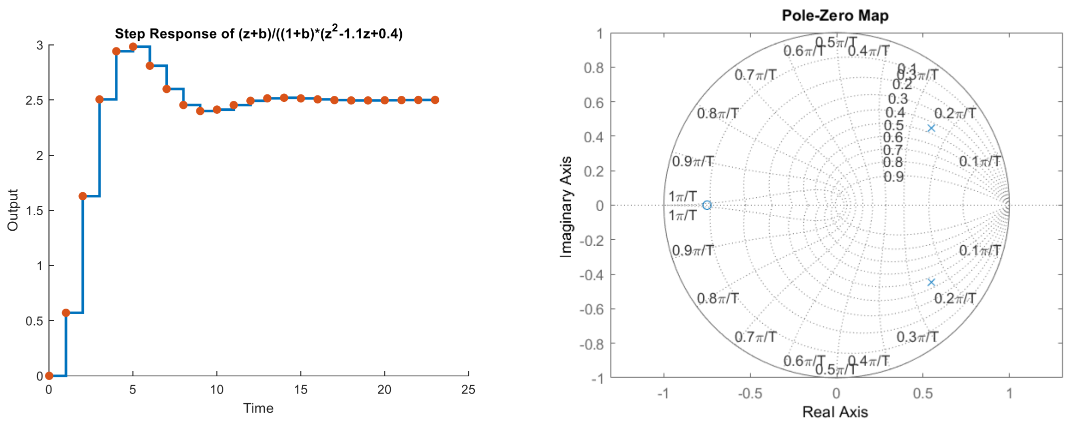
(a) (b)

Figure 2-15. When a = 0.5 and b = 0, (a) notes that step response, and (b) notes the poles and zeros in z-Transform.



(a) (b)

Figure 2-16. When a = 0.5 and b = 0.5, (a) notes that step response, and (b) notes the poles and zeros in z-Transform.



(a) (b)

Figure 2-17. When a = 0.5 and b = 0.75, (a) notes that step response, and (b) notes the poles and zeros in z-Transform.

The overshoot corresponding to different pairs of a and b is shown in TABLE 2-1:

TABLE 2-1

OVERSHOOT CORRESPONDENCE

Value of a	Value of b	Overshoot (%)
0.3	-0.75	14.8
0.3	-0.5	0
0.3	0	0
0.3	0.5	0
0.3	0.75	0
0.4	-0.75	62.2
0.4	-0.5	14.3
0.4	0	6.2
0.4	0.5	6.1
0.4	0.75	6.1
0.5	-0.75	115.6
0.5	-0.5	41.1
0.5	0	21.8
0.5	0.5	19.0
0.5	0.75	19.4

The locations of poles and zeros differ because of the different pairs of a and b. Therefore, the systems result in the different step responses. The TABLE 2-1 shows that the increment of a makes the overshoot larger and the increment of b makes the overshoot smaller.

Problem 3-3

[Ref: HW3_劉仲恩 Chung-En Liu_Digital Control_HW3_20190412]

The main issue of this problem is to find the magnitude of the spectrum of a signal that is sampled at different rates.

The horizontal axis refers to the frequency of the signal, while the vertical axis refers to the magnitude of the signal. The given signal is a triangular wave, with magnitude one and frequency of 20 rad/s. It is also known that $\omega = 2\pi f = \frac{2\pi}{T}$. F

In part (a), the sampling time is $T = \frac{2\pi}{10}$. Which means that (rad/s). From the graph given in the problem, we can roughly sketch 3 triangle waves as in Figure 1.4-1:

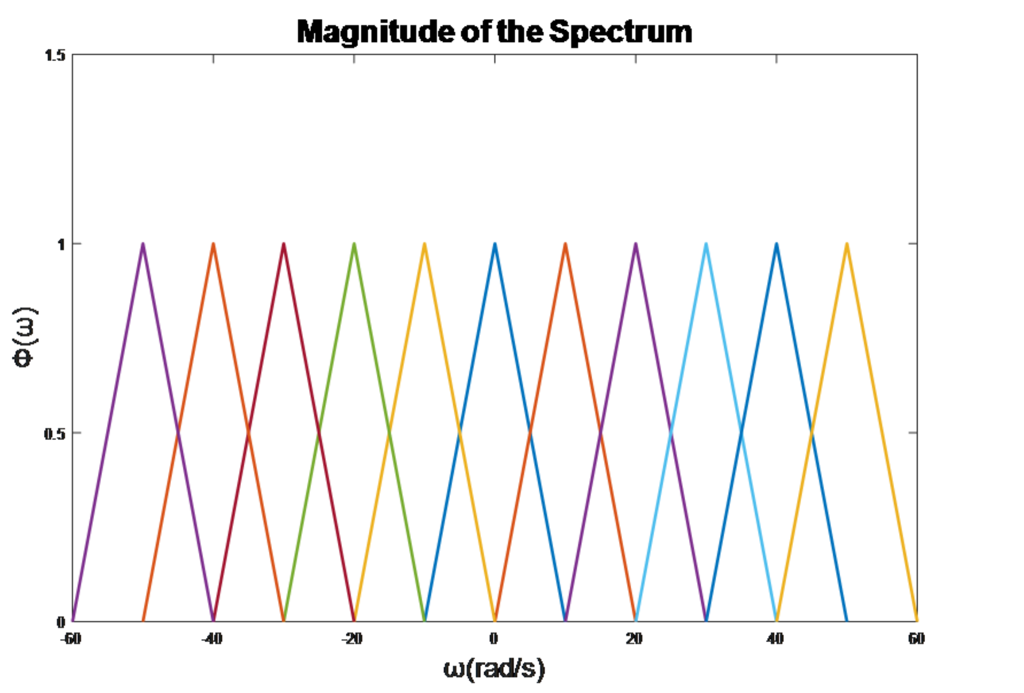


Figure 1.4-1. Three triangle waves are produced, each of magnitude 1 and a period of 20 rad/s. Aliasing occurs as the sampling time is shorter than the period of each triangular wave. The overlapped regions all have summed magnitude of one.

Since the period for each cycle is longer than the sampling rate, we see that aliasing occurs between adjacent triangular waves. At $\omega = 0$, we can see that the peak of the second wave meets with the lowest value, which is zero, of the first and third waves. At $\omega = 5$, we can see that both the second and the third wave have values at 0.5, which means the sum of the value at that point is equal to one. Similarly, for each value of ω, the sum of the values are equal to one. So, by summing up the overlapped areas, we get the magnitude of the spectrum as in Figure 1.4-2:

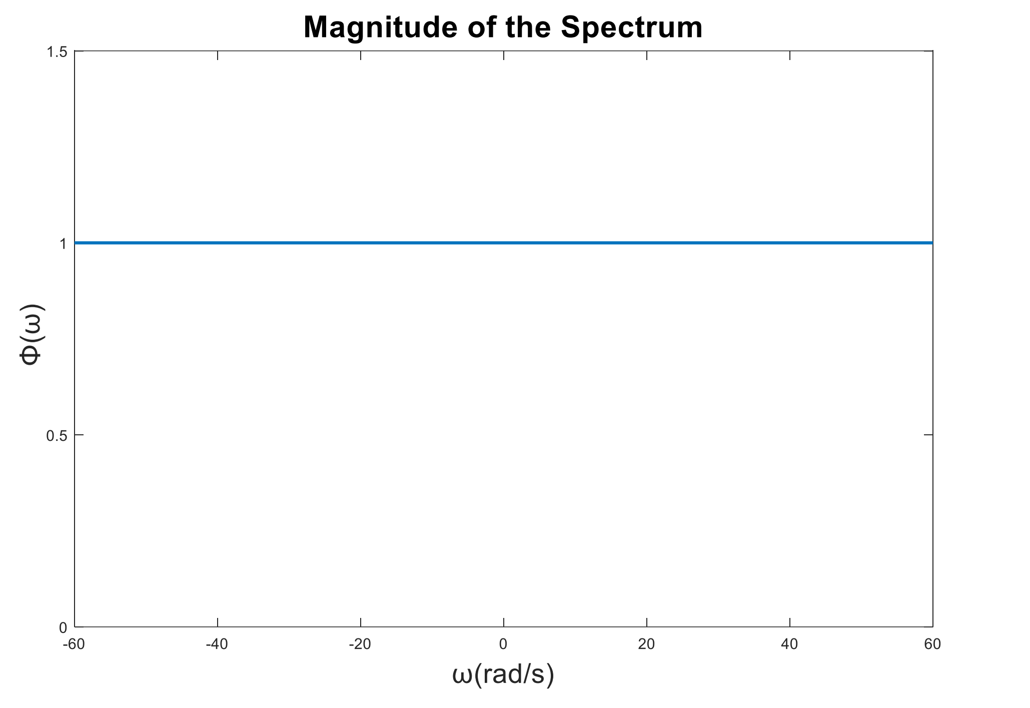


Figure 1.4-2. The aliased result of the magnitude of the spectrum for part (a).

In part (b), the sampling time is $T=\frac{2\pi}{20}$. This means that $\omega=\frac{2\pi}{(\frac{2\pi}{20})}=20\,(\text{rad/s})$, which is equal to the period of each triangular wave provided in the problem set. Thus, we can know that the start and end point of adjacent triangular waves meet at value 0. The peak value of the triangular waves is one and situated at ω that are multiples of 20. This sampling frequency is also known as the Nyquist Frequency. We can then roughly sketch the magnitude of the spectrum as below in Fig 1.4-3:

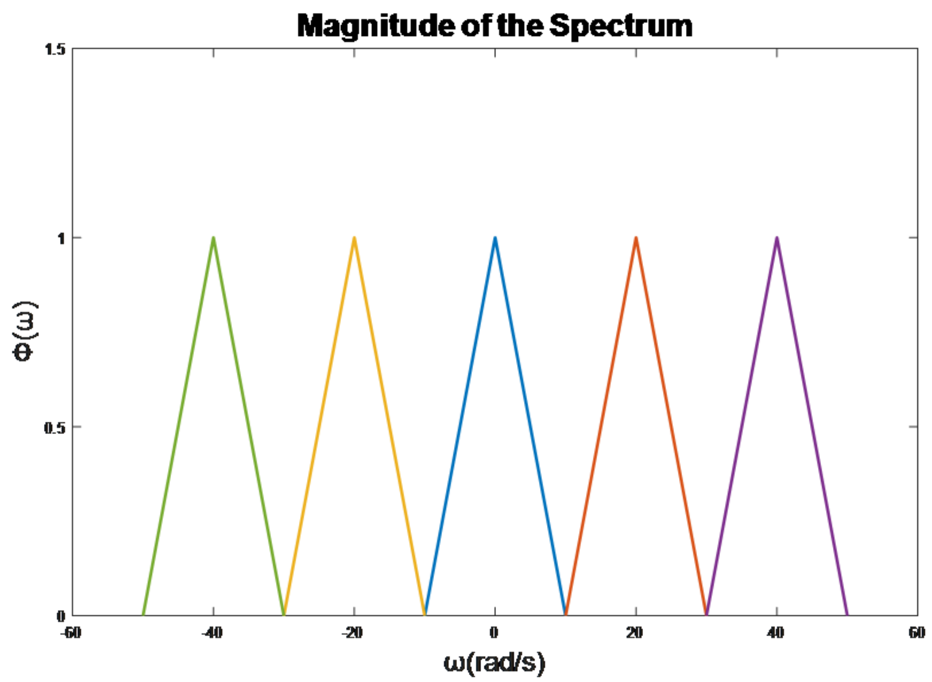


Figure 1.4-3. Three triangle waves are produced, each of magnitude 1 and a period of 20 rad/s. Since the sampling distance is equal to the period of each triangular wave, the start and end point of adjacent waves meet at 0. The peak value of the triangular waves is one and situated at ω that are multiples of 20.

As In part (c), the sampling time is $T=\frac{2\pi}{50}$. This means that $\omega=\frac{2\pi}{(\frac{2\pi}{50})}=50$ (rad/s), which is way larger than the period of each triangular wave. Thus, adjacent triangular waves are separated, with the peak value being one and situated at ω that are multiples of 50. We can then roughly sketch the magnitude of the spectrum as below in Fig 1.4-4:

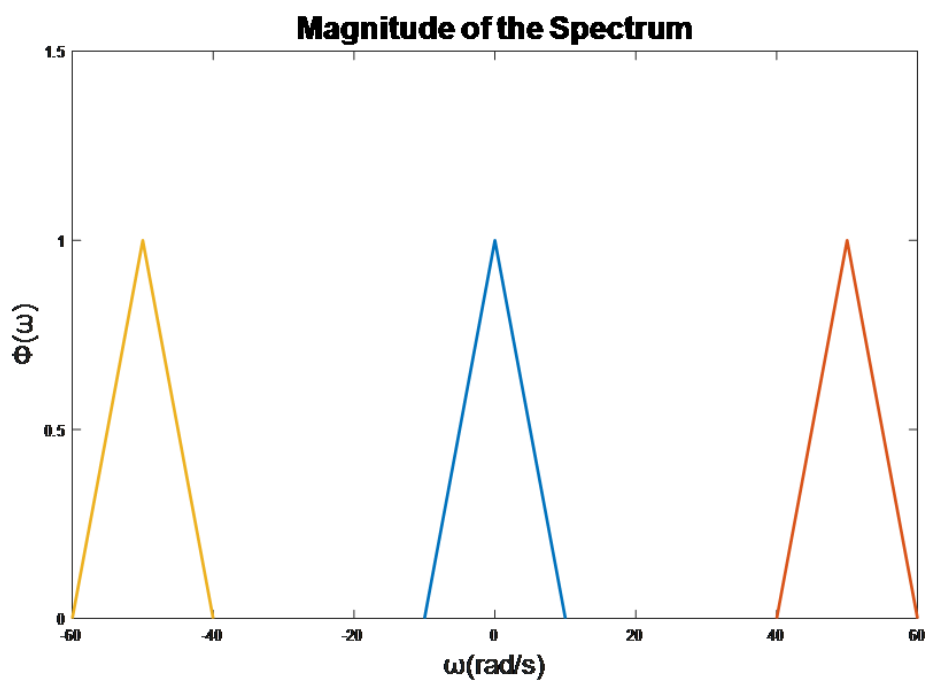


Figure 1.4-4. Three triangle waves are produced, each of magnitude 1 and a period of 20 rad/s. Since the sampling distance is much larger than the period of each triangular wave, adjacent waves are far apart from each other, with the peak value being one and situated at ω that are multiples of 50.

From the above results, we can see how sampling rates effect the magnitude of the spectrum. If the sampling rate is too high, aliasing occurs, resulting in a change in the magnitude of the spectrum. As for lower sampling rates, effects will not be visible.