

HW 2: Discrete-Time Models	Digital Control Systems, Spring 2021, NTU-EE
Name: 参考答案	Date: 3/21, 2021

Problem 2-1:

[Ref: HW2_林柏宇_林柏宇_DCS_HW02_20190322_Discrete-Time Models]

Derive the formulas of the DT State-Space System with Inner Time Delay.

Consider the LTI system with inner time delay:

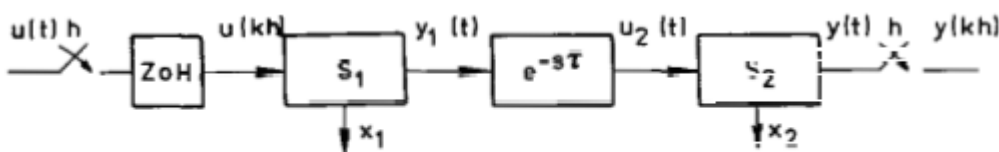


Fig. 1. The time delay system.

[1: Wittenmark 1985]

$S_1: \begin{cases} \dot{x}_1(t) = A_1 x_1(t) + B_1 u_1(t) \\ y_1(t) = C_1 x_1(t) + D_1 u_1(t) \end{cases}$	(2.1)
$S_2: \begin{cases} \dot{x}_2(t) = A_2 x_2(t) + B_2 u_2(t) \\ y_2(t) = C_2 x_2(t) + D_2 u_2(t) \end{cases}$	(2.2)

$u_2(t) = y_1(t - \tau)$	(2.3)
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Where τ is the delay time

And let

$\tau = (d - 1)h + \tau'$	(2.4)
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Where d is an integer, h is the sampling period, and τ' is a fraction of the sampling interval, i.e.,

$0 < \tau' \leq h$	
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Three cases will be considered:

Case 1 – time delay before the system, i.e., $A_1 = B_1 = C_1 = D_2 = 0$, $D_1 = I$

Case 2 – time delay after the system, i.e., $A_2 = B_2 = C_2 = D_1 = 0$, $D_2 = I$

Case 3 – time delay between the subsystems. Assume $D_1 = D_2 = 0$

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Case 1 – time delay before the system:

Assume subsystem S_1 has a unity transfer function:

$$\rightarrow A_1 = B_1 = C_1 = D_2 = 0, D_1 = I$$

Assume state of S_2 is known at $t = kh$,

The state at $kh + h$ is given by solving S_2 :

$x_2(kh + h) = e^{A_2 h} x_2(kh) + \int_{kh}^{kh+h} e^{A_2(kh+h-s)} B_2 u_2(s) ds$	
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From $u_2(t) = y_1(t - \tau) = u(t - \tau)$

The integral can be separated into $u_2(t) = u(kh - dh)$ & $u_2(t) = u(kh - (d - 1)h)$

$x_2(kh + h) = e^{A_2 h} x_2(kh) + \int_0^{t'} e^{A_2(h-\tau')} e^{A_2 s'} B_2 ds' \cdot u(kh - dh)$ $+ \int_0^{h-t'} e^{A_2 s'} B_2 ds' \cdot u(kh - (d - 1)h)$	
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Introduce the notations:

$\Phi_i(t) = e^{A_i t}$	(3.1)
$\Gamma_i(t) = \int_0^t e^{A_i s} B_i ds$	(3.2)

Then,

$x_2(kh + h) = \Phi_2(h)x_2(kh) + \Phi_2(h - \tau')\Gamma_2(\tau')u(kh - dh)$ $+ \Gamma_2(h - \tau')u(kh - (d - 1)h)$	Ans. of Case 1 (3.3)
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Case 2 – time delay after the system:

Assume subsystem S_2 has a unity transfer function.

$$\rightarrow A_2 = B_2 = C_2 = D_1 = 0, D_2 = I$$

$x_1(kh + h) = \Phi_1(h)x_1(kh) + \Gamma_1(h)u(kh)$	Ans. of Case 2 (3.4)
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Where

$\Phi_1(h) = e^{A_1 h}$ $\Gamma_1(t) = \int_0^t e^{A_1 s} B_1 ds$	
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Case 3 – time delay between the subsystems:

(1) For $\tau = 0$, system =

$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} A_1 & 0 \\ A_{21} & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u \\ y &= [0 \quad C_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$	(3.6)
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Where

$A_{21} = B_2 C_1$	
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Then,

$x_2(kh + h) = e^{A_2 h} x_2(kh) + \int_{kh}^{kh+h} e^{A_2(kh+h-s)} A_{21} x_1(s) ds$	(3.7)
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In the interval $kh < s \leq kh + h$

$x_1(s) = e^{A_1(s-kh)} x_1(kh) + \int_{kh}^s e^{A_1(s-s')} B_1 ds' \cdot u(kh)$	(3.8)
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Thus,

$\begin{aligned} x_2(kh + h) &= \Phi_2(h) x_2(kh) \\ &+ \int_{kh}^{kh+h} e^{A_2(kh+h-s)} A_{21} e^{A_1(s-kh)} ds x_1(kh) \\ &+ \int_{kh}^{kh+h} e^{A_2(kh+h-s)} A_{21} \int_{kh}^s e^{A_1(s-s')} B_1 ds' ds u(kh) \\ &= \Phi_2(h) x_2(kh) + \Phi_{21}(h) x_1(kh) + \Gamma'_2(h) u(kh) \end{aligned}$	(3.9)
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Where

$\Phi_{21}(h) = \int_0^h e^{A_2 s'} A_{21} e^{A_1(h-s')} ds'$	(3.10)
$\Gamma'_2(h) = \int_0^h e^{A_2 s''} A_{21} \Gamma_1^{(h-s'')} ds''$	(3.11)

The sampled version of (3.6):

$\begin{aligned} \begin{bmatrix} x_1(kh + h) \\ x_2(kh + h) \end{bmatrix} &= \begin{bmatrix} \Phi_1(h) & 0 \\ \Phi_{21}(h) & \Phi_2(h) \end{bmatrix} \begin{bmatrix} x_1(kh) \\ x_2(kh) \end{bmatrix} + \begin{bmatrix} \Gamma_1(h) \\ \Gamma'_2(h) \end{bmatrix} u(kh) \\ y(kh) &= [0 \quad C_2] \begin{bmatrix} x_1(kh) \\ x_2(kh) \end{bmatrix} \end{aligned}$	(3.12)
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If there is a delay in the system, then $x_1(s)$ in (3.7) is replaced by $x_1(s - \tau)$, i.e., if the delay is a multiple of sampling interval, then (3.9) is changed to:

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$$x_2(kh + h) = \Phi_2(h)x_2(kh) + \Phi_{21}(h)x_1(kh - (d - 1)h) + \Gamma'_2(h)u(kh - (d - 1)h)$$

When delay is not a multiple of sampling period, it can be assumed that

$$0 < \tau \leq h$$

Assume the states of S_1 are denoted x'_1 and are related to the state of S_1 through

$$x'_1(t) = x_1(t - \tau)$$

Then,

$$\begin{aligned} \begin{bmatrix} x'_1(kh + h) \\ x_2(kh + h) \end{bmatrix} &= \begin{bmatrix} \Phi_1(h) & 0 \\ \Phi_{21}(h) & \Phi_2(h) \end{bmatrix} \begin{bmatrix} x'_1(kh) \\ x_2(kh) \end{bmatrix} \\ &+ \begin{bmatrix} \Phi_1(h - \tau) & 0 \\ \Phi_{21}(h - \tau) & \Phi_2(h - \tau) \end{bmatrix} \begin{bmatrix} \Gamma_1(\tau) \\ \Gamma'_2(\tau) \end{bmatrix} u(kh - h) \\ &+ \begin{bmatrix} \Gamma_1(h - \tau) \\ \Gamma'_2(h - \tau) \end{bmatrix} u(kh) \end{aligned}$$

Since $x'_1(kh) = x_1(kh - \tau)$

$$\begin{aligned} x_2(kh + h) &= \Phi_2(h)x_2(kh) \\ &+ \Phi_{21}(h)[\Phi_1(h - \tau)x_1(kh - h) \\ &+ \Gamma_1(h - \tau)u(kh - h)] + \Gamma'_2(h - \tau)u(kh) \\ &+ [\Phi_{21}(h - \tau)\Gamma_1(\tau) + \Phi_2(h - \tau)\Gamma'_2(\tau)]u(kh - h) \end{aligned}$$

Therefore, we have:

$$\begin{aligned} x_1(kh + h) &= \Phi_1(h)x_1(kh) + \Gamma_1(h)u(kh) \\ x_2(kh + h) &= \Phi_2(h)x_2(kh) \\ &+ \Phi_{21}(h)[\Phi_1(h - \tau)x_1(kh - h) \\ &+ \Gamma_1(h - \tau)u(kh - h)] + \Gamma'_2(h - \tau)u(kh) \\ &+ [\Phi_{21}(h - \tau)\Gamma_1(\tau) + \Phi_2(h - \tau)\Gamma'_2(\tau)]u(kh - h) \end{aligned}$$

Final Answer

Where

$$\begin{aligned} \Phi_1(h) &= e^{A_1 h} \\ \Gamma_1(t) &= \int_0^t e^{A_1 s} B_1 ds \\ \Phi_2(h) &= e^{A_2 h} \\ \Gamma_2(t) &= \int_0^t e^{A_2 s} B_2 ds \\ \Phi_{21}(h) &= \int_0^h e^{A_2 s'} A_{21} e^{A_1(h-s')} ds' \end{aligned}$$

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$\Gamma_2'(h) = \int_0^h e^{A_2 s''} A_{21} \Gamma_1^{(h-s'')} ds''$	
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Problem 2-2:

[Ref: HW2_張立揚_數位控制 HW2_R07921008_張立揚]

From the given equation of y and $\frac{d}{dt}x$, we get A , B , C and D in (2-1), while our goal is to get F and H in (2-2).

<p>where</p> $\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$ $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $C = 1$ $D = 0$	(2-1)
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$\begin{aligned} x[k+1] &= Fx[k] + Hu[k] \\ y[k] &= Cx[k] + Du[k] \end{aligned}$	(2-2)
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Let “h” be the sampling period, we can calculate F and H with the equations in Lecture Note: DCS-11 page 11, as shown in (2-3) and (2-4).

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$F = e^{Ah} = I + Ah + \frac{A^2 h^2}{2!} + \dots$ $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} h + \frac{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^2 h^2}{2!} + \dots$ $= \begin{bmatrix} 1 - \frac{h^2}{2!} + \frac{h^4}{4!} - \dots & h - \frac{h^3}{3!} + \frac{h^5}{5!} + \dots \\ -h + \frac{h^3}{3!} - \frac{h^5}{5!} + \dots & 1 - \frac{h^2}{2!} + \frac{h^4}{4!} - \dots \end{bmatrix}$ <p>approximated F to the inverse of a Taylor series, we get:</p> $F \approx \begin{bmatrix} \cos(h) & \sin(h) \\ -\sin(h) & \cos(h) \end{bmatrix}$	<p>Ans. of F (2-3)</p>
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$H = \left(\int_0^h e^{A\eta} d\eta \right) B = \left(\sum_{k=0}^{\infty} \frac{A^k h^{k+1}}{(k+1)!} \right) B$ $= \left(Ih + A \frac{h^2}{2!} + A^2 \frac{h^3}{3!} + \dots \right) B$ <p>with similar approximation in (2-3), we get:</p> $H \approx \begin{bmatrix} \sin(h) & 1 - \cos(h) \\ \cos(h) - 1 & \sin(h) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 - \cos(h) \\ \sin(h) \end{bmatrix}$	<p>Ans. of H (2-4)</p>
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With F and H being calculated, we can present the discrete-time system as (2-2).

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Problem 2-3

The difference equation:

$y[k + 2] - 1.5y[k + 1] + 0.5y[k] = u[k + 1]$	(1)
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For initial condition, let $k=-1$:

$y[1] - 1.5y[0] + 0.5y[-1] = u[0]$	(2)
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Substitute $y[0]$, $y[-1]$ into (2), derive $y[1]$:

$y[1] = 1.25$	
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Consider the initial condition, the zTransform will be:

$y[k + 2] \rightarrow z^2(Y(z) - y[0] - z^{-1}y[1])$ $y[k + 1] \rightarrow z(Y(z) - y[0])$ $y[k] \rightarrow Y(z)$ $u[k + 1] \rightarrow z(U(z) - u[0])$	(3)
--	-----

where

$y[1] = 1.25, y[0] = 0.5, y[-1] = 1, u[0] = 1$	
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Substitute (3) into (1) and simplify, then we can derive the equation:

$Y(z) = z \frac{1}{(z - 0.5)(z - 1)} U(z) + 0.5z \frac{1}{z - 0.5}$	(4)
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$= \left[\frac{-2z}{z - 0.5} + \frac{2z}{z - 1} \right] U(z) + 0.5 \frac{z}{z - 0.5}$	(5)
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$= \left[\frac{z}{(z - 0.5)(z - 1)} \right] U(z) + 0.5 \frac{z}{z - 0.5}$	(6)
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Finally, the pulse-transfer function is:

$G(z) = \frac{U(z)}{U(z)} = \frac{z}{(z - 0.5)(z - 1)}$	(7)
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And, its poles and zeros are: poles: $z=0.5, 1$, zeros: $z=0$.

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Problem 2-4

[Ref: HW2_張峻豪_r07921012 張峻豪_DCS_HW2_107 0322_Discrete-Time Models]

From the state space equation:

$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, h = 0.3, \tau = 0.2$	
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Substitute A into the formulation of F:

$F = e^{Ah} = \sum_{k=0}^{\infty} \frac{A^k h^k}{k!} = \sum_{k=0}^{\infty} \frac{h^k}{k!} \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{h^k}{k!} & 0 \\ h \sum_{k=0}^{\infty} \frac{h^k}{k!} & \sum_{k=0}^{\infty} \frac{h^k}{k!} \end{bmatrix}$	
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F is derived:

$F = \begin{bmatrix} e^h & 0 \\ he^h & e^h \end{bmatrix}$	(1)
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Substitute h = 0.3 into (1):

$F = \begin{bmatrix} 1.3499 & 0 \\ 0.4050 & 1.3499 \end{bmatrix} \approx \begin{bmatrix} 1.350 & 0 \\ 0.405 & 1.350 \end{bmatrix}$	Ans. of F (2)
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For H_1 :

$H_1 = e^{A(h-\tau)} \int_0^{\tau} e^{A\eta} B d\eta$	
$H_1 = \begin{bmatrix} e^{h-\tau} & 0 \\ (h-\tau)e^{h-\tau} & e^{h-\tau} \end{bmatrix} \sum_{k=0}^{\infty} \frac{A^k \tau^{k+1}}{(k+1)!} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$	
$H_1 = \begin{bmatrix} e^{h-\tau} & 0 \\ (h-\tau)e^{h-\tau} & e^{h-\tau} \end{bmatrix} \begin{bmatrix} \sum_{k=0}^{\infty} \frac{\tau^k}{k!} - 1 & 0 \\ \tau \sum_{k=0}^{\infty} \frac{\tau^k}{k!} - \sum_{k=0}^{\infty} \frac{\tau^k}{k!} + 1 & \sum_{k=0}^{\infty} \frac{\tau^k}{k!} - 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$	

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$H_1 = \begin{bmatrix} e^{h-\tau} & 0 \\ (h-\tau)e^{h-\tau} & e^{h-\tau} \end{bmatrix} \begin{bmatrix} e^\tau - 1 & 0 \\ \tau e^\tau - e^\tau + 1 & e^\tau - 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $= \begin{bmatrix} e^{h-\tau}(-1 + e^\tau) \\ e^{h-\tau}(1 - h + \tau) + (h-1)e^h \end{bmatrix}$	(3)
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Substitute $h = 0.3$, $d = 0.2$ into (3):

$H_1 = \begin{bmatrix} 0.2447 \\ 0.0498 \end{bmatrix} \approx \begin{bmatrix} 0.245 \\ 0.050 \end{bmatrix}$	Ans. of H₁ (4)
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Similarly, for H_0 :

$H_0 = \int_0^{h-\tau} e^{A\eta} B d\eta$	
$H_0 = \sum_{k=0}^{\infty} \frac{A^k (h-\tau)^{k+1}}{(k+1)!} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$	
$H_0 = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{(h-\tau)^k}{k!} - 1 & 0 \\ (h-\tau) \sum_{k=0}^{\infty} \frac{(h-\tau)^k}{k!} - \sum_{k=0}^{\infty} \frac{(h-\tau)^k}{k!} + 1 & \sum_{k=0}^{\infty} \frac{(h-\tau)^k}{k!} - 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$	
$H_0 = \begin{bmatrix} e^{h-\tau} - 1 & 0 \\ (h-\tau)e^{h-\tau} - e^{h-\tau} + 1 & e^{h-\tau} - 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$	(5)

Substitute $h = 0.3$, $d = 0.2$ into (5):

$H_0 = \begin{bmatrix} 0.1052 \\ 0.0053 \end{bmatrix} \approx \begin{bmatrix} 0.105 \\ 0.005 \end{bmatrix}$	Ans. of H₀ (6)
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For the formulation of the pulse-transfer operator:

$G(q) = C(qI - F)^{-1}(H_0 + H_1 q^{-1})$	(7)
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Combine the (2), (4), (6), and (7), the pulse-transfer operator $G(q)$ is derived:

$G(q) = C \frac{\begin{bmatrix} 0.1052q^2 + 0.1027q - 0.3303 \\ 0.0053q^2 + 0.0853q + 0.0319 \end{bmatrix}}{q(q - 1.3499)^2}$	(8)
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References

[1: Wittenmark 1985]

Bjorn Wittenmark, "Sampling of a system with a time delay," IEEE Transactions on Automatic Control, Vol. 30, No. 5, pp. 507-510, May 1985.