Discretized Controller – Techniques for Enhancing Performance (T/2-Delay Compensation)

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Feb – Jun, 2021

Figures and images used in these lecture notes are adopted from
Introduction: CT and DT Plant-Controller

- **Transform CT plant into DT plant**
- **By DT plant, design DT controller**
- **Discrete Design**

- **Emulation**
- **By CT plant, design CT controller**
- **Transform CT controller into DT controller**

- **Direct Design**
Basic Design Concept

- Basic principles of low-order controller design

![Diagram of basic design concept]

Fig. 1. Basic principles of low order controller design.

Study in Digital Control Systems

- Controller Design of Digital Control Systems

  - Design Process

    > Discrete Design:
      » CT plant -> DT plant -> DT controller

    > Emulation:
      » CT plant -> CT controller -> DT controller

    > Direct Design: (B.D.O. Anderson, 1992 Bode Prize Lecture)
      » CT plant -> DT controller
Outline

- Discrete Design
  - By Transfer Function
  - By State Space

- Design by Emulation
  - Tustin’s Method or bilinear approximation
  - Matched Pole-Zero method (MPZ)
  - Modified Matched Pole-Zero method (MMPZ)
  - Digital PID-Controllers

- Techniques for Enhancing the Performance
Basic Design Concept

- Textbook schemes for replacing a CT controller by a DT one

With $C(s)$ continuous time and $C_d(z)$ discrete time,

$$C_d(z) = C \left( \frac{z-1}{T} \right)$$

Euler or forward difference

$$C_d(z) = C \left( \frac{z-1}{zT} \right)$$

Balanced difference

$$C_d(z) = C \left( \frac{2(z-1)}{Tz+1} \right)$$

Tustin or bilinear

$$C_d(z) = C \left( \frac{\omega_1 z-1}{\tan \left( \frac{\omega_1 T}{2} \right) Z+1} \right)$$

Tustin with prewarping

$$C_d(z) = \frac{(z-1)}{Tz} \frac{1}{2\pi j} \int_{\gamma-j\infty}^{\gamma+j\infty} \frac{e^{sT} G(s)}{s} ds$$

Step-invariance

$$C_d(z) = \frac{(z-1)^2}{Tz} \frac{1}{2\pi j} \int_{\gamma-j\infty}^{\gamma+j\infty} \frac{e^{sT} G(s)}{s^2} ds$$

Ramp-invariance

Poles and zeros of $C_d(z)$ are images under $z = e^{sT}$ of those of $C(s)$, with $C_d(1) = C(0)$.

Zero-order hold equivalence.

First-order hold equivalence.

Triangular-hold equivalence.

Anderson 1993
Measuring the difference between using Continuous and Discrete controllers

- Data
  Shaping or Weighting Filter $W(s)$, rational, stable and strictly proper

- Assumption
  $C_d(z)$ (unknown) is stabilizing

- Operator from $r \in L_2(0, \infty)$ to $e \in L_2(0, \infty)$ is
  - bounded
  - dependent on $C_d(z)$
  - not describable via a transfer function
  - periodically time-varying

Fig. 7. Measuring the difference between using continuous and discrete controllers.

Anderson 1993
Problem Formulation

- Digital control design through discretizing an analog controller

**Fig. 1.** (a) The analog closed-loop control system, (b) The digital closed-loop control system.
Problem Formulation

- **Given**
  - A process $G_p(s)$
  - A sensor $H(s)$
  - A presumably well designed analog controller $G_c(s)$

- **Find**
  - A digital controller $D(z)$ which produces closed-loop behavior similar to the analog system both in the time and frequency domains

Raviv & Djaja 1999
Problem Formulation

- **Solutions:**
  - Analog control design followed by controller discretization
    - More convenient
    - Deal with sampling time $T$ at the final phase
  - Direct digital control design
  - To enhance the performance by the first method
    - Add a pole-zero pair in the $z$-plane
    - To compensate for the low-frequencies and mid-frequencies phase and gain response effects contributed by ZOH

Raviv & Djaja 1999
Problem Formulation

- Potential problem:
  - The ZOH causes a delay of approximately $T/2$

*Fig. 2. A reconstructed signal using ZOH and its smoothed approximation.*
A pole-zero compensation for delay:

\[ C(z) = \frac{2z}{z + 1} \]

- Provides a phase of \((\omega T/2)\)
- Which exactly cancels the frequency phase response of the ZOH obtained from Raviv & Djaja 1999
A pole-zero compensation for delay:

- The ZOH transfer function:
  \[
  \frac{1 - e^{-sT}}{s} \approx \frac{T}{1 + \frac{sT}{2}}
  \]
  1st-order Pade approximation

  \[
  \Rightarrow \frac{T}{1 + \frac{sT}{2}} \bigg|_{s = \frac{2}{T} \frac{z-1}{z+1}} = \frac{T}{2} \frac{z + 1}{z}
  \]  
  Tustin transformation

- The characteristic polynomial

  \[
  1 + \left( \frac{2z}{z + 1} \right) D'(z) \left( 1 - z^{-1} \right) \mathcal{Z} \left\{ \frac{G_p(s)}{s} \right\} = 0
  \]

  \[D'(z) : \text{any discretized } D(s)\]

Raviv & Djaja 1999
A pole-zero compensation for delay:

- IF the proposed compensation causes instability
  a modified ZOH compensation

\[ C'(z) = \frac{2(z-\varepsilon)}{z + 1-2\varepsilon} \]

- The characteristic polynomial

\[
1 + \left( \frac{2(z-\varepsilon)}{z + 1-2\varepsilon} \right) D'(z) (1-z^{-1}) Z \left\{ \frac{G_p(s)}{s} \right\} = 0
\]

Raviv & Djaja 1999
Pole-Zero Compensation

- A pole-zero compensation for delay:

(a) Zero-Pole Pair from ZOH Compensation

(b) The New Location of Zero-Pole Pair

Raviv & Djaja 1999
Examples – Lag

Lag Compensator:

\[ G_p(s) = \frac{4 \times 10^6}{s(s + 20)(s + 200)} \]

\[ H(s) = 1 \]

• Design specifications:
  1. Velocity error constant \( K_v \) at least 1000 \( \text{s}^{-1} \)
  2. Attenuation of all sinusoidal inputs of frequency above 400 rad/sec by at least 16
  3. Steady-state error of (up to) 1% for sinusoidal inputs for frequencies less than 1 rad/sec

Raviv & Djaja 1999
Examples – Lag

\[ G_c(s) = \frac{1}{80(s + 0.1)} (s + 8) \]

- D'(z): by Tustin

<table>
<thead>
<tr>
<th>( T )</th>
<th>D'(z)</th>
<th>Multiplier</th>
<th>D(z)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01 s</td>
<td>( 0.0130z - 0.0120 \frac{z}{z - 0.9990} )</td>
<td>( 2z \frac{z}{z + 1} )</td>
<td>( 0.0260z^2 - 0.0240z \frac{z}{z^2 + 0.0010z - 0.9990} )</td>
</tr>
<tr>
<td>0.05 s</td>
<td>( 0.0150z - 0.0100 \frac{z}{z - 0.9950} )</td>
<td>( 2z \frac{z}{z + 1} )</td>
<td>( 0.0299z^2 - 0.0200z \frac{z}{z^2 + 0.0050z - 0.9950} )</td>
</tr>
<tr>
<td>0.1 s</td>
<td>Unstable ( \frac{0.0174z - 0.0075}{z - 0.9900} )</td>
<td>( 2z \frac{z}{z + 1} )</td>
<td>Unstable ( \frac{0.0348z^2 - 0.0150z}{z^2 + 0.0100z - 0.9900} )</td>
</tr>
<tr>
<td>0.1 s</td>
<td>Unstable ( \frac{0.0174z - 0.0075}{z - 0.9900} )</td>
<td>( 2(z - 0.2) \frac{z + 0.6}{z^2 - 0.3900z - 0.5940} )</td>
<td>( 0.0348z^2 - 0.0219z + 0.0030 \frac{z}{z^2 - 0.3900z - 0.5940} )</td>
</tr>
</tbody>
</table>

Raviv & Djaja 1999
Examples – Lag

\[ T = 0.01s \]

\[ T = 0.05s \]

\[ T = 0.1s \]

Raviv & Djaja 1999
Examples – Lag

\[ T = 0.01s \]

\[ T = 0.05s \]

\[ T = 0.1s \]

Raviv & Djaja 1999
**Examples – Lead-Lag**

- **Lead-Lag Compensator:**

\[
G_p(s) = \frac{1000}{s(1 + \frac{s}{10})(1 + \frac{s}{250})}
\]

\[
H(s) = 1
\]

- **Design specifications:**
  1. Phase margin of at least 50°
  2. Velocity error constant \( K_v \) at least 1000 s\(^{-1}\)
  3. Attenuation of the input noise at 60 Hz and above by a factor of 100
  4. Steady-state error for frequencies less than 1 rad/sec less than 1%

*Raviv & Djaja 1999*
Examples – Lead-Lag

\[ G_c(s) = \frac{(1 + \frac{s}{4.5})(1 + \frac{s}{10})}{(1 + \frac{s}{0.1})(1 + \frac{s}{110})} \]

\[ T = 0.01s \]

\[ D'(z) = \frac{0.6597z^2 - 1.2897z + 0.6300}{1.00z^2 - 1.2893z + 0.2900} \]

With the ZOH compensation of \( \frac{2z}{z + 1} \)

\[ D(z) = \frac{1.3194z^3 - 2.5793z^2 + 1.26z}{z^3 - 0.2893z^2 - 0.9993z + 0.2900} \]

Raviv & Djaja 1999
Examples – Lead-Lag

Step Response

Time (s)

continous

$D_p(z)$

$D(z)$

Raviv & Djaja 1999
Examples – Lead-Lag

Fig. 6. Closed-loop step response of Example (b). $T = 0.01$ s.
Katz’s Example:

\[ G_p(s) = \frac{863.3}{s^2} \]

- Design specifications:
  1. Max phase lag at \( f = 3 \text{ Hz} \) should not be more than \( 13^\circ \)
  2. At any given frequency the CL gain should not exceed 5 dB beyond the CL dc gain
  3. Max tracking error due to an input disturbance moment of 0.028Nm should not be 0.01 rad
$G_c(s) = \frac{2940(s + 29.4)}{(s + 294)^2}$

$T = 0.03s$

$\Rightarrow D'(z) = \frac{1.8958z^2 + 1.1685z - 0.7273}{z^2 + 1.1653z + 0.3395}$

With the ZOH compensation of $\frac{2z}{z + 1}$

$\Rightarrow D(z) = \frac{3.7916z^3 + 2.3369z^2 - 1.4546z}{z^3 + 2.1653z^2 + 1.5047z + 0.3395}$
Examples – Katz

\[ T = 0.03s \]

\[ \Delta t = 0.03s \]

\[ \Delta t = 0.01s \]

\[ \Delta t = 0.003s \]

\[ >> \text{plot( time, data(:,1), 'k-') } \]

\[ >> \text{stairs( time, data(:,1), 'k-') } \]

\[ >> \text{stairs( time, data(:,1), 'k-') } \]

\[ >> \text{stairs( time, data(:,1), 'k-') } \]

\[ >> \text{stairs( time, data(:,1), 'k-') } \]
$T = 0.03 \text{s}$

$\Delta t = 0.003 \text{s}$

Sample time $= -1$ (inherited)

Raviv & Djaja 1999
Rattan’s Example:

\[ G_p(s) = \frac{10}{s(s + 1)} \]

\[ G_c(s) = \frac{1 + 0.416s}{1 + 0.319s} \]

\[ T = 0.15s \]

\[ D_{\text{Rattan}}(z) = \frac{3.436z - 2.191}{z + 0.2390} \]

\[ D'(z) = \frac{2.294z - 1.5935}{z - 0.2991} \]

\[ D(z) = \frac{4.5888z^2 - 3.6459z + 0.3187}{z^2 + 0.5009z - 0.2393} \]

Tustin transformation

Raviv & Djaja 1999
Examples – Rattan

Step Response

- **Continuous**
- **$D^p(z)$**
- **$D(z)$**
- **Rattan**

![Graph showing step response for different systems with labels and axes](image)

Raviv & Djaja 1999
Examples – Rattan

Fig. 8. Closed-loop step response of Example (d), $T = 0.15$ s.