Spring 2021

數位控制系統
Digital Control Systems

DCS-31
State Space Design

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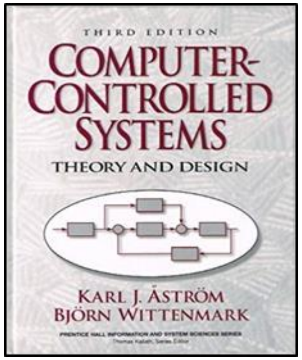
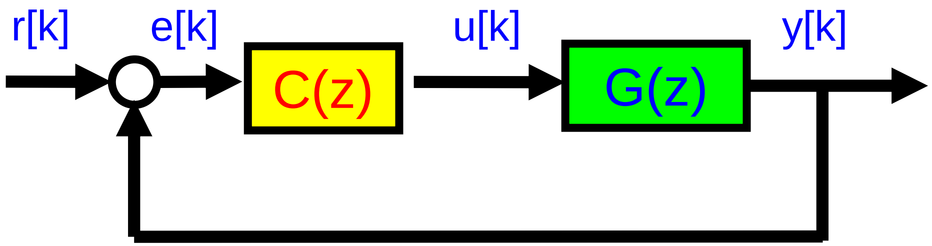
NTU-EE

Feb – Jun 2021





- Plant (DT):
  - Input-Output Model:

$$\frac{Y_d(z)}{\boxed{U_d(z)}} = G_d(z) = \frac{B_d(z)}{A_d(z)}$$

  - State-Space Model:

$$\begin{aligned} \mathbf{x}[k+1] &= \mathbf{F}\mathbf{x}[k] + \mathbf{H}\boxed{u[k]} \\ y[k] &= \mathbf{C}\mathbf{x}[k] + \mathbf{D}\boxed{u[k]} \end{aligned}$$

- design $u[k] \Rightarrow$

- System Properties:
  - Stability — unstable $\rightarrow$ stable; stable $\rightarrow$ more stable
  - Controllability (and Reachability)
  - Observability (and Detectability) — from $y[k]$ to estimate $\mathbf{x}[k]$

Outline

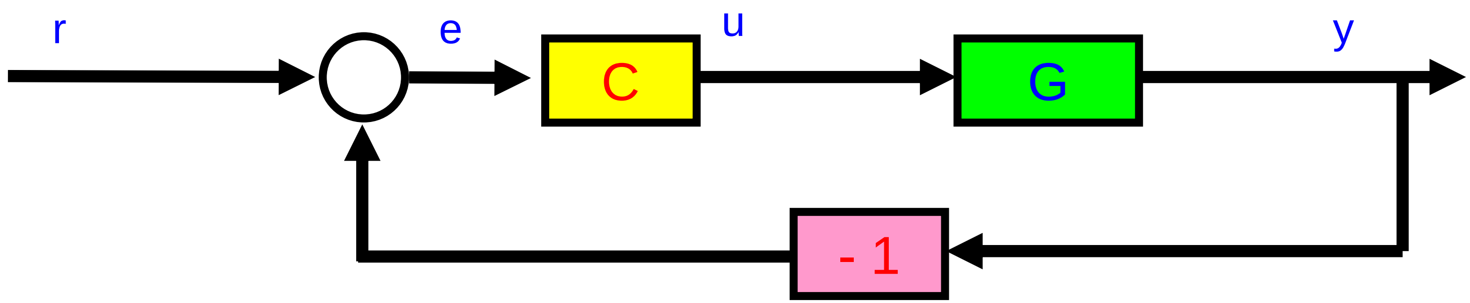
- Control System Design
- Regulation by State Feedback
- Observer Design
- Output Feedback
- Tracking Problem

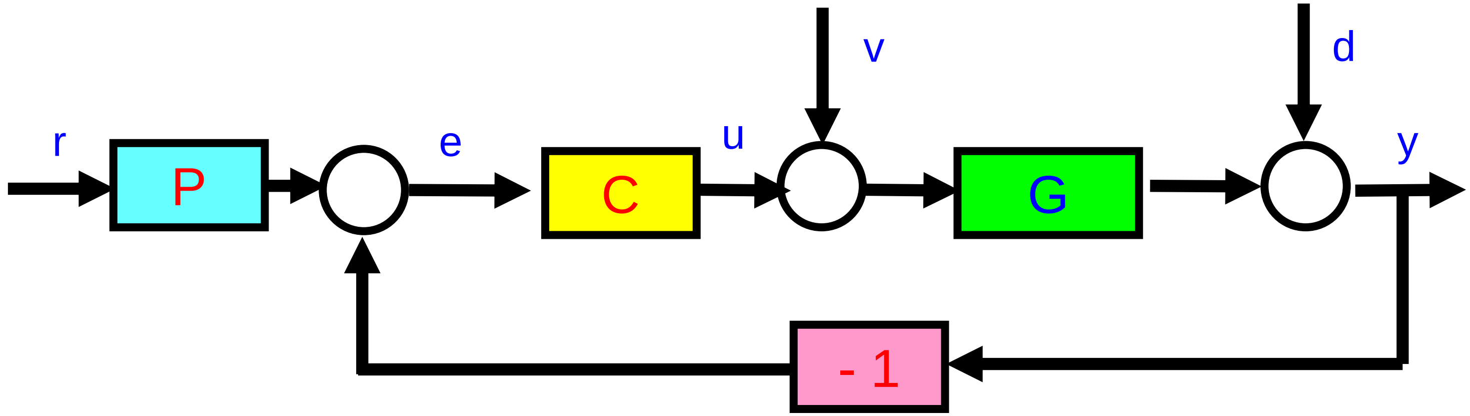
- Control System Design:
  - Load disturbance (actuator)
  - Measurement noise (sensor)
  - Process disturbance (un-modeled dynamics)

- Control Objects:
  - Regulation:
    - ✓ Reduction of load disturbances
    - ✓ Fluctuations by measure noise
  - Tracking:

- Major Ingredients of a Design Problem:
  - Purpose of the system
  - Process model
  - Model for disturbance
  - Model variants and uncertainties
  - Admissible control strategies
  - Design parameters

- Block Diagram of a Typical Control System:





- The Process:

$$\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

- Discrete-time model with *h*:

$$\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{H}u(k)$$

where $\mathbf{F} = e^{\mathbf{A}h}$ and $\mathbf{H} = \int_0^h e^{\mathbf{A}s} ds \mathbf{B}$

- Disturbances:
  - Impulse signals: irregularly, widely spread, etc.
  - Step signals:
  - Ramp signals:
  - Sinusoidal signals:

- Process Uncertainties:
  - In the elements of A, B, C, D or F, H, C, D

- Design Criteria:
  - Regulation:
    - Bring the state to zero after perturbations
    - The rate of decay of state $\rightarrow$ C.L. poles
  - Tracking:
    - From commands to states $\rightarrow$ "model"
- Admissible Controls:
  - When all states are measured w/o errors:
    - Linear feedback control law

$$u(k) = -\mathbf{K}\,\mathbf{x}(k) = -\begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- Design Parameters :

  - ✓Sampling period
  - ✓Desired C.L. poles

  - ✓Time histories of states and controls
  - ✓Magnitude of control signals
  - ✓Speed at which the system recovers from a disturbance

- The C.T. Model:

$$\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

- The D.T. Model: (given *h*)

$$\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{H}u(k)$$

- Characteristic Equation:

$$\mathbf{F} \quad \Rightarrow \quad z^n + a_1 z^{n-1} + \cdots + a_n = 0$$

- System Model and Characteristic Equation:

$$\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{H}u(k)$$

$$\mathbf{A}(z) = z^n + a_1 z^{n-1} + \cdots + a_n = 0$$

- If the system is "Controllable"
  the system is in the Controllable Canonical Form:

$$\mathbf{x}(k+1) = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & & & 0 \\ 0 & 1 & & & \\ \vdots & & \ddots & & \\ 0 & & & 1 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u(k)$$

$$y(k) = [\, b_1 \; b_2 \; \cdots \; b_n \,]\, \mathbf{x}(k)$$

▪ The State Feedback Law:

$$u(k) = -\begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = -\mathbf{K}\,\mathbf{x}(k)$$

▪ Then:

$$\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{H}u(k) = \mathbf{F}\mathbf{x}(k) + \mathbf{H}(\,-\mathbf{K}\mathbf{x}(k)\,)$$

$$= (\,\mathbf{F} - \mathbf{H}\,\mathbf{K}\,)\,\mathbf{x}(k)$$

$$= \left\{ \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & & & 0 \\ 0 & 1 & & & \\ \vdots & & \ddots & & \\ 0 & & & 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix} \right\} \mathbf{x}(k)$$

$$= \begin{bmatrix} -(a_1+k_1) & -(a_2+k_2) & \cdots & -(a_{n-1}+k_{n-1}) & -(a_n+k_n) \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \mathbf{x}(k)$$

- Closed-Loop Characteristic Equation:

$$\det\left(\lambda\mathbf{I} - (\mathbf{F} - \mathbf{HK})\right)$$

$$= \lambda^n + (a_1 + k_1)\lambda^{n-1} + \cdots + (a_n + k_n)$$

- Desired Characteristic Equation:

$$\mathbf{P}(z) = \lambda^n + p_1\lambda^{n-1} + \cdots + p_n$$

- Then

$$p_i = a_i + k_i$$

$$k_i = p_i - a_i$$

$$u(k) = -\mathbf{K}\,\mathbf{x}(k)$$

- This is the Pole Placement

$$\mathbf{A}(z) \to \mathbf{P}(z)$$

- OR, the Eigenvalue Assignment

$$\text{eig}(\mathbf{F}) \to \text{eig}(\mathbf{F} - \mathbf{HK})$$

- That is,

$$\mathbf{x}(k+1) = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & & & 0 \\ 0 & 1 & & & \\ \vdots & & \ddots & & \\ 0 & & & 1 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u(k)$$

$$u(k) = -\begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\mathbf{x}(k+1) = \begin{bmatrix} -p_1 & -p_2 & \cdots & -p_{n-1} & -p_n \\ 1 & 0 & & & 0 \\ 0 & 1 & & & \\ \vdots & & \ddots & & \\ 0 & & & 1 & 0 \end{bmatrix} \mathbf{x}(k)$$

- Now, we have (in controllable canonical form):

$$\mathbf{x}(k+1) = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & & & 0 \\ 0 & 1 & & & \\ \vdots & & \ddots & & \\ 0 & & & 1 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u(k)$$

$$\mathbf{x}_c(k+1) = \mathbf{F}_c \mathbf{x}_c(k) + \mathbf{H}_c u(k)$$

- For any other SS forms:

$$\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{H}u(k)$$

- Assume that exiting a nonsingular matrix **T**:

$$\begin{aligned} \mathbf{x}_c(k) &= \mathbf{T}\,\mathbf{x}(k) \\ \mathbf{x}(k) &= \mathbf{T}^{-1}\mathbf{x}_c(k) \end{aligned}$$

$$\mathbf{x}_c(k) = \mathbf{T}\,\mathbf{x}(k)$$
$$\mathbf{x}(k) = \mathbf{T}^{-1}\mathbf{x}_c(k)$$

- Then:

$$\mathbf{x}(k+1) = \mathbf{F}\,\mathbf{x}(k) + \mathbf{H}\,u(k)$$

$$\mathbf{T}^{-1}\,\mathbf{x}_c(k+1) = \mathbf{F}\,\mathbf{T}^{-1}\,\mathbf{x}_c(k) + \mathbf{H}\,u(k)$$

$$\mathbf{x}_c(k+1) = \mathbf{T}\,\mathbf{F}\,\mathbf{T}^{-1}\,\mathbf{x}_c(k) + \mathbf{T}\,\mathbf{H}\,u(k)$$

$$\mathbf{x}_c(k+1) = \mathbf{F}_c\,\mathbf{x}_c(k) + \mathbf{H}_c\,u(k)$$

$$\mathbf{F}_c = \mathbf{T}\,\mathbf{F}\,\mathbf{T}^{-1}$$

$$\mathbf{H}_c = \mathbf{T}\,\mathbf{H}$$

- If the system is "Controllable"

$$\exists \; \mathbf{T} : \; \text{nonsingular}$$

$$\text{such that } \mathbf{x}_c(k) = \mathbf{T}\,\mathbf{x}(k)$$

$$\Rightarrow \; \text{Controllable Canonical Form:}$$

$$\mathbf{x}_c(k+1) = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & & & 0 \\ 0 & 1 & & & \\ \vdots & & \ddots & & \\ 0 & & & 1 & 0 \end{bmatrix} \mathbf{x}_c(k) + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \\ 0 \end{bmatrix} u(k)$$

$$y(k) = [\, b_1 \; b_2 \; \cdots \; b_n \,]\, \mathbf{x}_c(k)$$

$$\mathbf{x}_c(k+1) = \mathbf{F}_c\,\mathbf{x}_c(k) + \mathbf{H}_c\,u(k)$$

$$y(k) = \mathbf{C}_c\,\mathbf{x}_c(k)$$

$$\mathbf{F}_c = \mathbf{T}\mathbf{F}\mathbf{T}^{-1}$$

$$\mathbf{H}_c = \mathbf{T}\mathbf{H}$$

$$\mathbf{C}_c = \mathbf{C}\mathbf{T}$$

- The State Feedback Law:

$$u(k) = -\mathbf{K}_c\,\mathbf{x}_c(k)$$

$$= -\mathbf{K}_c\,\mathbf{T}\,\mathbf{x}(k)$$

$$= -\mathbf{K}\,\mathbf{x}(k)$$

$$\Rightarrow\ \mathbf{K} = \mathbf{K}_c\,\mathbf{T}$$

$$\mathbf{F}_c = \mathbf{T}\mathbf{F}\mathbf{T}^{-1}$$

$$\mathbf{H}_c = \mathbf{T}\mathbf{H}$$

$$\mathbf{C}_c = \mathbf{C}\mathbf{T}$$

$$\mathbf{x}_c(k) = \mathbf{T}\,\mathbf{x}(k)$$
$$\mathbf{x}(k) = \mathbf{T}^{-1}\mathbf{x}_c(k)$$

$$\mathbf{F}_c = \mathbf{T}\mathbf{F}\mathbf{T}^{-1}$$

$$\mathbf{H}_c = \mathbf{T}\mathbf{H}$$

- How to find **T**:

$$\mathbf{x}(k+1) = \mathbf{F}\,\mathbf{x}(k) + \mathbf{H}\,u(k)$$

$$\mathbf{x}_c(k+1) = \mathbf{F}_c\,\mathbf{x}_c(k) + \mathbf{H}_c\,u(k)$$

$$\mathbf{W_c} = \begin{bmatrix} \mathbf{H} & \mathbf{F}\mathbf{H} & \cdots & \mathbf{F}^{n-1}\mathbf{H} \end{bmatrix}$$

$$\begin{aligned}
\mathbf{W_{c}}_c &= \begin{bmatrix} \mathbf{H}_c & \mathbf{F}_c\mathbf{H}_c & \cdots & \mathbf{F}_c^{n-1}\mathbf{H}_c \end{bmatrix} \\
&= \begin{bmatrix} (\mathbf{T}\mathbf{H}) & (\mathbf{T}\mathbf{F}\mathbf{T}^{-1})(\mathbf{T}\mathbf{H}) & \cdots & (\mathbf{T}\mathbf{F}\mathbf{T}^{-1})^{n-1}(\mathbf{T}\mathbf{H}) \end{bmatrix} \\
&= \begin{bmatrix} (\mathbf{T}\mathbf{H}) & (\mathbf{T}\mathbf{F}\mathbf{H}) & \cdots & (\mathbf{T}\mathbf{F}^{n-1}\mathbf{H}) \end{bmatrix} \\
&= \mathbf{T}\begin{bmatrix} (\mathbf{H}) & (\mathbf{F}\mathbf{H}) & \cdots & (\mathbf{F}^{n-1}\mathbf{H}) \end{bmatrix} \\
&= \mathbf{T}\,\mathbf{W_c}
\end{aligned}$$

- If the system is "Controllable"

$$\rightarrow \quad \mathbf{W_c} \text{ and } \mathbf{W}_{\mathbf{c}c} \text{ are nonsingular}$$

$$\rightarrow \quad \mathbf{T} = (\mathbf{W}_{\mathbf{c}c})\,(\mathbf{W_c})^{-1}$$

- In Summary:

$$\mathbf{x}(k+1) = \mathbf{F}\,\mathbf{x}(k) + \mathbf{H}\,u(k)$$

$$\mathbf{x}_c(k+1) = \mathbf{F}_c\,\mathbf{x}_c(k) + \mathbf{H}_c\,u(k)$$

$$\mathbf{x}_c(k) = \mathbf{T}\,\mathbf{x}(k)$$

$$u(k) = -\mathbf{K}_c\,\mathbf{x}_c(k) = -\mathbf{K}\,\mathbf{x}(k)$$

$$\mathbf{T} = (\mathbf{W}_{\mathbf{c}c})\,(\mathbf{W_c})^{-1}$$

$$\mathbf{F}_c = \mathbf{T}\mathbf{F}\mathbf{T}^{-1}$$

$$\mathbf{H}_c = \mathbf{T}\mathbf{H}$$

$$\mathbf{C}_c = \mathbf{C}\mathbf{T}$$

$$\mathbf{K} = \mathbf{K}_c\,\mathbf{T}$$

- Ackermann's formula

Deadbeat Control:

- IF all C.L. poles $= 0$, i.e., $P(z) = z^n$
- By Cayley-Hamilton Theorem:

$$(\mathbf{F}_{cl})^n = 0, \quad \mathbf{F}cl = \mathbf{F} - \mathbf{HK}$$

- $\mathbf{x}(k) = 0$ at most n steps

$$b/c: \; \mathbf{x}(n) = (\mathbf{F}_{cl})^n \mathbf{x}(0)$$

- h : one design parameter
- $x = 0$: at most n steps
- settling time: at most nh
- $h \downarrow \; \Rightarrow \; u \uparrow$

- Example:

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u(k)$$

$$\mathbf{W_c} = \begin{bmatrix} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0.5 & 1.5 \\ 1 & 1 \end{bmatrix}$$

- Characteristic Polynomial:

$$\begin{aligned} \det(\, z\mathbf{I} - \mathbf{F} \,) &= \det(\, z\mathbf{I} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \,) \\ &= \det(\, \begin{bmatrix} z-1 & -1 \\ 0 & z-1 \end{bmatrix} \,) \\ &= (z-1)(z-1) \qquad = z^2 - 2z + 1 \end{aligned}$$

- Desired Characteristic Polynomial:

$$\mathbf{P}(z) = (z-0.2)(z-0.5) \qquad = z^2 - 0.7z + 0.1$$

- Example:
- Eigenvalue Assignment:

$$\det\left(z\mathbf{I} - (\mathbf{F} - \mathbf{H}\mathbf{K}) \right) = z^2 - 0.7z + 0.1$$

$$= \det\left(z\mathbf{I} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \right)$$

$$= \det\left(\begin{bmatrix} z - 1 + 0.5k_1 & -1 + 0.5k_2 \\ k_1 & z - 1 + k_2 \end{bmatrix} \right)$$

$$= (z - 1 + 0.5k_1)(z - 1 + k_2) - k_1(-1 + 0.5k_2)$$

$$= z^2 + (-2 + 0.5k_1 + k_2)z + (-1 + 0.5k_1)(-1 + k_2) - k_1(-1 + 0.5k_2)$$

$$= z^2 - 0.7z + 0.1$$

$$\Rightarrow k_1 = 0.4, \; k_2 = 1.1$$

- Example:
- Matlab code:
  - F = [1 1; 0 1]
  - H = [0.5; 1]
  - Wc = [H F*H]
  - rank(Wc)
  - poly(F)
  - eig(F)
  - P = conv([1 -0.5], [1 -0.2])
  - roots(P)
  - K = place(F, H, roots(P))

- Example:
- Matlab code:
  - Fc = [2 -1; 1 0]
  - Hc = [1; 0]
  - Wcc = [Hc Fc*Hc]
  - rank(Wcc)
  - poly(Fc)
  - eig(Fc)
  - P = conv([1 -0.5], [1 -0.2])
  - roots(P)
  - Kc = place(Fc, Hc, roots(P))

- Example:
- Matlab code:
  - Fbf = Fc-Hc*Kc
  - T = Wcc * Wc^(-1)
  - T*F*T^(-1)
  - T*H
  - K*T^(-1)
  - Kc*T

$$\mathbf{T} = (\mathbf{W_{c}}_{c})\,(\mathbf{W_c})^{-1}$$

$$\mathbf{F}_c = \mathbf{T}\mathbf{F}\mathbf{T}^{-1}$$

$$\mathbf{H}_c = \mathbf{T}\mathbf{H}$$

$$\mathbf{C}_c = \mathbf{C}\mathbf{T}$$

$$\mathbf{K} = \mathbf{K}_c\,\mathbf{T}$$

- Example: (Choice of Design Parameters)
- The desired DT system is obtained by sampling:

$$s^2 + 2\,\zeta\,w\,s + w^2$$

$$\rightarrow P(z) = z^2 + p_1\,z + p_2$$

$$\rightarrow p_1 = -\,2\,e^{-\,\zeta\,w\,h}\,\cos\left(\,w\,h\,\sqrt{1-\zeta^2}\,\right)$$

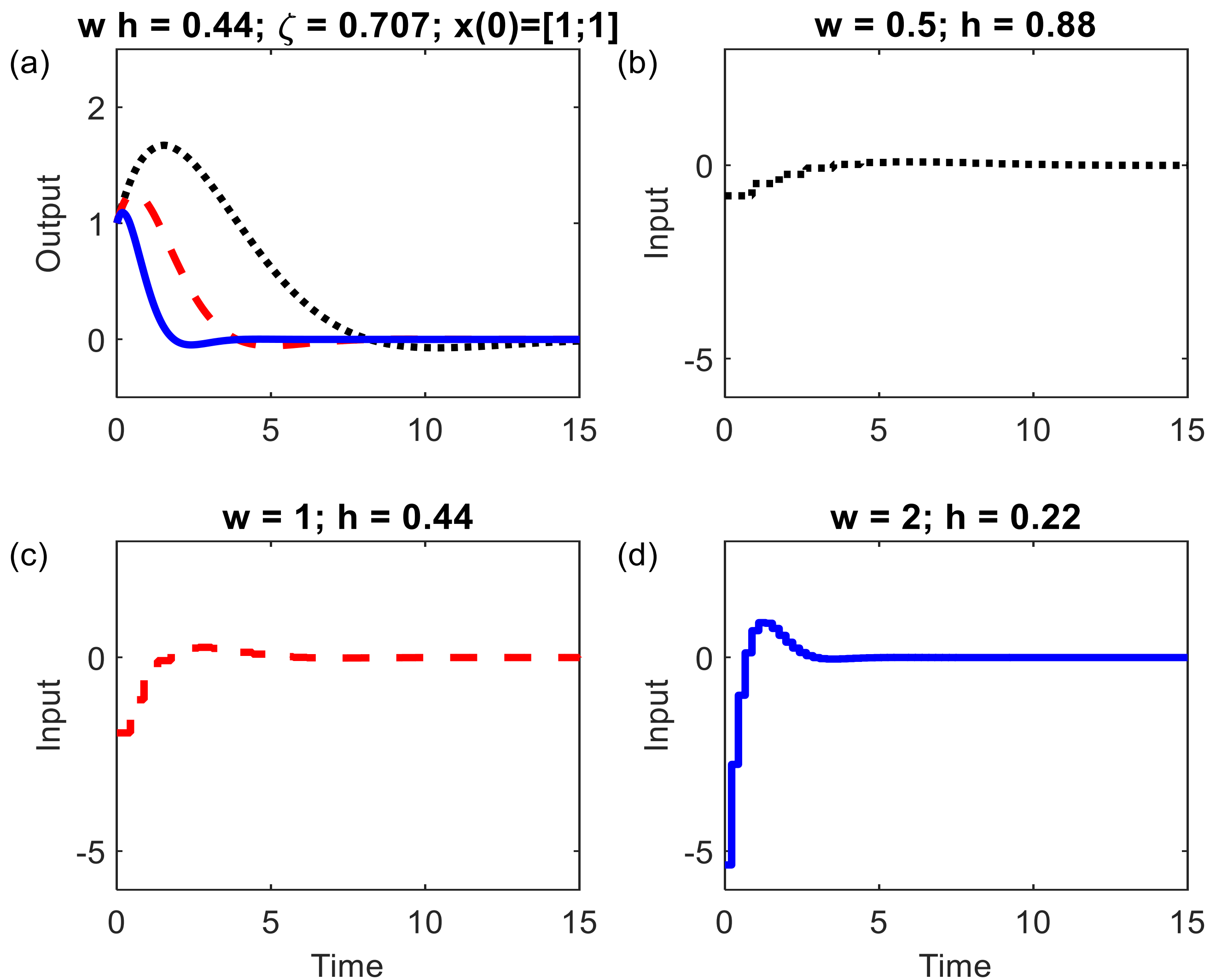
$$\rightarrow p_2 = \,e^{-2\,\zeta\,w\,h}$$

$$\Rightarrow k_1 \;=\; f_1(\,p_1,\;p_2\,) \;=\; f(\,w,\;\zeta,\;h\,)$$

$$\Rightarrow k_2 \;=\; f_2(\,p_1,\;p_2\,) \;=\; f(\,w,\;\zeta,\;h\,)$$

- Example: (Choice of Design Parameters)

$$\Rightarrow h = \ 0.88, \ \ 0.44, \ \ 0.22$$



- D.T. Model:

$$\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{H}u(k)$$

$$y(k) = \mathbf{C}\mathbf{x}(k)$$

$$u(k) = -\mathbf{K}\,\mathbf{x}(k)$$

- Given:

$$y(k), y(k-1), \cdots$$

$$u(k), u(k-1), \cdots$$

- To calculate or reconstruct: $\mathbf{x}(k) = ?$
  - Direct calculation
  - Using dynamic model

- Direct calculation for *n* samples: *k, k-1, …, k-n+1*:

$$y(k) = \mathbf{C}\mathbf{x}(k)$$

$$\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{H}u(k)$$

$$y(k-n+1) = \mathbf{C}\mathbf{x}(k-n+1)$$

$$y(k-n+2) = \mathbf{C}\mathbf{x}(k-n+2)$$

$$= \mathbf{C}\ [\ \mathbf{F}\mathbf{x}(k-n+1) + \mathbf{H}u(k-n+1)\ ]$$

$$= \mathbf{C}\mathbf{F}\mathbf{x}(k-n+1) + \mathbf{C}\mathbf{H}u(k-n+1)$$

$$y(k-n+3) = \mathbf{C}\mathbf{x}(k-n+3)$$

$$= \mathbf{C}\ \{\ \mathbf{F}\mathbf{x}(k-n+2) + \mathbf{H}u(k-n+2)\ \}$$

$$= \mathbf{C}\ \{\ \mathbf{F}\ [\ \mathbf{F}\mathbf{x}(k-n+1) + \mathbf{H}u(k-n+1)\ ]\ + \mathbf{H}u(k-n+2)\ \}$$

$$= \mathbf{C}\mathbf{F}^2\mathbf{x}(k-n+1) + \mathbf{C}\mathbf{F}\mathbf{H}u(k-n+1) + \mathbf{C}\mathbf{H}u(k-n+2)$$

$$y(k)$$

$$= \mathbf{C}\mathbf{F}^{n-1}\mathbf{x}(k-n+1) + \mathbf{C}\mathbf{F}^{n-2}\mathbf{H}u(k-n+1) + \cdots + \mathbf{C}\mathbf{H}u(k-1)$$

Observer Design: Direct Calculation

$$y(k-n+1) = \mathbf{C}\mathbf{x}(k-n+1)$$

$$y(k-n+2) = \mathbf{CF}\mathbf{x}(k-n+1) + \mathbf{CH}u(k-n+1)$$

$$y(k) = \mathbf{CF}^{n-1}\mathbf{x}(k-n+1) + \mathbf{CF}^{n-2}\mathbf{H}u(k-n+1) + \cdots + \mathbf{CH}u(k-1)$$

$$\begin{bmatrix} y(k-n+1) \\ y(k-n+2) \\ \vdots \\ y(k) \end{bmatrix} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CF} \\ \vdots \\ \mathbf{CF}^{n-1} \end{bmatrix} \mathbf{x}(k-n+1)$$

$$+ \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \mathbf{CH} & 0 & \cdots & 0 \\ \mathbf{CFH} & \mathbf{CH} & & 0 \\ \vdots & & & \\ \mathbf{CF}^{n-2}\mathbf{H} & \mathbf{CF}^{n-3}\mathbf{H} & & 0 \end{bmatrix} \begin{bmatrix} u(k-n+1) \\ u(k-n+2) \\ \vdots \\ u(k-1) \end{bmatrix}$$

$$\Rightarrow \mathbf{Y}_k = \mathbf{W}_o\, \mathbf{x}(k-n+1) + \mathbf{W}_u\, \mathbf{U}_{k-1}$$

$$\Rightarrow \mathbf{x}(k-n+1) = \mathbf{W}_o{}^{-1}\mathbf{Y}_k - \mathbf{W}_o{}^{-1}\mathbf{W}_u\mathbf{U}_{k-1}$$

- From state-state form:

$$\mathbf{x}(k) = \mathbf{F}^{n-1}\mathbf{x}(k-n+1) + \mathbf{F}^{n-2}\mathbf{H}u(k-n+1) + \cdots + \mathbf{H}u(k-1)$$

$$\Rightarrow \quad \mathbf{x}(k-n+1) = \mathbf{W}_o{}^{-1}\mathbf{Y}_k - \mathbf{W}_o{}^{-1}\mathbf{W}_u\mathbf{U}_{k-1}$$

$$\mathbf{x}(k) = \mathbf{F}^{n-1}\mathbf{W}_o{}^{-1}\mathbf{Y}_k - \mathbf{F}^{n-1}\mathbf{W}_o{}^{-1}\mathbf{W}_u\mathbf{U}_{k-1}$$

$$+ \begin{bmatrix} \mathbf{F}^{n-2}\mathbf{H} & \mathbf{F}^{n-3}\mathbf{H} & \cdots & \mathbf{H} \end{bmatrix} \begin{bmatrix} u(k-n+1) \\ u(k-n+2) \\ \vdots \\ u(k-1) \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{x}(k) \;=\; \mathbf{A}_y \, \mathbf{Y}_k \;+\; \mathbf{B}_u \, \mathbf{U}_{k-1}$$

$$\mathbf{A}_y = \mathbf{F}^{n-1}\mathbf{W}_o{}^{-1}$$

$$\mathbf{B}_u = \begin{bmatrix} \mathbf{F}^{n-2}\mathbf{H} & \mathbf{F}^{n-3}\mathbf{H} & \cdots & \mathbf{H} \end{bmatrix} - \mathbf{F}^{n-1}\mathbf{W}_o{}^{-1}\mathbf{W}_u$$

- Estimation using dynamic model:

$$\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{H}u(k)$$

$$\hat{\mathbf{x}}(k+1) = \mathbf{F}\hat{\mathbf{x}}(k) + \mathbf{H}u(k)$$

- If $\hat{\mathbf{x}}(0) = \mathbf{x}(0) \quad \Rightarrow \quad \hat{\mathbf{x}}(k) = \mathbf{x}(k)$

- If $\hat{\mathbf{x}}(0) \neq \mathbf{x}(0) \quad \Rightarrow \quad \hat{\mathbf{x}}(k) \to \mathbf{x}(k)$

  only if asymptotically stable

- Improvement by using measured outputs:

$$\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{H}u(k)$$

$$\hat{\mathbf{x}}(k+1|k) = \mathbf{F}\hat{\mathbf{x}}(k|k-1) + \mathbf{H}u(k)$$

$$+\mathbf{L}\,[\,y(k) - \mathbf{C}\hat{\mathbf{x}}(k|k-1)\,]$$

- Estimation error:

$$\tilde{\mathbf{x}} = \mathbf{x} - \hat{\mathbf{x}}$$

$$\tilde{\mathbf{x}}(k+1|k) = \mathbf{F}\tilde{x}(k|k-1)$$

$$-\mathbf{L}\,[\,\mathbf{C}\mathbf{x}(k) \;-\; \mathbf{C}\hat{\mathbf{x}}(k|k-1)\,]$$

$$-\mathbf{L}\mathbf{C}\,[\,\mathbf{x}(k) \;-\; \hat{\mathbf{x}}(k|k-1)\,]$$

$$-\mathbf{L}\mathbf{C}\,\tilde{x}(k|k-1)$$

$$= [\,\mathbf{F} \;-\; \mathbf{L}\,\mathbf{C}\,]\,\tilde{x}(k|k-1)$$

- Hence,
  - If L is chosen such that the system is asymptotically stable

$$\Rightarrow \tilde{\mathbf{x}} \rightarrow 0$$

$$\Rightarrow \mathbf{x} \rightarrow \hat{\mathbf{x}}$$

$$\tilde{\mathbf{x}}(k+1|k) = [\ \mathbf{F} - \mathbf{LC}\ ]\ \tilde{x}(k|k-1)$$

- That is,
  - Given $\mathbf{F}$ and $\mathbf{C}$
  - To find $\mathbf{L}$

$\Rightarrow$ $\mathbf{F} - \mathbf{LC}$ has prescribed eigenvalues

In Summary: Observer Dynamics

$$\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{H}u(k)$$
$$y(k) = \mathbf{C}\mathbf{x}(k)$$

- If the system is completely observable

$\exists$ $\mathbf{L}$ such that $\mathbf{F} - \mathbf{L}\mathbf{C}$ of

$$\hat{\mathbf{x}}(k+1|k) = \mathbf{F}\hat{\mathbf{x}}(k|k-1) + \mathbf{H}u(k)$$
$$+\mathbf{L}\left[\, y(k) - \mathbf{C}\hat{\mathbf{x}}(k|k-1) \,\right]$$

has the following desired char. poly.

$$P(z) = z^n + p_1 z^{n-1} + \cdots + p_n$$

$$\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{H}u(k)$$
$$y(k) = \mathbf{C}\mathbf{x}(k)$$

- If the system is "Controllable"

$$\mathbf{W_c} = \begin{bmatrix} \mathbf{H} & \mathbf{FH} & \cdots & \mathbf{F}^{n-1}\mathbf{H} \end{bmatrix} \quad \text{is nonsingular}$$

$$\rightarrow \; u(k) = -\mathbf{K}\,\mathbf{x}(k)$$

$\rightarrow$ Desired eigenvalues: $\text{eig}(\mathbf{F} - \mathbf{HK})$

- If the system is "Observable"

$$\mathbf{W_o} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CF} \\ \vdots \\ \mathbf{CF}^{n-1} \end{bmatrix} \quad \text{is nonsingular}$$

$$\rightarrow \; \mathbf{L}\,[\, y(k) - \mathbf{C}\hat{\mathbf{x}}(k|k-1) \,]$$

$\rightarrow$ Desired eigenvalues: $\text{eig}(\mathbf{F} - \mathbf{LC})$

Deadbeat Observer:

- Choose $\mathbf{L}$, such that eig($\mathbf{F} - \mathbf{LC}$) $= 0$
- $\tilde{\mathbf{x}}(k) = 0$ at most n steps
- It is equivalent to the direct calculation!

- D.T. Model:

$$\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{H}u(k)$$
$$y(k) = \mathbf{C}\mathbf{x}(k)$$

- If all states are measured directly:

$$u(k) = -\mathbf{K}\,\mathbf{x}(k) \quad \text{gives desired C.L. poles}$$

- If the states CANNOT measured:

$$\hat{\mathbf{x}}(k|k-1) = \mathbf{F}\hat{\mathbf{x}}(k-1|k-2) + \mathbf{H}u(k-1)$$
$$+\,\mathbf{L}\,[\,y(k-1) - \mathbf{C}\hat{\mathbf{x}}(k-1|k-2)\,]$$
$$u(k) = -\mathbf{K}\,\hat{\mathbf{x}}(k|k-1)$$

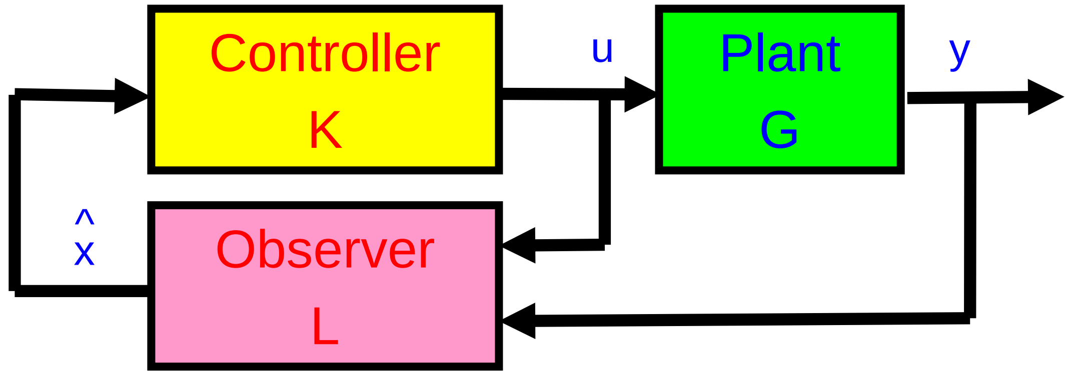
Output Feedback

$$\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{H}u(k)$$

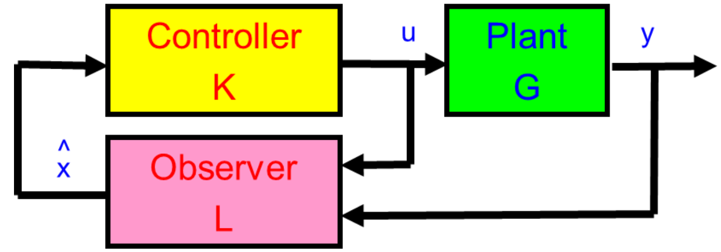
$$y(k) = \mathbf{C}\mathbf{x}(k)$$

$$\hat{\mathbf{x}}(k|k-1) = \mathbf{F}\hat{\mathbf{x}}(k-1|k-2) + \mathbf{H}u(k-1) + \mathbf{L}\left[\, y(k-1) - \mathbf{C}\hat{\mathbf{x}}(k-1|k-2) \,\right]$$

$$u(k) = -\mathbf{K}\,\hat{\mathbf{x}}(k|k-1)$$



- Closed-Loop System:



$$\tilde{\mathbf{x}} = \mathbf{x} - \hat{\mathbf{x}}$$

$$\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{H}u(k)$$

$$\hat{\mathbf{x}}(k+1|k) = \mathbf{F}\hat{\mathbf{x}}(k|k-1) + \mathbf{H}u(k)$$

$$+\mathbf{L}\,[\,y(k) - \mathbf{C}\hat{\mathbf{x}}(k|k-1)\,]$$

$$u(k) = -\mathbf{K}\,\hat{\mathbf{x}}(k|k-1)$$

$$-\mathbf{L}\mathbf{C}\,\tilde{x}(k|k-1)$$

$$\tilde{\mathbf{x}}(k+1|k) = \mathbf{F}\tilde{x}(k|k-1)$$

$$-\mathbf{L}\,[\,\mathbf{C}\mathbf{x}(k) - \mathbf{C}\hat{\mathbf{x}}(k|k-1)\,]$$

$$= [\,\mathbf{F} - \mathbf{L}\,\mathbf{C}\,]\,\tilde{x}(k|k-1)$$

$$\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) - \mathbf{H}\,\mathbf{K}\,\hat{\mathbf{x}}(k|k-1)$$

Output Feedback

$$\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) - \mathbf{H}\,\mathbf{K}\,\hat{\mathbf{x}}(k|k-1)$$

$$\tilde{\mathbf{x}} = \mathbf{x} - \hat{\mathbf{x}}$$

$$= \mathbf{F}\mathbf{x}(k) - \mathbf{H}\,\mathbf{K}\,(\,\mathbf{x}(k) - \tilde{\mathbf{x}}(k|k-1)\,)$$

$$= [\,\mathbf{F} - \mathbf{H}\,\mathbf{K}\,]\,\mathbf{x}(k) + \mathbf{H}\,\mathbf{K}\,\tilde{\mathbf{x}}(k|k-1)$$

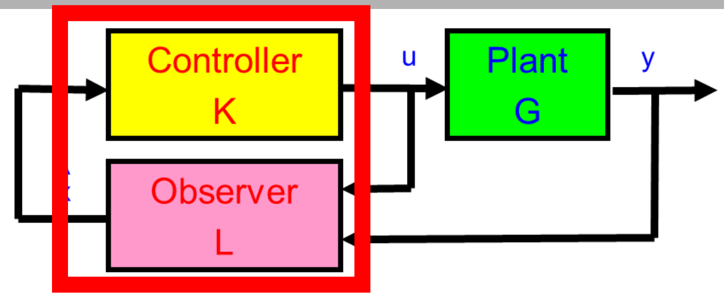
$$\tilde{\mathbf{x}}(k+1|k) = [\,\mathbf{F} - \mathbf{L}\,\mathbf{C}\,]\,\tilde{\mathbf{x}}(k|k-1)$$

$\Rightarrow$ system order of $2n$

$\Rightarrow$ eig(C.L. system) $=$

eig($\mathbf{F} - \mathbf{H}\mathbf{K}$) & eig($\mathbf{F} - \mathbf{L}\mathbf{C}$)

Output Feedback



$$u(k) = -\mathbf{K}\,\hat{\mathbf{x}}(k|k-1)$$

$$\begin{aligned}
\hat{\mathbf{x}}(k+1|k) &= \mathbf{F}\hat{\mathbf{x}}(k|k-1) + \mathbf{H}u(k) \\
&\quad + \mathbf{L}\,[\,y(k) - \mathbf{C}\hat{\mathbf{x}}(k|k-1)\,] \\
&= \mathbf{F}\hat{\mathbf{x}}(k|k-1) - \mathbf{H}\,\mathbf{K}\,\hat{\mathbf{x}}(k|k-1) \\
&\quad + \mathbf{L}\,[\,y(k) - \mathbf{C}\hat{\mathbf{x}}(k|k-1)\,] \\
&= [\,\mathbf{F} - \mathbf{H}\,\mathbf{K} - \mathbf{L}\,\mathbf{C}\,]\,\hat{\mathbf{x}}(k|k-1) \\
&\quad + \mathbf{L}\,y(k)
\end{aligned}$$

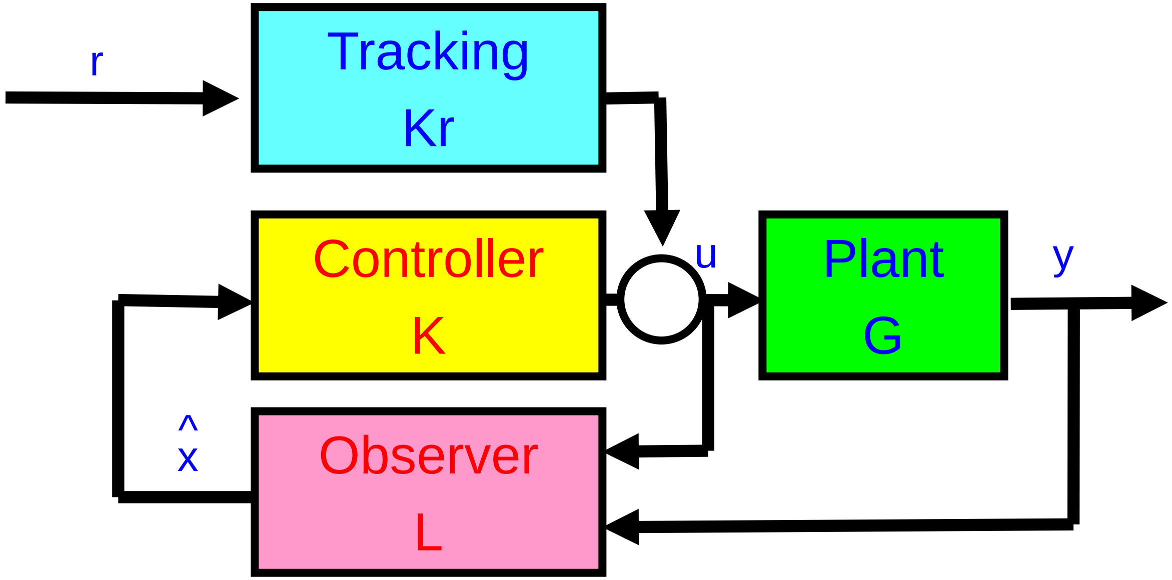
$$\rightarrow \text{eig}(\,\mathbf{F} - \mathbf{H}\,\mathbf{K} - \mathbf{L}\,\mathbf{C}\,)$$

Objective of Tracking Problem:

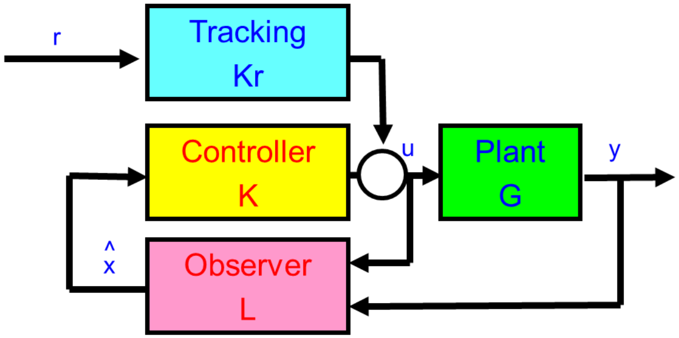
- Make the states and the outputs of the system respond to command signals in a specific way

Regulation & Tracking:

- Regulation: $u(k) = -\mathbf{K}\,\hat{\mathbf{x}}(k)$
- Tracking: $u(k) = -\mathbf{K}\,\hat{\mathbf{x}}(k) + \mathbf{K_r}\,r(k)$



- Plant, Observer & Controller Models:



$$\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{H}u(k)$$

$$y(k) = \mathbf{C}\mathbf{x}(k)$$

$$\hat{\mathbf{x}}(k+1|k) = \mathbf{F}\hat{\mathbf{x}}(k|k-1) + \mathbf{H}u(k)$$
$$+\mathbf{L}\,[\,y(k) - \mathbf{C}\hat{\mathbf{x}}(k|k-1)\,]$$

$$u(k) = -\mathbf{K}\,\hat{\mathbf{x}}(k) + \mathbf{K_r}\,r(k)$$

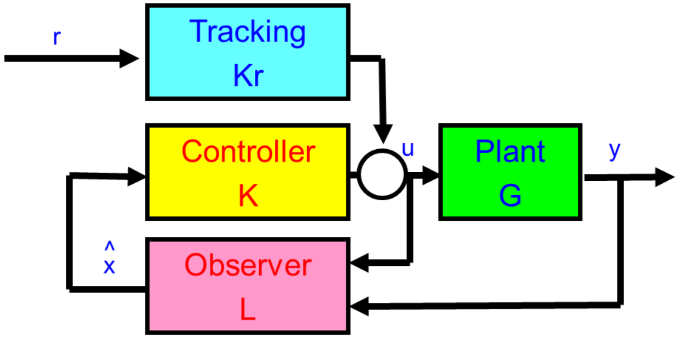
$$\tilde{\mathbf{x}} = \mathbf{x} - \hat{\mathbf{x}} \qquad \hat{\mathbf{x}} = \mathbf{x} - \tilde{\mathbf{x}}$$

$$u(k) = -\mathbf{K}\,\mathbf{x} + \mathbf{K}\,\tilde{\mathbf{x}} + \mathbf{K_r}\,r(k)$$

$$\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) - \mathbf{H}\mathbf{K}\,\mathbf{x}(k) + \mathbf{H}\mathbf{K}\,\tilde{\mathbf{x}} + \mathbf{H}\mathbf{K_r}\,r(k)$$
$$= [\,\mathbf{F} - \mathbf{H}\mathbf{K}\,]\,\mathbf{x}(k) + \mathbf{H}\mathbf{K}\,\tilde{\mathbf{x}} + \mathbf{H}\mathbf{K_r}\,r(k)$$

- Plant, Observer & Controller Models:

$$u(k) = -\mathbf{K}\,\hat{\mathbf{x}}(k) + \mathbf{K_r}\,r(k)$$



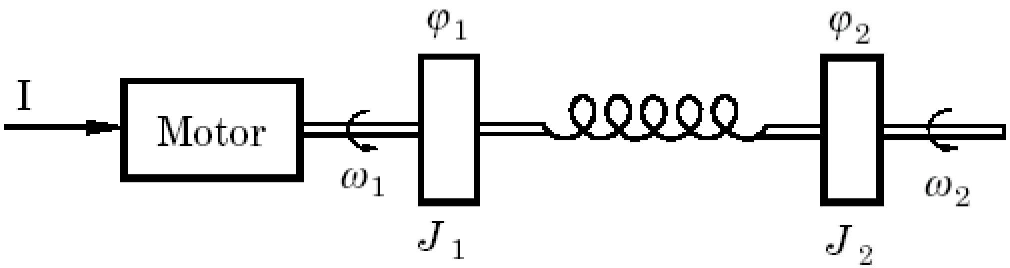
$$\mathbf{x}(k+1) = [\,\mathbf{F} - \mathbf{HK}\,]\,\mathbf{x}(k) + \mathbf{HK}\,\tilde{\mathbf{x}} + \mathbf{HK_r}\,r(k)$$

$$y(k) = \mathbf{C}\mathbf{x}(k)$$

$$\tilde{\mathbf{x}}(k+1|k) = [\,\mathbf{F} - \mathbf{L}\,\mathbf{C}\,]\,\tilde{\mathbf{x}}(k|k-1)$$

$$\hat{\mathbf{x}}(k+1|k) = [\,\mathbf{F} - \mathbf{H}\,\mathbf{K} - \mathbf{L}\,\mathbf{C}\,]\,\hat{\mathbf{x}}(k|k-1) + \mathbf{L}\,y(k) + \mathbf{HK_r}\,r(k)$$

A Flexible Robot Arm:



- States:

$$\begin{aligned} x_1 &= \phi_1 - \phi_2 \\ x_2 &= w_1 / w_0 \\ x_3 &= w_2 / w_0 \end{aligned}$$

wehre $w_0 = \sqrt{k(J_1 + J_2)/(J_1 J_2)}$

C.T. Model of a Flexible Robot Arm:

$$\frac{dx}{dt} = w_0 \begin{bmatrix} 0 & 1 & -1 \\ \alpha - 1 & -\beta_1 & \beta_1 \\ \alpha & \beta_2 & -\beta_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ \gamma \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ \delta \\ 0 \end{bmatrix} v$$

$$y = \begin{bmatrix} 0 & 0 & w_0 \end{bmatrix} x$$

$$\text{wehre} \left\{ \begin{array}{lcl} \alpha & = & J_1/(J_1 + J_2) \\ \beta_1 & = & d/J_1 w_0 \\ \beta_2 & = & d/J_2 w_0 \\ \gamma & = & k_I/J_1 w_0 \\ \delta & = & 1/J_1 w_0 \end{array} \right.$$

- Numerical Values used in simulation:

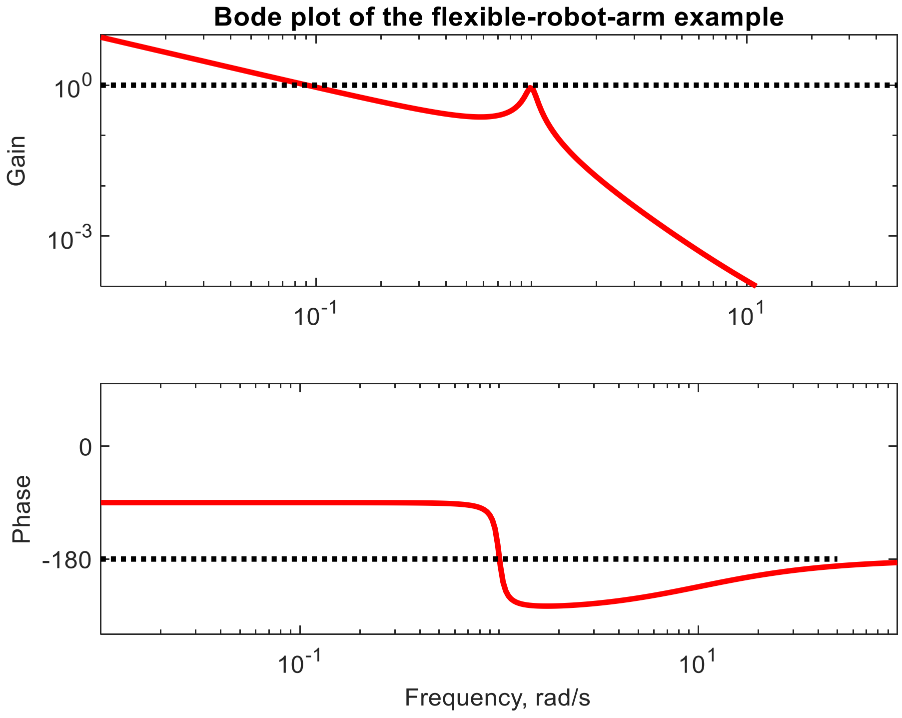
$$\begin{cases} J_1 & = & 10/9 \\ J_2 & = & 10 \\ k & = & 1 \\ d & = & 0.1 \\ k_I & = & 1 \\ w_0 & = & 1 \end{cases}$$

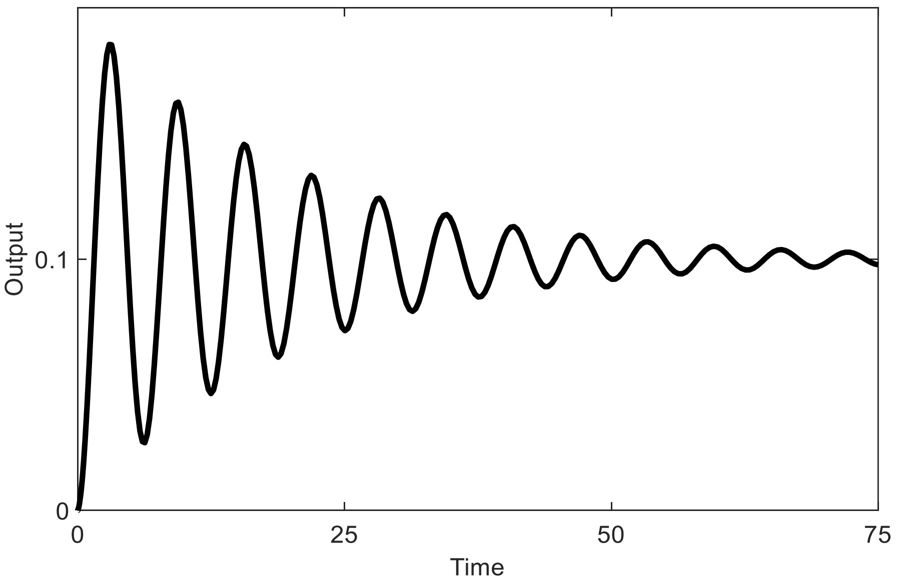
- Poles & Zeros:

$$\begin{cases} \text{Zeros:} & z_1 = -10 \\ \text{Poles:} & p_1 = 0, p_{23} = -0.05 \pm 0.999i \end{cases}$$

$$\Rightarrow \text{ a pure integrator, } \zeta_p = 0.05, \; w_p = 1 \text{ rad/s}$$

- Bode plot & impulse response:





Design Example

- Specifications:

  - $\zeta_m = 0.7, \ w_m = 0.5 \text{ rad/s}$

- Sampling Time:

  - $wh = 0.1 - 0.6$

    $\Rightarrow \ h = 0.5 \text{ s}$

    $\Rightarrow \ w_N = \pi/h = 6 \text{ rad/s}$

State Feedback Design:

- $u(k) = -\mathbf{K}\mathbf{x}(k) + \mathbf{K}_c u_c(k)$

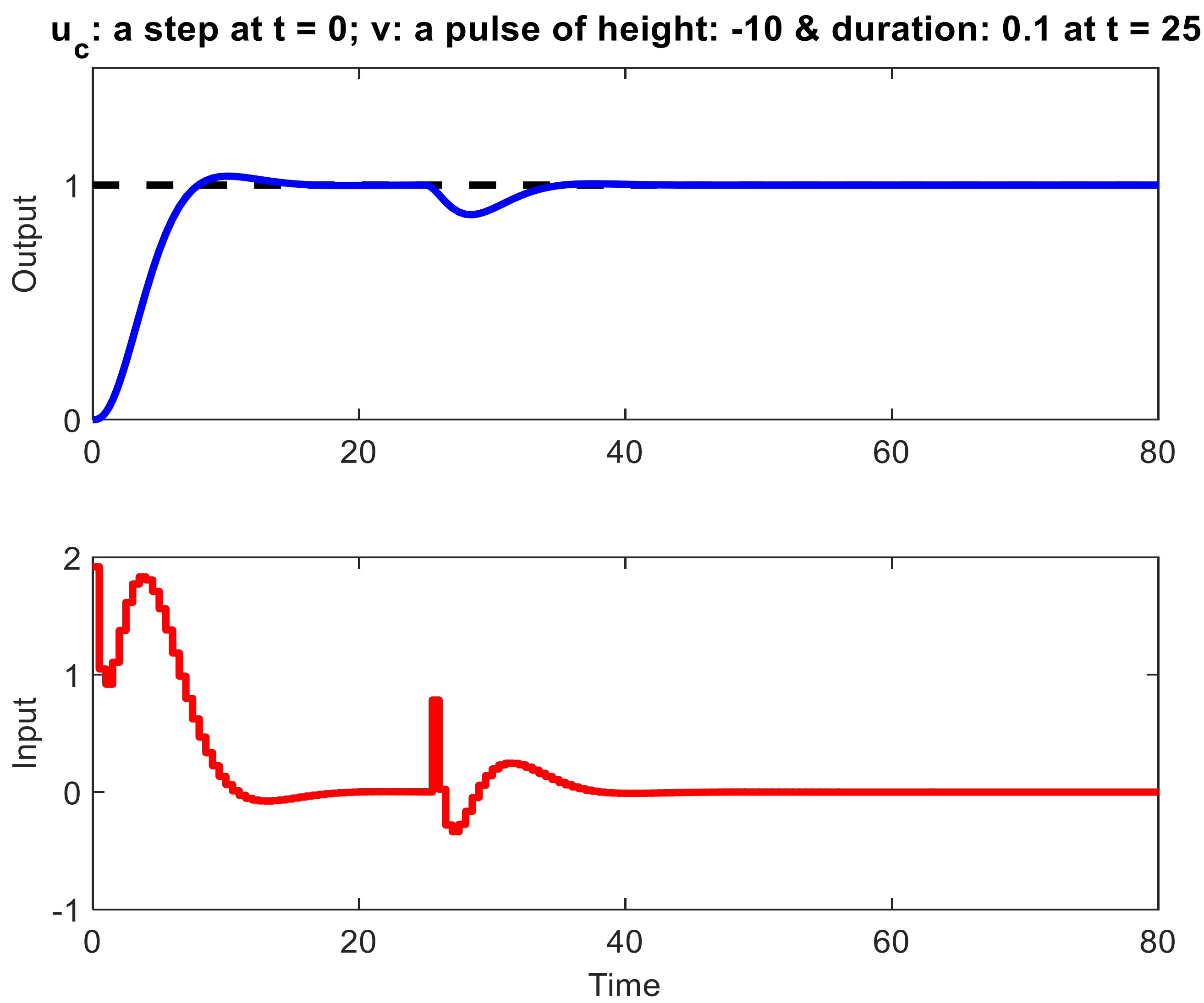
Desired Poles:

- $(s^2 + 2\zeta_m w_m s + w_m^2)(s + \alpha_1 w_m) = 0$

  $\Rightarrow$ Sampled form with $h = 0.5$ s

  $\Rightarrow \alpha_1 = 2$

Design Example



■ Observer Design:

- eig($\mathbf{F} - \mathbf{L}\mathbf{C}$): similar to C.L. Poles

- $(s^2+2\zeta_m\alpha_0 w_m s+(\alpha_0 w_m)^2)(s+\alpha_0\alpha_1 w_m) = 0$

$\Rightarrow \alpha_0$ farther away from the origin

$\Rightarrow \alpha_0 = 2, \alpha_1 = 2$

Design Example