Spring 2021

數位控制系統
Digital Control Systems

DCS-23
Controllability and Observability

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NTU-EE

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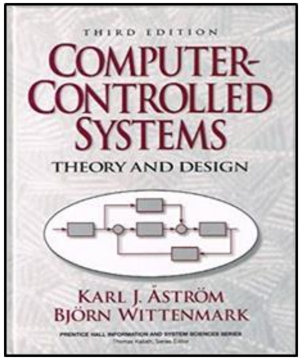
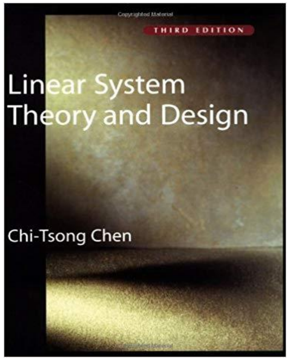
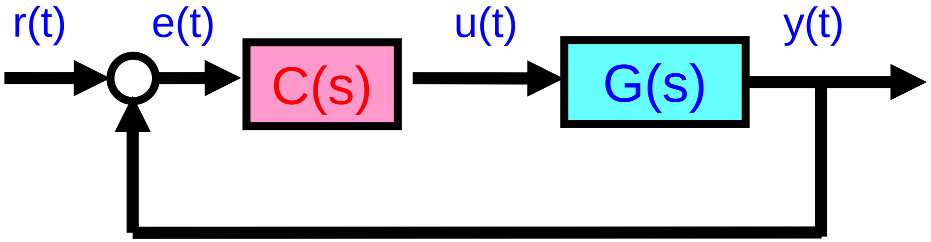
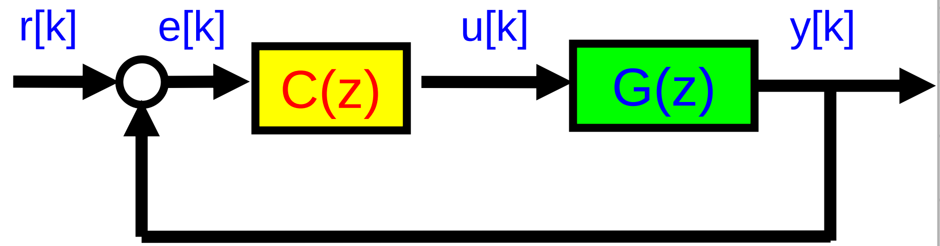




Introduction: Model and Analysis





- Plant (CT):
  - Input-Output Model:

$$\frac{Y_c(s)}{U_c(s)} = G_c(s) = \frac{B_c(s)}{A_c(s)}$$

  - State-Space Model:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\ y(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t) \end{aligned}$$

- Plant (DT):
  - Input-Output Model:

$$\frac{Y_d(z)}{U_d(z)} = G_d(z) = \frac{B_d(z)}{A_d(z)}$$

  - State-Space Model:

$$\begin{aligned} \mathbf{x}[k+1] &= \boxed{\mathbf{F}}\mathbf{x}[k] + \boxed{\mathbf{H}}u[k] \\ y[k] &= \boxed{\mathbf{C}}\mathbf{x}[k] + \boxed{\mathbf{D}}u[k] \end{aligned}$$

- System Properties:
  - Stability
  - Controllability
  - Observability

Outline

- Controllability
- Controllable Canonical Form
- Observability
- Observable Canonical Form
- Kalman's Decomposition
- Loss of Controllability & Observability through Sampling

Controllability

- Whether it is possible to steer a system

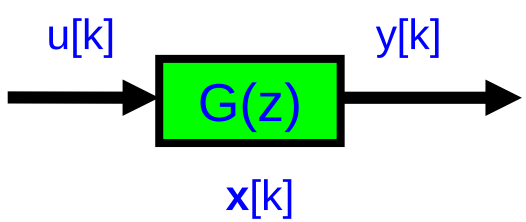
  from a given initial state to another state?

$x(k_0)$ $x(k)$

Observability

- How to determine the state of a dynamic system

  from the observations of inputs and outputs



- Some Examples

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$\begin{cases} x_1(k+1) = 2x_1(k) + u(k) \\ x_2(k+1) = 3x_2(k) + u(k) \end{cases}$$

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$$\begin{cases} x_1(k+1) = 2x_1(k) + u(k) \\ x_2(k+1) = 3x_2(k) + x_1(k) \end{cases}$$

- Definition: Controllability
  - The system is controllable
  - if it is possible to find a control sequence
  - such that an arbitrary state can be reached
  - from any initial state in finite time.

- Consider the system:

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{H}u(k) \\ y(k) = \mathbf{C}\mathbf{x}(k) \end{cases}$$

$$\mathbf{x} \in R^n$$

Initial state: $\mathbf{x}_0 = \mathbf{x}(0)$

- The state at time *n*

$$\begin{aligned} \mathbf{x}(n) &= \mathbf{F}^n\mathbf{x}(0) + \mathbf{F}^{n-1}\mathbf{H}u(0) + \cdots + \mathbf{H}u(n-1) \\ &= \mathbf{F}^n\mathbf{x}(0) + \begin{bmatrix} \mathbf{H} & \mathbf{F}\mathbf{H} & \cdots & \mathbf{F}^{n-1}\mathbf{H} \end{bmatrix} \begin{bmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \end{bmatrix} \\ &= \mathbf{x}_0^{\mathbf{n}} + \mathbf{W}_\mathbf{c}\,\mathbf{U} \end{aligned}$$

$$\Rightarrow \quad \mathbf{x}(n) - \mathbf{x}_0^{\mathbf{n}} = \mathbf{W}_\mathbf{c}\,\mathbf{U}$$

$$\Rightarrow \quad \mathbf{U} = \mathbf{W}_\mathbf{c}^{-1}\left[\mathbf{x}(n) - \mathbf{x}_0^{\mathbf{n}}\right]$$

IF $\text{rank}(\,\mathbf{W}_\mathbf{c}\,) = n$

- That is, exit some control signals, such that:

Initial state: $\mathbf{x}_0 = \mathbf{x}(0) \quad \rightarrow \quad \mathbf{x}(n)$

- Theorem: Controllability
  - The system is controllable
  - if and only the matrix **Wc** has rank *n*.

Controllability Matrix

$$\mathbf{W_c} = \begin{bmatrix} \mathbf{H} & \mathbf{FH} & \cdots & \mathbf{F}^{n-1}\mathbf{H} \end{bmatrix}$$

- Example: Controllable and Un-controllable systems

$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)$$

$$\mathbf{W_c} = \begin{bmatrix} \mathbf{H} & \mathbf{FH} & \cdots & \mathbf{F}^{n-1}\mathbf{H} \end{bmatrix}$$

$$\mathbf{W_c} = \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix}$$

- It is controllable

$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$\mathbf{W_c} = \begin{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

- It is not controllable

Controllable Canonical Form

- Assume that **F** has the characteristic polynomial:

$$\det\left(\lambda\mathbf{I}-\mathbf{F}\right) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n$$

- Assume that **Wc** is nonsingular.

- Then, the system can be described by the following

  Controllable Canonical From:

$$\mathbf{z}(k+1) = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \mathbf{z}(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} b_1 & \cdots & b_n \end{bmatrix} \mathbf{z}(k)$$

- The advantage of using the Controllable Canonical From:

$$\mathbf{z}(k+1) = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \mathbf{z}(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u(k)$$

- IF the input is:

$$u(k) = -\begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

$$\mathbf{z}(k+1) = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \mathbf{z}(k)$$

$$\begin{bmatrix} -k_1 & -k_2 & \cdots & -k_{n-1} & -k_n \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \mathbf{z}(k) \qquad - \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

Controllable Canonical Form

$$\mathbf{z}(k+1) = \begin{bmatrix} -(a_1+k_1) & -(a_2+k_2) & \cdots & -(a_{n-1}+k_{n-1}) & -(a_n+k_n) \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \mathbf{z}(k)$$

- The characteristic polynomial of the controlled system is:

$$\det\left(\lambda \mathbf{I} - (\mathbf{F} - \mathbf{H}\mathbf{K}) \right)$$

$$= \lambda^n + (a_1+k_1)\lambda^{n-1} + \cdots + (a_n+k_n)$$

$$\mathbf{z}(k+1) = \mathbf{F}\mathbf{z}(k) + \mathbf{H}u(k) \qquad = \mathbf{F}\mathbf{z}(k) - \mathbf{H}\mathbf{K}\mathbf{z}(k)$$

$$u(k) = -\mathbf{K}\mathbf{z}(k) \qquad = (\mathbf{F} - \mathbf{H}\mathbf{K})\,\mathbf{z}(k)$$

Controllable Canonical Form

- Example:

$$\mathbf{x}(k+1) = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} b_1 & b_2 \end{bmatrix} \mathbf{x}(k)$$

- The input-output transfer function is:

$$G(z) = \mathbf{C}(z\mathbf{I} - \mathbf{F})^{-1}\mathbf{H} + \mathbf{D} \qquad = \frac{B(z)}{A(z)}$$

$$= \begin{bmatrix} b_1 & b_2 \end{bmatrix} \left(z\mathbf{I} - \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} z + a_1 & a_2 \\ -1 & z \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{z^2 + a_1 z + a_2} \begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} z & -a_2 \\ 1 & z + a_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{b_1 z + b_2}{z^2 + a_1 z + a_2}$$

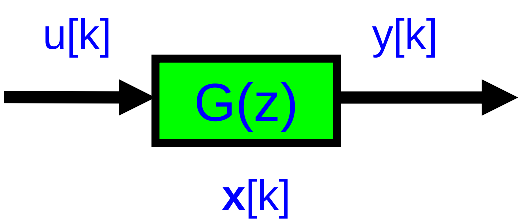
▪ Controllability

- Whether it is possible to steer a system
  from a given initial state to another state?

$x(k_0)$ $\qquad\qquad\qquad$ $x(k)$

▪ Observability

- How to determine the state of a dynamic system
  from the observations of inputs and outputs



Observability

- ## Definition: Un-observable States

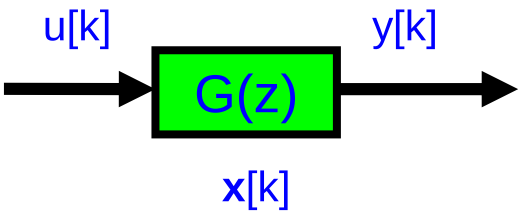
$$\begin{cases} \mathbf{x}(k+1) &= \mathbf{F}\mathbf{x}(k) + \mathbf{H}u(k) \\ y(k) &= \mathbf{C}\mathbf{x}(k) \end{cases}$$

$$\mathbf{x} \in R^n$$

Initial state: $\mathbf{x}_0 = \mathbf{x}(0)$

$x_0 \neq 0$ is un-observable
if $\exists$ a finite $k_1 \geq n-1$
such that
when $x(0) = x_0$ & $u(k) = 0$, for $0 \leq k \leq k_1$
then $y(k) = 0$, for $0 \leq k \leq k_1$

- ## Definition: Observable



A system is observable
if $\exists$ a finite k
such that
the knowledge of $\begin{cases} u(0), \ldots, u(k-1) \\ y(0), \ldots, y(k-1) \end{cases}$
is sufficient to
determine the initial state of the system

Observability

$$\begin{cases} \mathbf{x}(k+1) &= \mathbf{F}\mathbf{x}(k) + \mathbf{H}u(k) \\ y(k) &= \mathbf{C}\mathbf{x}(k) \end{cases}$$

$$\mathbf{x} \in R^n$$

Initial state: $\mathbf{x}_0 = \mathbf{x}(0)$

Let $u(k) \equiv 0$

and $y(0), y(1), \ldots, y(k-1)$ are given:

$$y(0) = \mathbf{C}\mathbf{x}(0)$$

$$y(1) = \mathbf{C}\mathbf{x}(1) = \mathbf{C}\,(\,\mathbf{F}\mathbf{x}(0)\,)$$

$$y(n-1) = \mathbf{C}\mathbf{x}(n-1) = \cdots = \mathbf{C}\,\mathbf{F}^{n-1}\mathbf{x}(0)$$

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(n-1) \end{bmatrix} = \begin{bmatrix} \mathbf{C}\mathbf{x}(0) \\ \mathbf{CF}\mathbf{x}(0) \\ \vdots \\ \mathbf{CF}^{n-1}\mathbf{x}(0) \end{bmatrix} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CF} \\ \vdots \\ \mathbf{CF}^{n-1} \end{bmatrix} \mathbf{x}(0)$$

$$\mathbf{Y} = \mathbf{W_o} \quad \mathbf{x}(0)$$

IF rank($\mathbf{W_o}$) $= n$ $\quad \Rightarrow \mathbf{x}(0) = \mathbf{W_o}^{-1}\,\mathbf{Y}$

- Theorem: Observability

$$\text{The system is observable} \iff \text{rank}(\mathbf{W_0}) = n$$

- Example: A system with unobservable states

$$\mathbf{x}(k+1) = \begin{bmatrix} 1.1 & -0.3 \\ 1 & 0 \end{bmatrix} \mathbf{x}(k)$$

$$y(k) = \begin{bmatrix} 1 & -0.5 \end{bmatrix} \mathbf{x}(k)$$

- The observability matrix is:

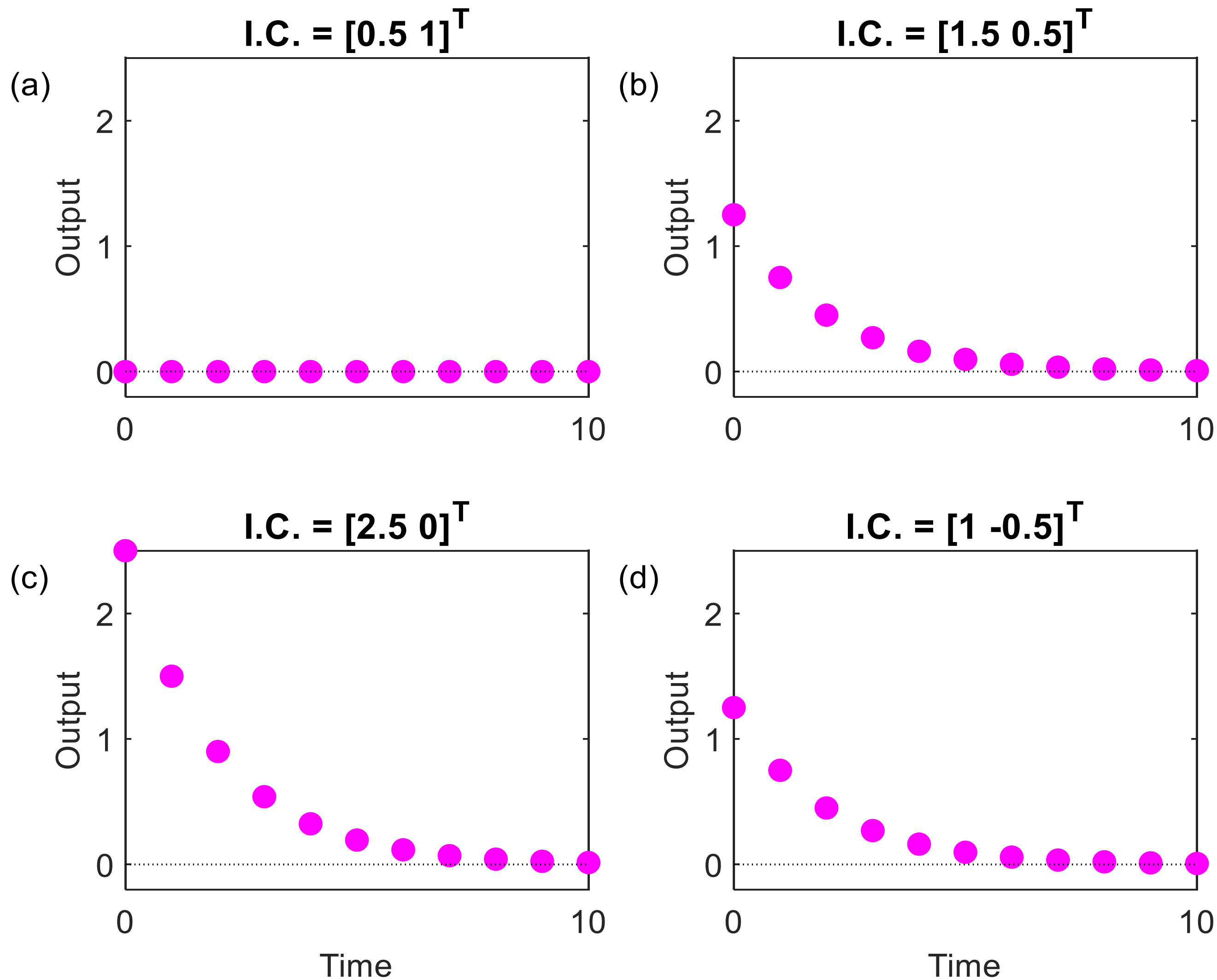
$$\mathbf{W_o} = \begin{bmatrix} \mathbf{C} \\ \mathbf{C\,F} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & -0.5 \end{bmatrix} \\ \begin{bmatrix} 1 & -0.5 \end{bmatrix} \begin{bmatrix} 1.1 & -0.3 \\ 1 & 0 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -0.5 \\ 0.6 & -0.3 \end{bmatrix} \quad \Rightarrow \text{rank}(\mathbf{W_o}) = 1$$

- The unobservable states belong to the null space of **Wo**:

that is, $\begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$

- Example 3.10: A system with unobservable states



- Observable Canonical Form

$$\mathbf{F}: \quad \det(\lambda I - \mathbf{F}) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n = 0$$

$$\mathbf{W_0}: \quad \text{nonsingular}$$

$$\mathbf{T}: \ \mathbf{x} \to \mathbf{z}, \text{ i.e., } \mathbf{z} = \mathbf{T}\mathbf{x}$$

- The transformed system is:

$$\mathbf{z}(k+1) = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & 0 \\ -a_{n-1} & \vdots & & & 1 \\ -a_n & 0 & & \cdots & 0 \end{bmatrix} \mathbf{z}(k) + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} u(k)$$

$$y(k) = [\, 1 \quad 0 \quad \cdots \quad 0 \,] \mathbf{z}(k)$$

- Observable Canonical Form

- Easy to find the observer gain:

$$\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{H}u(k) \qquad y(k) = \mathbf{C}\mathbf{x}(k)$$

$$\mathbf{x}_o(k+1) = \mathbf{F}\mathbf{x}_o(k) + \mathbf{H}u(k) + u_o(k)$$

$$u_o(k) = +\mathbf{L}[y(k) - \mathbf{C}\,\mathbf{x}_o(k)]$$

$$\mathbf{x}_o(k+1) = \mathbf{F}\mathbf{x}_o(k) + \mathbf{H}u(k) + \mathbf{L}[y(k) - \mathbf{C}\,\mathbf{x}_o(k)]$$

$$\mathbf{x}(k+1) - \mathbf{x}_o(k+1) = \mathbf{F}[\mathbf{x}(k) - \mathbf{x}_o(k)]$$

$$- \mathbf{L}[\mathbf{C}\,\mathbf{x}(k) - \mathbf{C}\,\mathbf{x}_o(k)]$$

$$\mathbf{x}_e(k) = \mathbf{x}(k) - \mathbf{x}_o(k) \qquad - \mathbf{L}\,\mathbf{C}[\,\mathbf{x}(k) - \mathbf{x}_o(k)\,]$$

$$\mathbf{x}_e(k+1) = \mathbf{F}\,\mathbf{x}_e(k) - \mathbf{L}\,\mathbf{C}\,\mathbf{x}_e(k)$$

$$= [\,\mathbf{F} - \mathbf{L}\,\mathbf{C}\,]\,\mathbf{x}_e(k)$$

- Observable Canonical Form
- Easy to find the observer gain:

$$\mathbf{x}_e(k+1) = [\,\mathbf{F} - \mathbf{L}\,\mathbf{C}\,]\,\mathbf{x}_e(k)$$

$$\mathbf{LC} = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{bmatrix} [\,1\ 0\ \cdots\ 0\,] = \begin{bmatrix} l_1 & 0 & \cdots & 0 \\ l_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & \\ l_n & 0 & \cdots & 0 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & 0 \\ -a_{n-1} & \vdots & & & 1 \\ -a_n & 0 & & \cdots & 0 \end{bmatrix}$$

$$\mathbf{F} - \mathbf{L}\,\mathbf{C}$$

Observable Canonical Form

$$\mathbf{x}_e(k+1) = [\,\mathbf{F} - \mathbf{L}\,\mathbf{C}\,]\,\mathbf{x}_e(k)$$

$$\mathbf{L}\mathbf{C} = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{bmatrix} [\,1\ 0\ \cdots\ 0\,] = \begin{bmatrix} l_1 & 0 & \cdots & 0 \\ l_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & \\ l_n & 0 & \cdots & 0 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & 0 \\ -a_{n-1} & \vdots & & & 1 \\ -a_n & 0 & & \cdots & 0 \end{bmatrix}$$

$$\mathbf{F} - \mathbf{L}\,\mathbf{C} = \begin{bmatrix} -(a_1 + l_1) & 1 & 0 & \cdots & 0 \\ -(a_2 + l_2) & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & 0 \\ -(a_{n-1} + l_{n-1}) & \vdots & & & 1 \\ -(a_n + l_n) & 0 & & \cdots & 0 \end{bmatrix}$$

$$\det(\lambda I - (\mathbf{F} - \mathbf{L}\,\mathbf{C})) = \lambda^n + (a_1 + l_1)\lambda^{n-1} + \cdots + (a_n + l_n) = 0$$

- Example: (in observable canonical form)

$$\mathbf{x}(k+1) = \begin{bmatrix} -a_1 & 1 \\ -a_2 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(k)$$

- The input-output transfer function is:

$$G(z) = \mathbf{C}(z\mathbf{I} - \mathbf{F})^{-1}\mathbf{H} + \mathbf{D} \qquad = \frac{B(z)}{A(z)}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \left(z\mathbf{I} - \begin{bmatrix} -a_1 & 1 \\ -a_2 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z + a_1 & -1 \\ a_2 & z \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad = \frac{b_1 z + b_2}{z^2 + a_1 z + a_2}$$

$$= \frac{1}{z^2 + a_1 z + a_2} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z & 1 \\ -a_2 & z + a_1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Observability

- Observable canonical form:

$$\mathbf{x}(k+1) = \begin{bmatrix} -a_1 & 1 \\ -a_2 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(k)$$

- The input-output transfer function is: $G(z) = \dfrac{b_1 z + b_2}{z^2 + a_1 z + a_2}$

- Controllable canonical form:

$$\mathbf{x}(k+1) = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} b_1 & b_2 \end{bmatrix} \mathbf{x}(k)$$

- The input-output transfer function is: $G(z) = \dfrac{b_1 z + b_2}{z^2 + a_1 z + a_2}$

- Kalman showed that:

$$\mathbf{x}(k+1) = \begin{bmatrix} F_{11} & F_{12} & 0 & 0 \\ 0 & F_{22} & 0 & 0 \\ F_{31} & F_{32} & F_{33} & F_{34} \\ 0 & F_{42} & 0 & F_{44} \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} H_1 \\ 0 \\ H_3 \\ 0 \end{bmatrix} u(k)$$

$$y(k) = [\, C_1 \;\; C_2 \;\; 0 \;\; 0 \,]\, \mathbf{x}(k)$$

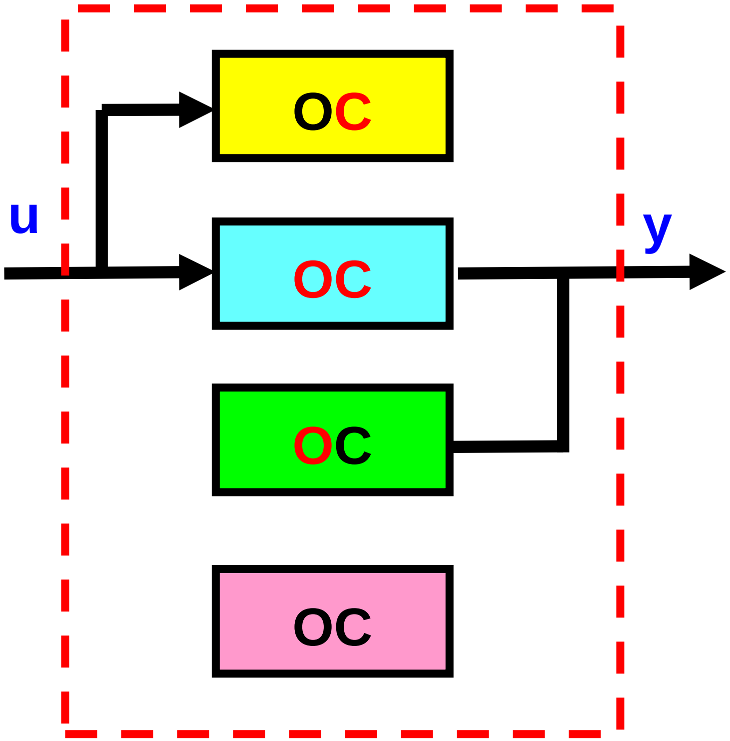
where $\mathbf{x}(k) = \begin{bmatrix} OC \\ O\bar{C} \\ \bar{O}C \\ \bar{O}\bar{C} \end{bmatrix}$ and, $\mathbf{G}(z) = \mathbf{C}\,(\,zI - \mathbf{F})^{-1}\,\mathbf{H}$

$$= C_1(zI - F_{11})^{-1}H_1$$

$$\mathbf{x}_1[k+1] = F_{11}\mathbf{x}_1[k] + H_1 u[k]$$

$$y[k] = C_1 \mathbf{x}_1[k]$$

Kalman's Decomposition



$$\mathbf{x}[k+1] = \mathbf{F}\mathbf{x}[k] + \mathbf{H}u[k]$$
$$y[k] = \mathbf{C}\mathbf{x}[k] + \mathbf{D}u[k]$$

$$\mathbf{x}_1[k+1] = F_{11}\mathbf{x}_1[k] + H_1 u[k]$$
$$y[k] = C_1 \mathbf{x}_1[k]$$

$$\frac{Y_d(z)}{U_d(z)} = G_d(z) = \frac{B_d(z)}{A_d(z)}$$
$$= C_1 (zI - F_{11})^{-1} H_1$$

$$\mathbf{x}(k) = \begin{bmatrix} OC \\ O\bar{C} \\ \bar{O}C \\ \bar{O}\bar{C} \end{bmatrix}$$

unstable $\to$ stable

design $u[k] \Rightarrow$

stable $\to$ more stable

from $y[k]$ to estimate $\mathbf{x}[k]$

- Loss of Controllability:

$$\text{C.T. system} \xrightarrow{\text{sampling}} \text{D.T. system}$$

  - If D.T. system is controllable, then C.T. system is controllable
  - But, if C.T. system is controllable, D.T. system may not!

- Loss of Observability:
  - Un-observability in C.T. system:
    - ✓ zero over a time interval
  - Un-observability in D.T. system:
    - ✓ zero only at sampling instants
    - ✓ May oscillate between sampling instants (hidden oscillation)

- Example 3.12: The harmonic oscillator
- The CT model is:

$$\frac{dx}{dt} = \begin{bmatrix} 0 & w \\ -w & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ w \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}$$

- The DT model (using zero-order hold) is:

$$\mathbf{x}(k+1) = \begin{bmatrix} \cos(wh) & \sin(wh) \\ -\sin(wh) & \cos(wh) \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1-\cos(wh) \\ \sin(wh) \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(k)$$

- Example 3.12: The harmonic oscillator
- The CT model is:

$$\frac{dx}{dt} = \begin{bmatrix} 0 & w \\ -w & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ w \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}$$

- The controllability matrix is:

$$\mathbf{W_c^c} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 \\ w \end{bmatrix} & \begin{bmatrix} 0 & w \\ -w & 0 \end{bmatrix} \begin{bmatrix} 0 \\ w \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & w \\ w^2 & 0 \end{bmatrix}$$

$$\Rightarrow \det(\mathbf{W_c^c}) = -w^3$$

- The observability matrix is:

$$\mathbf{W_o^c} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & w \\ -w & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & w \end{bmatrix}$$

$$\Rightarrow \det(\mathbf{W_o^c}) = w$$

- Example 3.12: The harmonic oscillator
- The DT model (using zero-order hold) is:

$$\mathbf{x}(k+1) = \begin{bmatrix} \cos(wh) & \sin(wh) \\ -\sin(wh) & \cos(wh) \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1-\cos(wh) \\ \sin(wh) \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(k)$$

- The controllability matrix is:

$$\mathbf{W}_{\mathbf{c}}^{\mathbf{d}} = \begin{bmatrix} \mathbf{H} & \mathbf{FH} \end{bmatrix}$$

$$= \begin{bmatrix} 1-\mathrm{c}wh & \mathrm{c}wh(1-\mathrm{c}wh)+(\mathrm{s}wh)^2 \\ \mathrm{s}wh & -\mathrm{s}wh(1-\mathrm{c}wh)+(\mathrm{c}wh)(\mathrm{s}wh) \end{bmatrix}$$

$$= \begin{bmatrix} 1-\mathrm{c}wh & \mathrm{c}wh-(\mathrm{c}wh)^2+(\mathrm{s}wh)^2 \\ \mathrm{s}wh & -\mathrm{s}wh+2(\mathrm{c}wh)(\mathrm{s}wh) \end{bmatrix}$$

$$\det(\ \mathbf{W}_{\mathbf{c}}^{\mathbf{d}}\ ) = \cdots = -2(\sin wh)(1-(\cos wh))$$

- Example 3.12: The harmonic oscillator
- The DT model (using zero-order hold) is:

$$\mathbf{x}(k+1) = \begin{bmatrix} \cos(wh) & \sin(wh) \\ -\sin(wh) & \cos(wh) \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1-\cos(wh) \\ \sin(wh) \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(k)$$

- The observability matrix is:

$$\mathbf{W}_{\mathbf{o}}^{\mathbf{d}} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CF} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \cos(wh) & \sin(wh) \\ -\sin(wh) & \cos(wh) \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ \cos(wh) & \sin(wh) \end{bmatrix}$$

$$\det(\ \mathbf{W}_{\mathbf{o}}^{\mathbf{d}}\ ) = \sin wh$$

- Example 3.12: D.T. model of the harmonic oscillator

$$\det(\ \mathbf{W_c^d}\ ) = -2(\sin wh)(1-\cos wh)$$

$$\Rightarrow \sin wh = 0 \quad \Rightarrow wh = 0, n\pi$$

$$\Rightarrow 1-\cos wh = 0 \quad \Rightarrow \cos wh = 1 \quad \Rightarrow wh = 2n\pi$$

$$\det(\ \mathbf{W_o^d}\ ) = \sin wh$$

$$\Rightarrow \sin wh = 0 \quad \Rightarrow wh = 0, n\pi \quad \Rightarrow w = \frac{\pi}{h} \quad \Rightarrow w_s = \frac{2\pi}{h}$$

$$\Rightarrow w_N = \frac{\pi}{h}$$

- SUMMARY

Models	**Controllability**	**Observability**
CT	OK	OK
DT	Lost when $wh = n\pi$	Lost when $wh = n\pi$