Spring 2021

數位控制系統

Digital Control Systems

DCS-22

Stability

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NTU-EE

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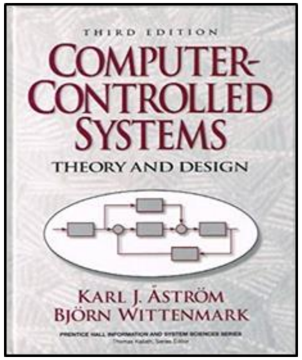

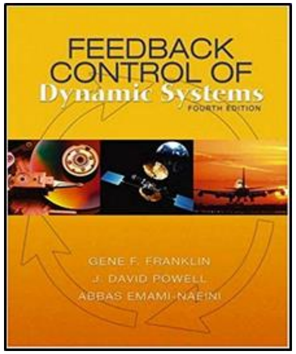

The Research Procedure in Control Science

Physical Process → Math Model → System Analysis → Control Design

Physical Process
- Plant
- Sensor
- Actuator
- Computer
- Communication
- Noise
- Disturbance

Math Model
- Differential eqn
- Laplace transform
- Transfer function
- State space form

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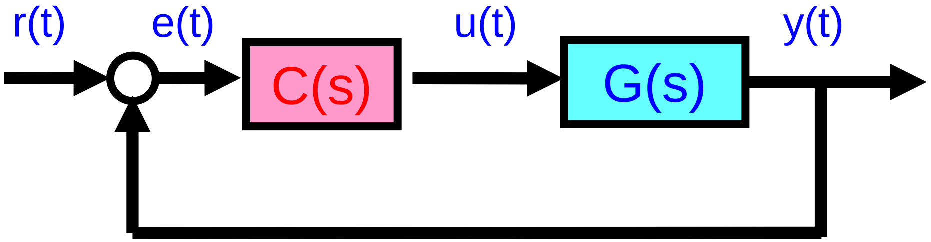
- Difference eqn
- z transform
- Transfer function
- State space form

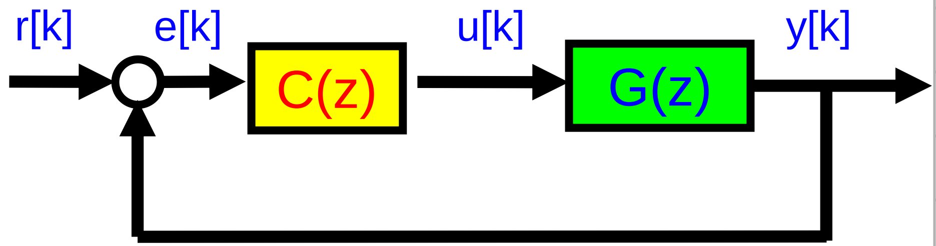
System Analysis
- Root locus
- Bode diagram
- Nyquist plot

- Stability
- Robustness
- Sensitivity
- Controllability
- Observability

Control Design
- Estimator
- Identification
- Regulation
- Tracking
- PID
- Pole placement
- Optimal Control LQR/LQG
- Adaptive control
- Robust control
- Decentralized (or Multi-person) Control

Introduction: From CT Plant to DT Plant





- Plant (CT):
  - Input-Output Model:

$$u(t)$$
$$y(t)$$

$$\xrightarrow{\;2\;}$$

- Plant (DT):
  - Input-Output Model:

$$u[k]$$
$$y[k]$$

$$G(s) = \frac{Y(s)}{U(s)} \quad \xrightarrow{\;3\;} \quad G(z) = \frac{Y(z)}{U(z)}$$

- State-Space Model:

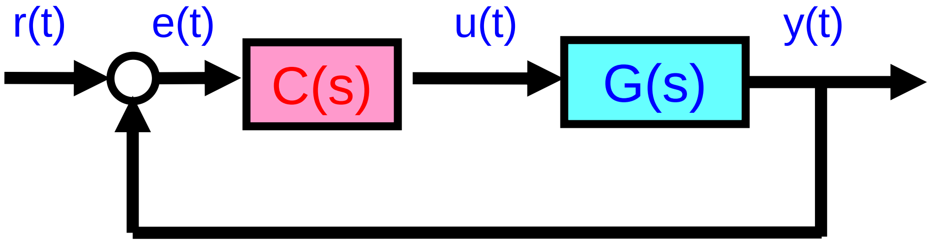
$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\ y(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t) \end{aligned}$$

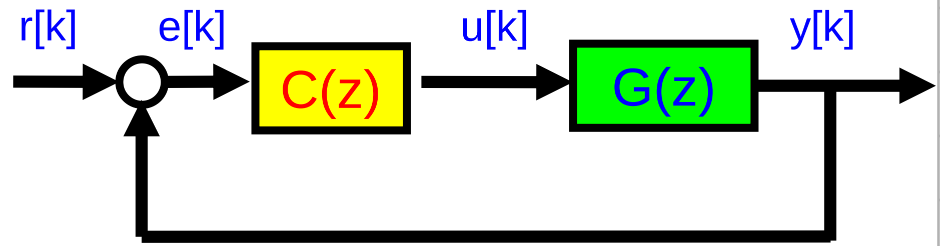
$$\xrightarrow{\;1\;}$$

- State-Space Model:

$$\begin{aligned} \mathbf{x}[k+1] &= \mathbf{F}\mathbf{x}[k] + \mathbf{H}u[k] \\ y[k] &= \mathbf{C}\mathbf{x}[k] + \mathbf{D}u[k] \end{aligned}$$

Introduction: Model and Analysis





- Plant (CT):
  - Input-Output Model:

$$\frac{Y_c(s)}{U_c(s)} = G_c(s) = \frac{B_c(s)}{A_c(s)}$$

  - State-Space Model:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\ y(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t) \end{aligned}$$

- Plant (DT):
  - Input-Output Model:

$$\frac{Y_d(z)}{U_d(z)} = G_d(z) = \boxed{\frac{B_d(z)}{A_d(z)}}$$

  - State-Space Model:

$$\begin{aligned} \mathbf{x}[k+1] &= \boxed{\mathbf{F}}\mathbf{x}[k] + \boxed{\mathbf{H}}u[k] \\ y[k] &= \boxed{\mathbf{C}}\mathbf{x}[k] + \boxed{\mathbf{D}}u[k] \end{aligned}$$

- System Properties:
  - ➢ Stability
  - ➢ Controllability and Reachability
  - ➢ Observability and Detectability

Outline

- Solution of a System
- Stability and Asymptotic Stability
- Input-Output Stability
- Stability Tests:
  - Jury’s Stability Criterion
  - Nyquist and Bode Diagrams
  - Nyquist Criterion
  - Relative Stability
- Lyapunov’s Second Stability

Solution of a System

- CT:

$$\frac{d}{dt}\mathbf{x}(t) = f(\mathbf{x}(t), t) \qquad \mathbf{x}(t) : R \rightarrow R^n$$

  - linear or nonlinear
  - time-invariant or time-varying

- DT:

$$\mathbf{x}(k+1) = f(\mathbf{x}(k), k) \qquad \mathbf{x}(k) : Z \rightarrow R^n$$

- Initial Condition: (for 2nd order system, n=2)

$$\mathbf{x}_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} = \begin{bmatrix} x_1(k_0) \\ x_2(k_0) \end{bmatrix}$$

- Solution: (for 2nd order system , n=2)

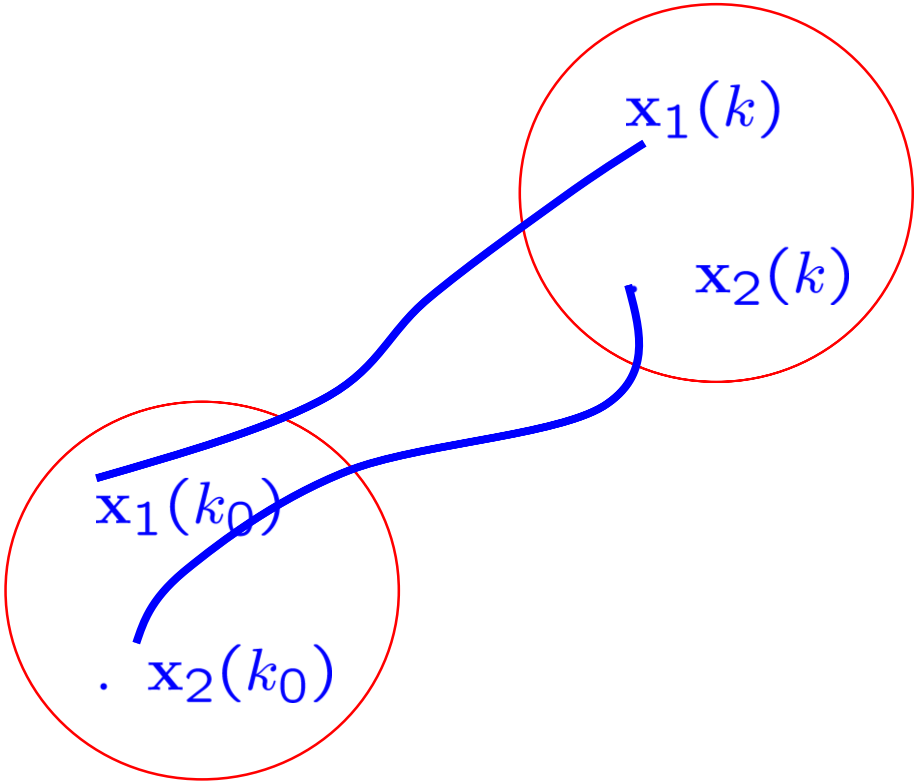
$$\mathbf{x}(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} p_1(k, k_0, x_{10}) \\ p_2(k, k_0, x_{20}) \end{bmatrix}$$

▪ Definition 3.1: Stability

- $\mathbf{x}_1(k)$ is stable

  if for a given $\epsilon > 0$

  there exists a $\delta(\epsilon, k_0)$



  such that
  all solutions with $||\mathbf{x}_2(k_0) - \mathbf{x}_1(k_0)|| < \delta$

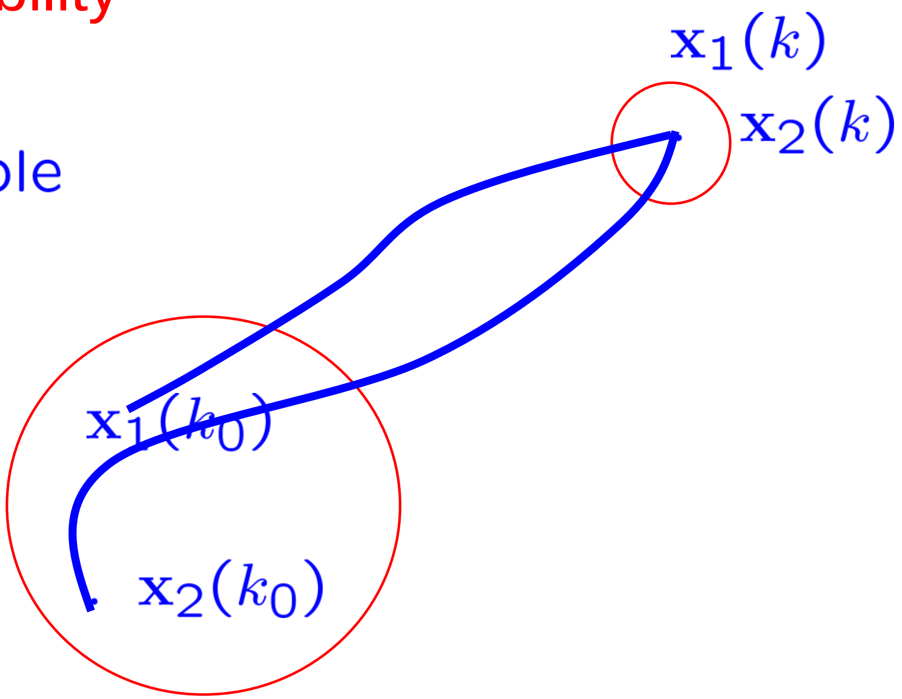
$$\Rightarrow \; ||\mathbf{x}_2(k) - \mathbf{x}_1(k)|| < \epsilon, \qquad \forall k \geq k_0$$

Definition 3.2: Asymptotic Stability

- $\mathbf{x}_1(k)$ is asymptotic stable

  if it is stable , and

  if δ can be chosen



  such that $||\mathbf{x}_2(k_0) - \mathbf{x}_1(k_0)|| < \delta$

  $\Rightarrow \; ||\mathbf{x}_2(k) - \mathbf{x}_1(k)|| \rightarrow 0,$ when $k \rightarrow \infty$

- Stability of Linear Discrete-Time Systems

$$\mathbf{x}_1(k+1) = \mathbf{F}\,\mathbf{x}_1(k), \quad \mathbf{x}_1(0) = \mathbf{a}_1$$

$$\mathbf{x}_2(k+1) = \mathbf{F}\,\mathbf{x}_2(k), \quad \mathbf{x}_2(0) = \mathbf{a}_2$$

$$\Rightarrow \tilde{\mathbf{x}} = \mathbf{x}_1 - \mathbf{x}_2$$

$$\Rightarrow \mathbf{x}_1(k+1) - \mathbf{x}_2(k+1) = \mathbf{F}\,\mathbf{x}_1(k) - \mathbf{F}\,\mathbf{x}_2(k)$$

$$\Rightarrow \tilde{\mathbf{x}}(k+1) = \mathbf{F}\,\tilde{\mathbf{x}}(k), \qquad \tilde{\mathbf{x}}(0) = \mathbf{a}_1 - \mathbf{a}_2$$

$\Rightarrow$ If $\mathbf{x}_1$ is stable

$\Rightarrow$ every other solution is also stable

$\Rightarrow$ Hence, for LTI systems,

stability is a property of the system and
not of a special solution

- Solution of LTI DT Systems

$$\tilde{\mathbf{x}}(k+1) = \mathbf{F}\,\tilde{\mathbf{x}}(k), \qquad \tilde{\mathbf{x}}(0) = \mathbf{a}_1 - \mathbf{a}_2$$

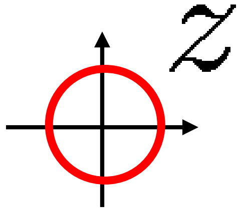
$$\Rightarrow \tilde{\mathbf{x}}(k) = \mathbf{F}^k\,\tilde{\mathbf{x}}(0)$$

$$\mathbf{F} = \mathbf{U}\begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}\mathbf{U}^{-1}$$

Let $\lambda_i = \text{eig}(\ \mathbf{F}\ )$

$$\mathbf{F}^k = \mathbf{U}\begin{bmatrix} \lambda_1^k & & * \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix}\mathbf{U}^{-1}$$

Asymptotic stable $\Rightarrow \quad |\lambda_i| < 1,\ i = 1, \cdots, n$



- Theorem 3.1: Asymptotic Stability of Linear Systems

  - A DT LTI system is asymptotic stable

    $\Leftrightarrow$ all eig($\mathbf{F}$) are strictly inside the unit disc

▪ Stability of Linear Continuous-Time Systems

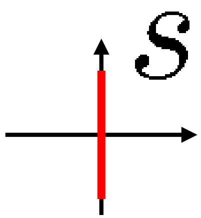
$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$$

$$\Rightarrow \mathbf{x}(t) = e^{(\mathbf{A}(t-t_0))}\,\mathbf{x}(t_0)$$

Let $\lambda_i = \text{eig}(\ \mathbf{A}\ )$ $\qquad \Rightarrow \mathbf{A} = \mathbf{U}\begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}\mathbf{U}^{-1}$

$$\Rightarrow \mathbf{x}(t) = \mathbf{U}\begin{bmatrix} e^{\lambda_1(t-t_0)} & & * \\ & \ddots & \\ 0 & & e^{\lambda_n(t-t_0)} \end{bmatrix}\mathbf{U}^{-1}\,\mathbf{x}(t_0)$$

Asymptotic stable $\Rightarrow$ $\text{Real}(\lambda_i) < 0,\ i = 1, \cdots, n$



- Definition 3.3: Bounded-Input-Bounded-Output Stability

  - A LTI system is defined as BIBO stable

    if a bounded input gives a bounded output
    for every initial value

- Theorem 3.2: Relation between Stability Concept

$$\text{Asymptotic stable} \;\Rightarrow\; \text{stable and BIBO stable}$$

Example 3.1: Harmonic Oscillator

$$\mathbf{x}(k+1) = \begin{bmatrix} \cos wh & \sin wh \\ -\sin wh & \cos wh \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1-\cos wh \\ \sin wh \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(k)$$

- $\text{mag}(\ \text{eig}(\mathbf{F})\ ) = 1$
- if $u(k) = 0$ $\Rightarrow$ $||\mathbf{x}(k+1)|| = ||\mathbf{x}(0)||$

  $\Rightarrow$ the system is stable
- But, if input is a cos or sin signal with $w\ rad/s$

  $\Rightarrow$ the output contains a sinusoidal function with growing amplitude

  $\Rightarrow$ the system is not BIBO stable

- Eigenvalues of $\mathbf{F}$

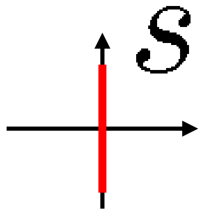
$$\lambda_i = \text{eig}(\mathbf{F})$$

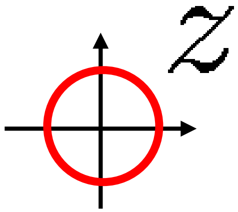
- Characteristic Polynomials

$$A(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$$

$$a_i \Leftrightarrow \lambda_i$$

- Root locus method





- Nyquist criterion

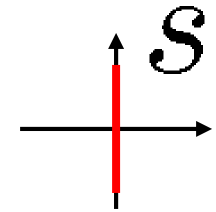
- Lyapunov's method

Stability Test: Routh's Stability Criterion (for CT)

3.6.2 Routh's Stability Criterion

- Routh in 1874
- Hurwitz in 1895

$$a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n. \tag{3.65}$$



A necessary (but not sufficient) condition for stability is that *all* the coefficients of the characteristic polynomial be positive.

A system is stable if and only if *all* the elements in the first column of the Routh array are positive.

We then add subsequent rows to complete the **Routh array**:

Row	n	s^n:	1	a_2	a_4	$\cdots$
Row	$n-1$	s^{n-1}:	a_1	a_3	a_5	$\cdots$
Row	$n-2$	s^{n-2}:	b_1	b_2	b_3	$\cdots$
Row	$n-3$	s^{n-3}:	c_1	c_2	c_3	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
Row	2	s^2:	$*$	$*$		
Row	1	s:	$*$			
Row	0	s^0:	$*$			

$$b_1 = -\frac{\det\begin{bmatrix} 1 & a_2 \\ a_1 & a_3 \end{bmatrix}}{a_1} = \frac{a_1 a_2 - a_3}{a_1},$$

$$b_2 = -\frac{\det\begin{bmatrix} 1 & a_4 \\ a_1 & a_5 \end{bmatrix}}{a_1} = \frac{a_1 a_4 - a_5}{a_1},$$

$$b_3 = -\frac{\det\begin{bmatrix} 1 & a_6 \\ a_1 & a_7 \end{bmatrix}}{a_1} = \frac{a_1 a_6 - a_7}{a_1},$$

$$c_1 = -\frac{\det\begin{bmatrix} a_1 & a_3 \\ b_1 & b_2 \end{bmatrix}}{b_1} = \frac{b_1 a_3 - a_1 b_2}{b_1},$$

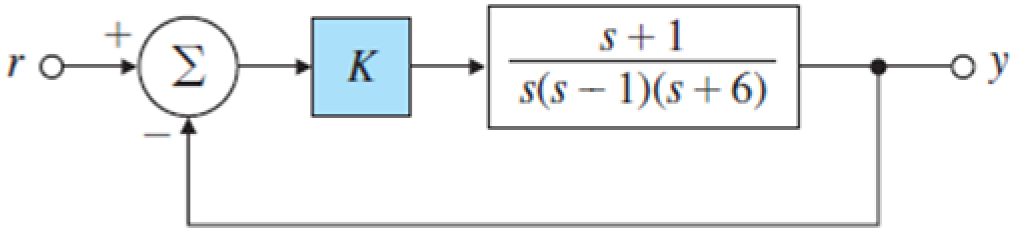
$$c_2 = -\frac{\det\begin{bmatrix} a_1 & a_5 \\ b_1 & b_3 \end{bmatrix}}{b_1} = \frac{b_1 a_5 - a_1 b_3}{b_1},$$

$$c_3 = -\frac{\det\begin{bmatrix} a_1 & a_7 \\ b_1 & b_4 \end{bmatrix}}{b_1} = \frac{b_1 a_7 - a_1 b_4}{b_1}.$$

Franklin, Powell, Emami-Naeini 2002

▪ Example 3.33: Stability versus Parameter Range

- A feedback system for testing stability



▪ The characteristic equation for the system:

$$1 + K\frac{s+1}{s(s-1)(s+6)} = 0 \qquad \boxed{s^3} + 5s^2 + \boxed{(K-6)s} + K = 0$$

$$\begin{array}{lcl} s^3 & : & \boxed{\begin{array}{c}1\\5\end{array}} \quad \begin{array}{c}K-6\\K\end{array} \\ s^2 & : & \\ s & : & \boxed{\begin{array}{c}(4K-30)/5\\K\end{array}} \\ s^0 & : & \end{array}$$

$$\Rightarrow \frac{(4K-30)}{5} > 0 \qquad \Rightarrow K > 7.5$$

$$\Rightarrow K > 0$$

Franklin, Powell, Emami-Naeini 2002

Stability Test: Jury's Stability Criterion (for DT)

(1918) (1922) (1961)

Schur-Cohn-Jury

$$A(z) = \boxed{a_0 z^n + a_1 z^{n-1} + \cdots + a_n} = 0$$

(Row 3) = (Row 1) – (Row 2) (α)

$$\begin{array}{ccccc}
a_0 & a_1 & \cdots & a_{n-1} & a_n \\
a_n & a_{n-1} & \cdots & a_1 & a_0 \\
\hline
a_0^{n-1} & a_1^{n-1} & \cdots & a_{n-1}^{n-1} & \\
a_{n-1}^{n-1} & a_{n-2}^{n-1} & \cdots & a_0^{n-1} & \\
\hline
\vdots & & & & \\
a_0^0 & & & &
\end{array}$$

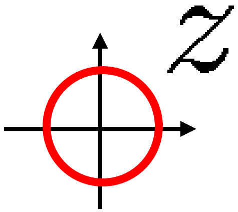
$$\alpha_n = \frac{a_n}{a_0}$$

$$\alpha_{n-1} = \frac{a_{n-1}^{n-1}}{a_0^{n-1}}$$

$$a_i^{k-1} = a_i^k - \alpha_k a_{k-i}^k$$

$$\alpha_k = a_k^k / a_0^k$$

Theorem 3.3: Jury's Stability Test



- If $a_0 > 0$,
  then, $A(z) = 0$ has all roots inside unit disc
  $\Longleftrightarrow$ all $a_0^k > 0$, $k = 0, 1, \cdots, n-1$

- If no a_0 is zero,
  then, the number of negative a_0^k
  $=$ the number of roots outside the unit disc

- Remark:

- If all $a_0^k > 0$,
  then,

$$a_0^0 > 0 \Longleftrightarrow \begin{cases} A(1) > 0 \\ (-1)^n A(-1) > 0 \end{cases}$$

Stability Test: Jury's Stability Criterion (for DT)

- Example: Jury's Stability Test

$$\alpha_n = \frac{a_n}{a_0} \qquad a_i^{k-1} = a_i^k - \alpha_k a_{k-i}^k$$

$$\alpha_k = a_k^k / a_0^k$$

$$A(z) = z^2 + a_1 z + a_2$$

$$\begin{array}{ccc}
\boxed{1} & a_1 & a_2 \\
a_2 & a_1 & 1 \\
\hline
\end{array} \qquad \alpha_2 = a_2$$

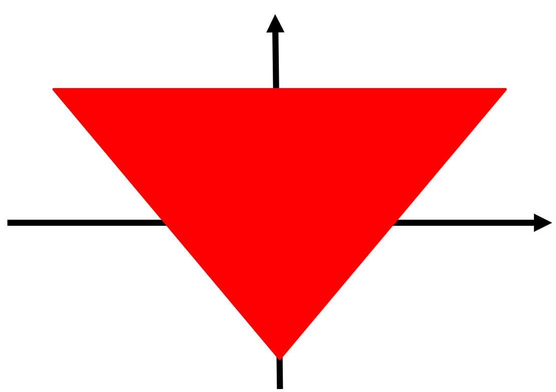
$$\begin{array}{cc}
\boxed{1 - a_2^2} & a_1(1 - a_2) \\
a_1(1 - a_2) & 1 - a_2^2 \\
\hline
\end{array} \qquad \alpha_1 = \frac{a_1}{1 + a_2}$$

$$\boxed{1 - a_2^2 - \frac{a_1^2(1-a_2)}{1+a_2}}$$

All the roots are inside the unit circle if

$$\begin{aligned}
1 - a_2^2 &> 0 \\
1 - a_2^2 - \frac{a_1^2(1-a_2)}{1+a_2} &> 0
\end{aligned}$$

$$\Rightarrow \quad \begin{aligned}
a_2 &< 1 \\
a_2 &> -1 + a_1 \\
a_2 &> -1 - a_1
\end{aligned}$$



- DT pulse-transfer function: $G(z)$
- Nyquist or Frequency curve

$$G(e^{jwh}), \quad \text{for } wh \in [0, \pi]$$

upto to the Nyquist frequency, $w_N = \pi / h$

- Note that it is sufficient
  to consider the map in $wh \in [-\pi, \pi]$

- Because $G(e^{jwh})$ is periodic
  with period $2\pi / h$

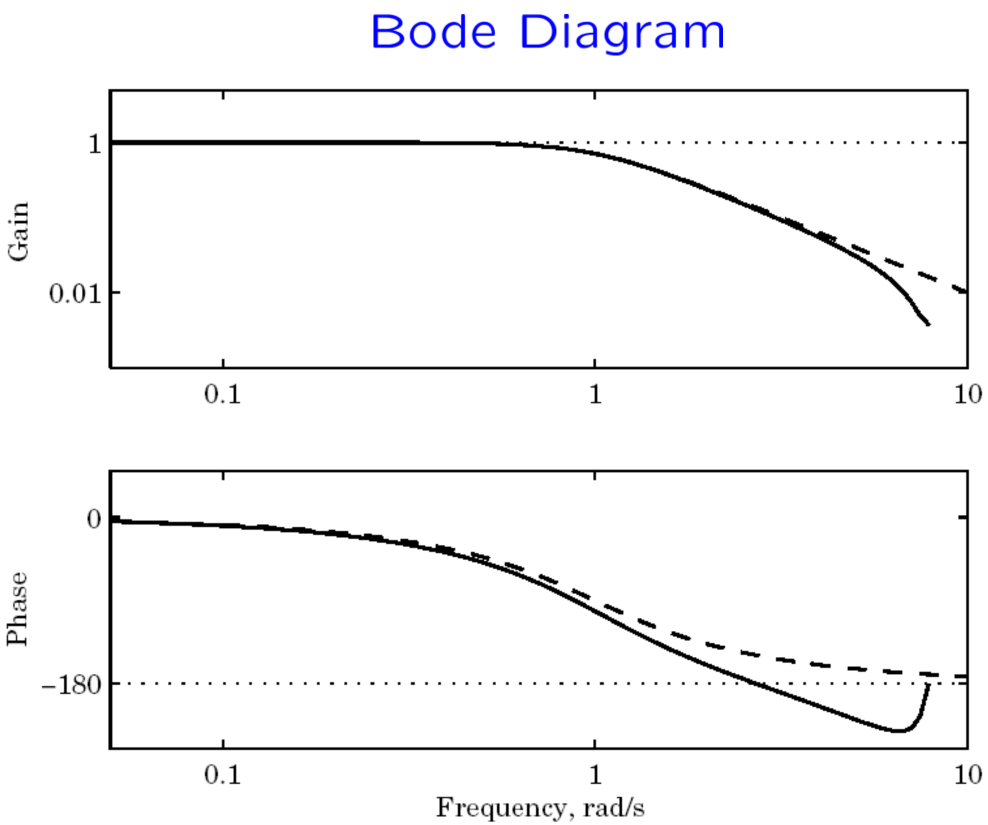
- Example 3.3: Frequency Responses

$$G(s) = \frac{1}{s^2 + 1.4s + 1}$$

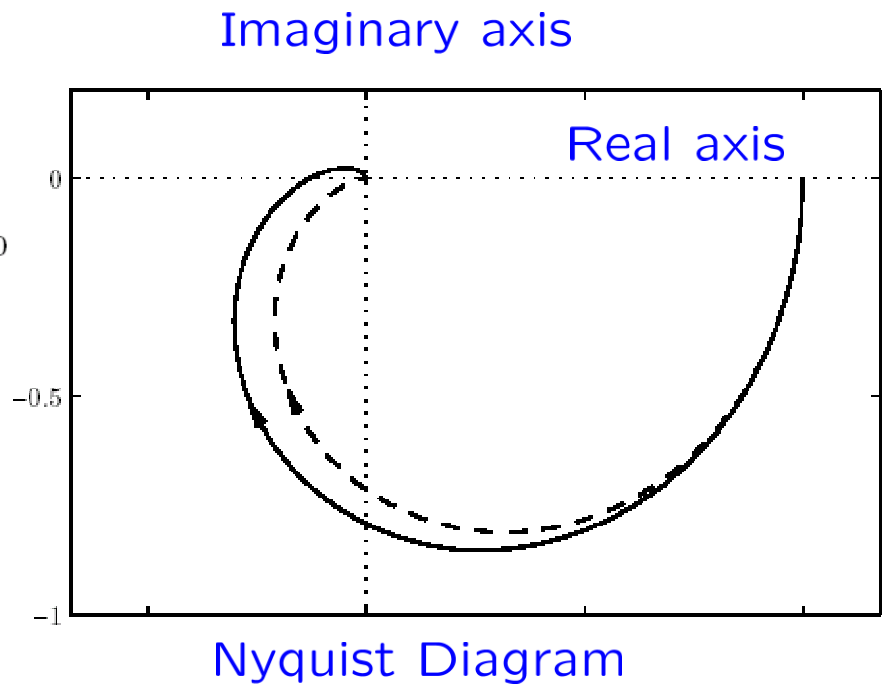
Zero-order hold sampling $h = 0.4$

$$G(z) = \frac{0.066z + 0.055}{z^2 - 1.450z + 0.571}$$

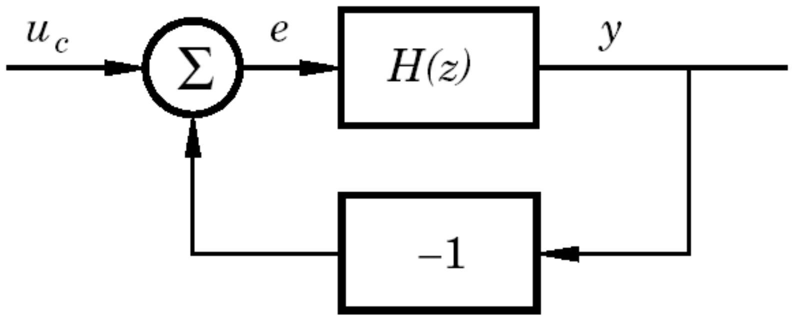
▪ Example 3.3: Frequency Responses



- CT: - - -
- DT: —



- Nyquist Criterion



Closed-loop system

$$Y(z) = H_{cl}(z)U_c(z) = \frac{H(z)}{1+H(z)}U_c(z)$$

Closed-loop system characteristic equation

$$1 + H(z) = 0$$

Stability Test: Nyquist Criterion

Franklin, Powell, Emami-Naeini 2002

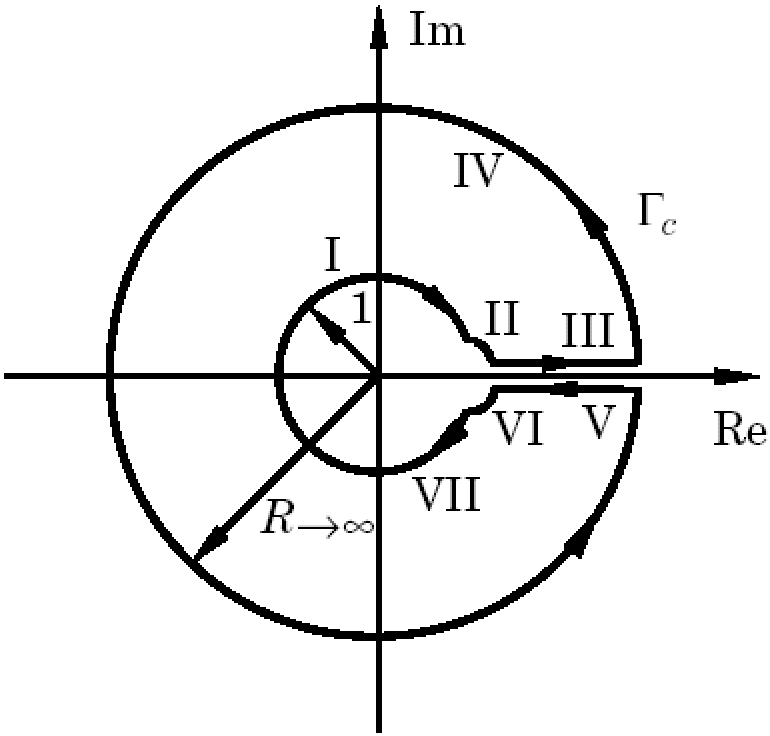
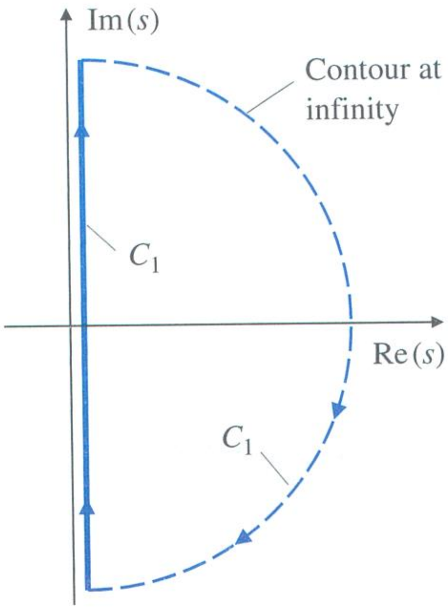


Figure 6.17
An s-plane plot of a contour C_1 that encircles the entire RHP



Principle of arguments states

$$N = Z - P$$

Z and P are the number of zeros and poles of $1 + H(z)$ outside the unit disc.

- Example 3.4: A Second-order system

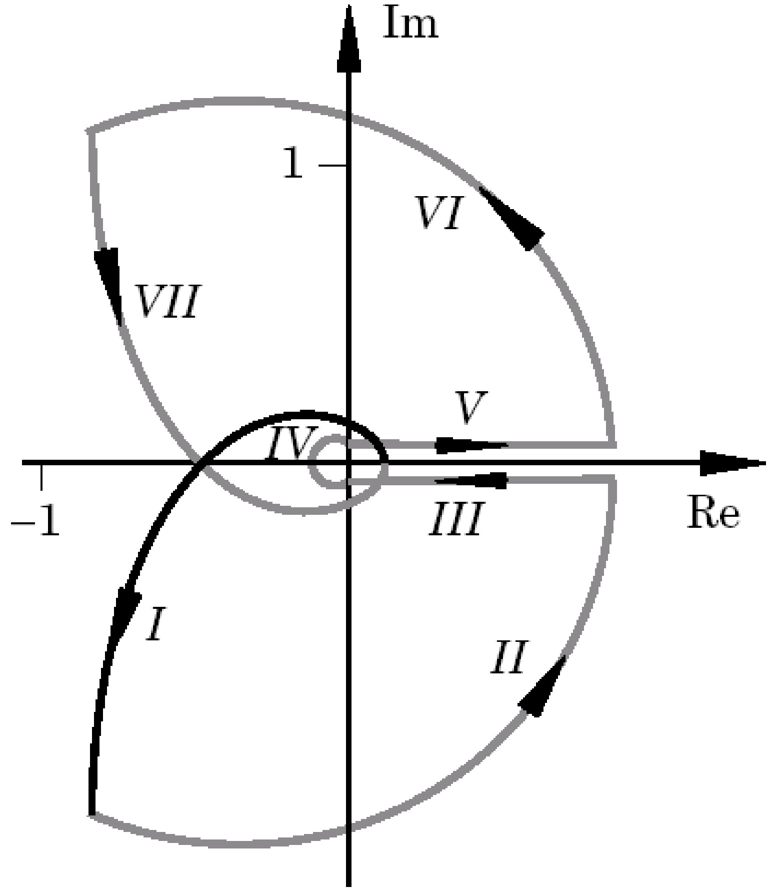
$$h = 1$$

$$H(z) = \frac{0.25K}{(z-1)(z-0.5)}$$

then

$$H(e^{i\omega}) = \frac{0.25K\left(1.5(1-\cos\omega) - 2\sin^2\omega - i\sin\omega(2\cos\omega - 1.5)\right)}{(2-2\cos\omega)(1.25+\cos\omega)}$$

- Example 3.4: A Second-order system



$H(e^{jw})$, for $w \in [0, \pi]$

- At some w, phase shift $> 180^0$
- Stable if $K < 2$

Definitions 3.4 & 3.5: Gain & Phase Margins

- The amplitude or gain margin:

$$\arg G(e^{i w_0 h}) = -\pi \qquad\qquad A_{\text{marg}} = \frac{1}{|G(e^{i w_0 h})|}$$

- The phase margin:

$$|G(e^{i w_c h})| = 1 \qquad\qquad \phi_{\text{marg}} = \pi + \arg G(e^{i w_c h})$$

Definition 3.6: Lyapunov Function

- $V(\mathbf{x})$ is a Lyapunov function for

$$\mathbf{x}(k+1) = f(\mathbf{x}(k)) \quad f(0) = 0$$

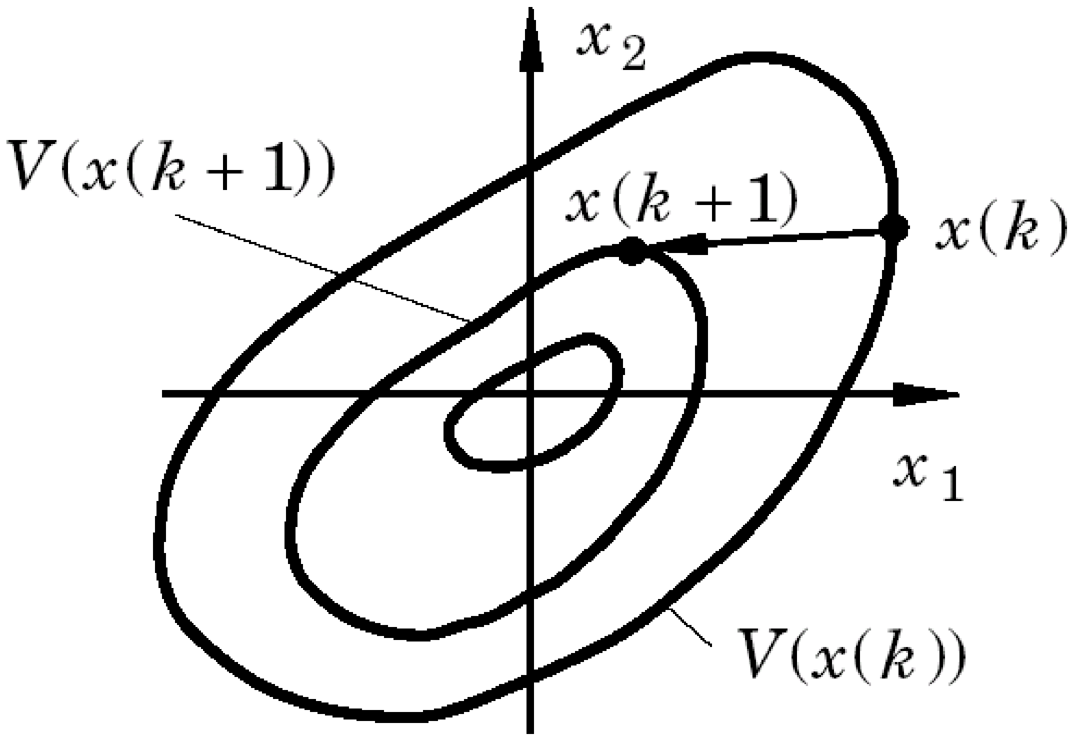
- If:
  1. $V(\mathbf{x})$ is continuous in x and $V(0) = 0$
  2. $V(\mathbf{x})$ is positive definite
  3. $\triangle V(\mathbf{x}) = V(f(\mathbf{x})) - V(\mathbf{x})$
     is negative definite
  4. $V(\mathbf{x}) \to \infty$ as $|\mathbf{x}| \to \infty$

- Existence of Lyapunov function implies asymptotic stability for the solution $x = 0$

- # Geometric Illustration

1. $V(\mathbf{x})$ is continuous in $\mathbf{x}$ and $V(0) = 0$
2. $V(\mathbf{x})$ is positive definite
3. $\triangle V(\mathbf{x}) = V(f(\mathbf{x})) - V(\mathbf{x})$ is negative definite
4. $V(\mathbf{x}) \to \infty$ as $|\mathbf{x}| \to \infty$

$$x(k+1) = f(x(k)), \qquad f(0) = 0$$



1. $V(\mathbf{x})$ is continuous in $\mathbf{x}$ and $V(0) = 0$
2. $V(\mathbf{x})$ is positive definite
3. $\triangle V(\mathbf{x}) = V(f(\mathbf{x})) - V(\mathbf{x})$ is negative definite
4. $V(\mathbf{x}) \to \infty$ as $|\mathbf{x}| \to \infty$

- Example 3.6: Lyapunov function

$$\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k)$$

$$V(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x} \qquad \mathbf{P} > 0$$

$$\begin{aligned}\Delta V(\mathbf{x}) &= V(\mathbf{x}(k+1)) - V(\mathbf{x}(k)) \\ &= V(\mathbf{F}\mathbf{x}(k)) - V(\mathbf{x}(k)) \\ &= (\mathbf{F}\mathbf{x}(k))^T\mathbf{P}(\mathbf{F}\mathbf{x}(k)) \quad -\mathbf{x}^T\mathbf{P}\mathbf{x} \\ &= \mathbf{x}^T\mathbf{F}^T\mathbf{P}\mathbf{F}\mathbf{x} \quad -\mathbf{x}^T\mathbf{P}\mathbf{x} \\ &= \mathbf{x}^T(\mathbf{F}^T\mathbf{P}\mathbf{F} - \mathbf{P})\mathbf{x} \quad = \mathbf{x}^T(-\mathbf{Q})\mathbf{x} \quad = -\mathbf{x}^T(\mathbf{Q})\mathbf{x}\end{aligned}$$

V is a Lyapunov function

iff there exists a $P > 0$

that satisfies the *Lyapunov equation*

$$\mathbf{F}^T\mathbf{P}\mathbf{F} - \mathbf{P} = -\mathbf{Q} \qquad \mathbf{Q} > 0$$

- Example 3.6: Lyapunov function for CT case

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$$

$$V(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x} \qquad \mathbf{P} > 0$$

$$\begin{aligned}\dot{V}(\mathbf{x}) &= \mathbf{x}^T\mathbf{P}\dot{\mathbf{x}} + \dot{\mathbf{x}}^T\mathbf{P}\mathbf{x} \\ &= \mathbf{x}^T\mathbf{P}\mathbf{A}\mathbf{x} + (\mathbf{A}\mathbf{x})^T\mathbf{P}\mathbf{x} \\ &= \mathbf{x}^T\mathbf{P}\mathbf{A}\mathbf{x} + \mathbf{x}^T\mathbf{A}^T\mathbf{P}\mathbf{x} \\ &= \mathbf{x}^T(\mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P})\mathbf{x} \\ &= \mathbf{x}^T(-\mathbf{Q})\mathbf{x} \\ &= -\mathbf{x}^T(\mathbf{Q})\mathbf{x}\end{aligned}$$

$$\mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P} = -\mathbf{Q}$$

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- Example 3.6: Lyapunov function

$$\Phi^T P \Phi - P = -Q \qquad Q > 0$$

$$\Phi = \begin{bmatrix} 0.4 & 0 \\ -0.4 & 0.6 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \Rightarrow P = \begin{bmatrix} 1.19 & -0.25 \\ -0.25 & 2.05 \end{bmatrix}$$