Spring 2021

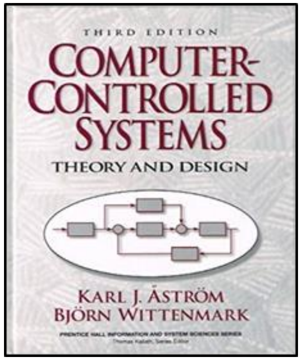
# 數位控制系統
# Digital Control Systems

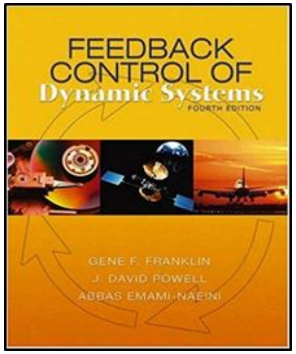
# DCS-14
# Sampling

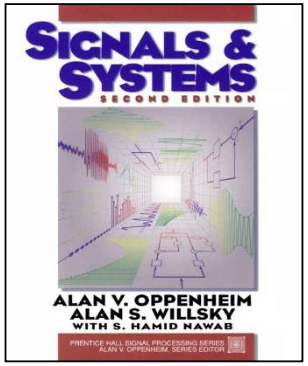

Feng-Li Lian

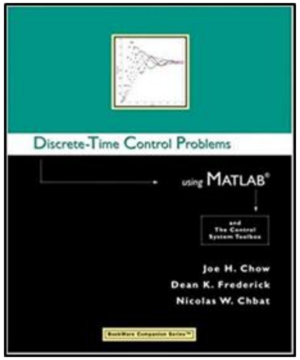
NTU-EE


Feb – Jun, 2021



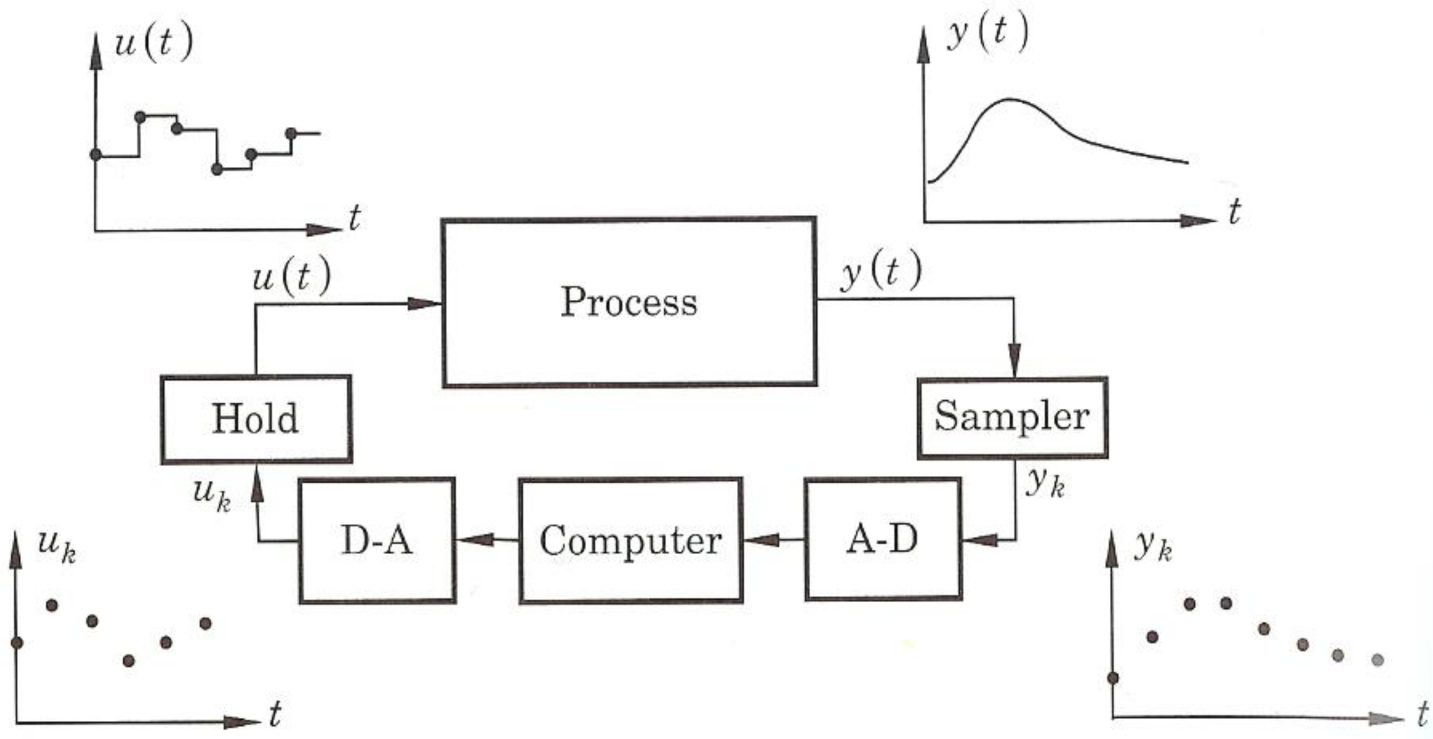



# Outline

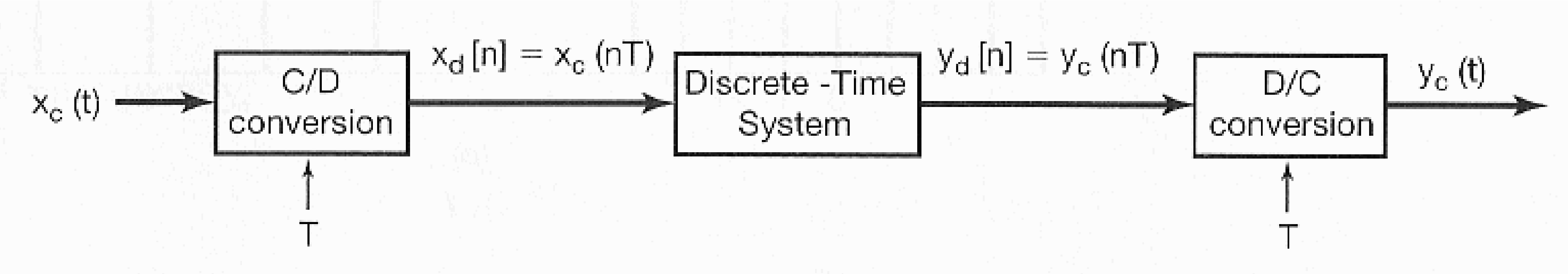
- Representation of a CT Signal by Its Samples:
  The Sampling Theorem
- Reconstruction of a Signal from Its Samples
  Using Interpolation
- The Effect of Under-sampling: Aliasing
- Discrete-Time Processing of Continuous-Time Signals

# Computer-Controlled System (7.2)




1. Wait for a clock pulse
2. Perform A/D conversion
3. Compute control variable
4. Perform D/A conversion
5. Update regulator state
6. Go to step 1

- C/D or A-to-D (ADC) and D/C or D-to-A (DAC):




C/D: continous-to-discrete-time conversion

A-to-D: analog-to-digital converter

D/C: discrete-to-continous-time conversion

D-to-A: digital-to-analog converter

- Representation of CT Signals by its Samples

$x_1(t)$

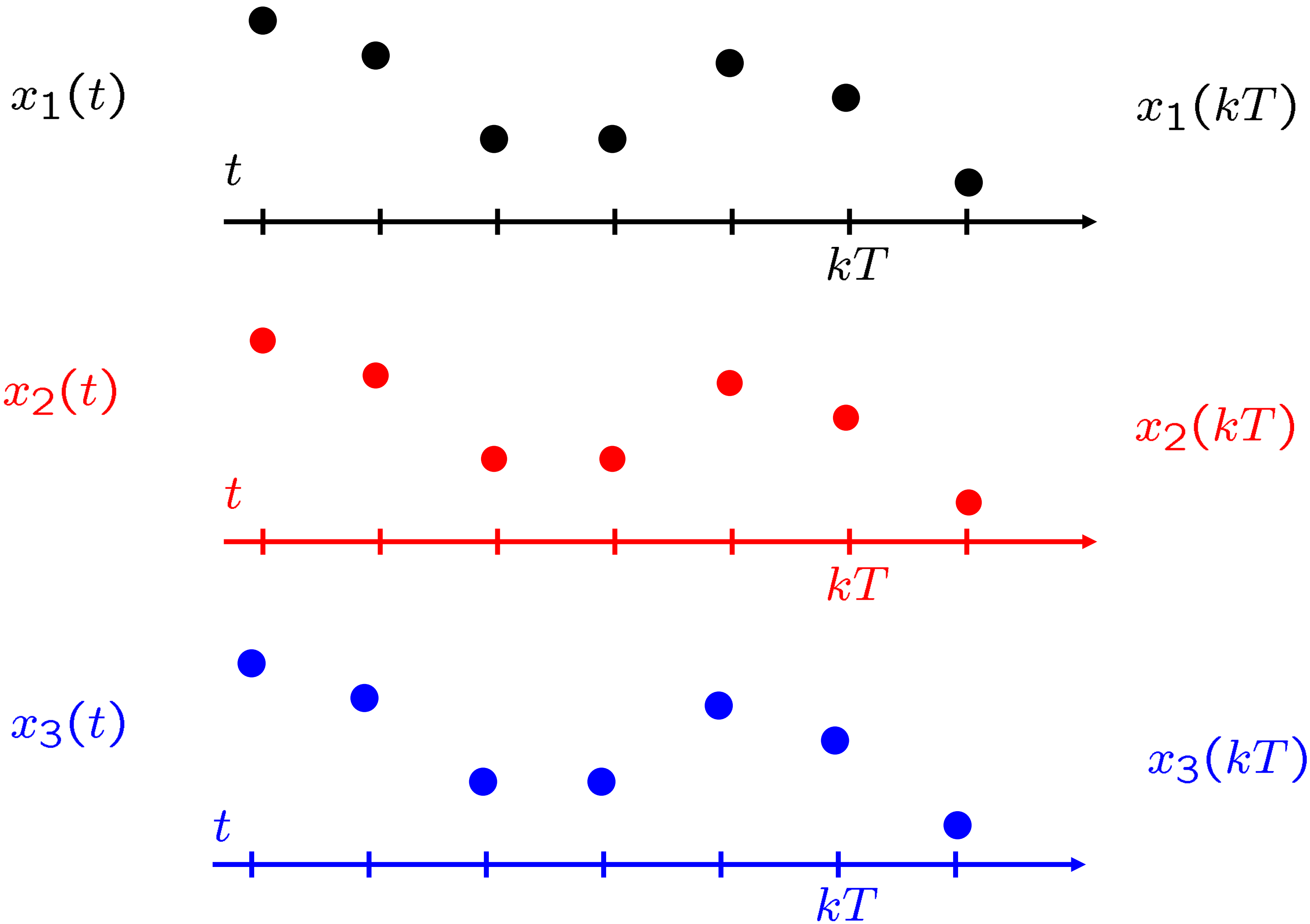
$x_2(t)$

$t$

$x_3(t)$

$t$

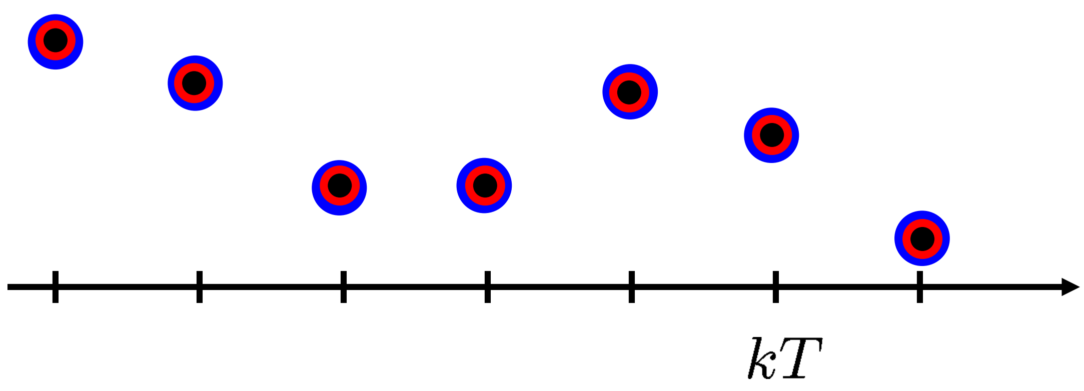
## Representation of CT Signals by its Samples



## Representation of CT Signals by its Samples

$$x_1(t) \neq x_2(t) \neq x_3(t)$$


$$x_1(kT) = x_2(kT) = x_3(kT)$$

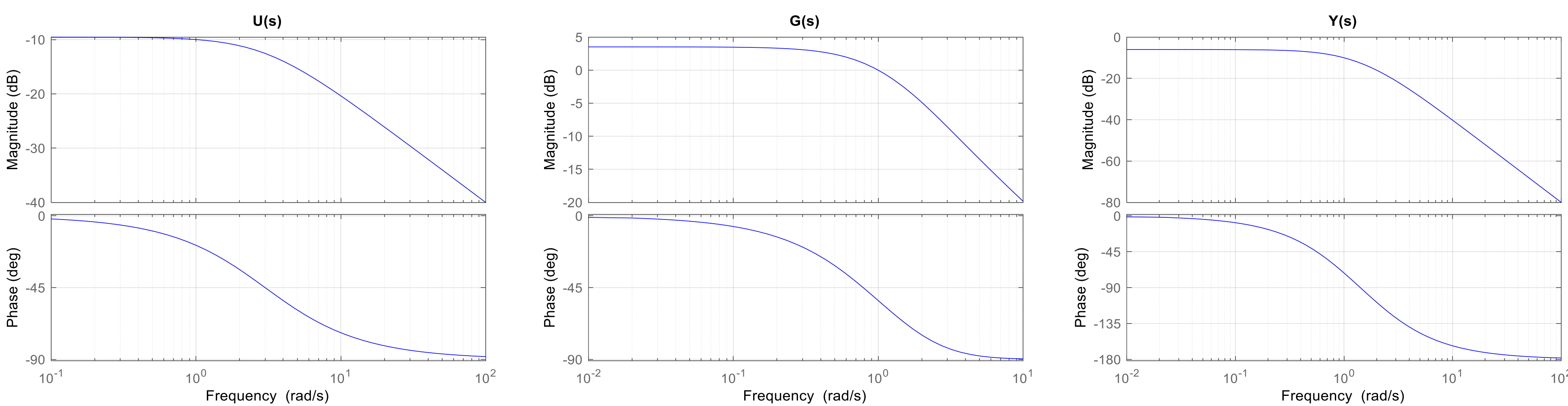


- for $t \geq 0$

$u(t) = e^{-3t}$ → **LTI System** → $y(t) = \left[e^{-t} - e^{-2t}\right]$

$$G(s) = \frac{(s+3)}{(s+1)(s+2)}$$

$$U(s) = \frac{1}{s+3} \qquad Y(s) = G(s)U(s) = \frac{1}{(s+1)(s+2)}$$



# The Sampling Theorem

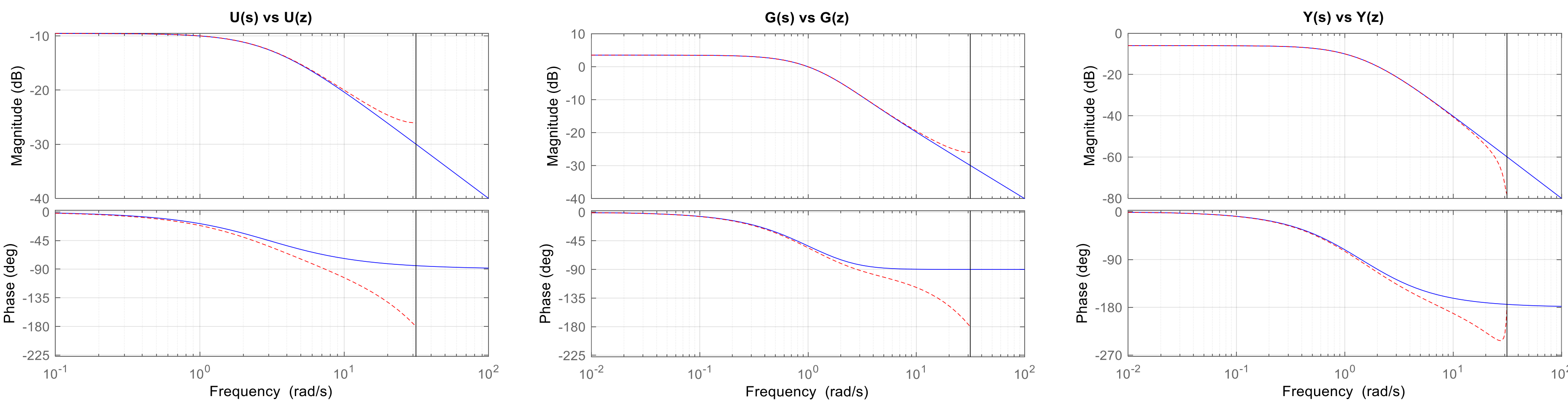


- sample at $h = 0.1(\mathrm{s})$
- $w_s = 2\pi/0.1 = 62.8(\mathrm{rad/s})$

$$u[k] = e^{-3kh} \longrightarrow \boxed{\text{LTI System}} \longrightarrow y[k] = \left[e^{-kh} - e^{-2kh}\right]$$

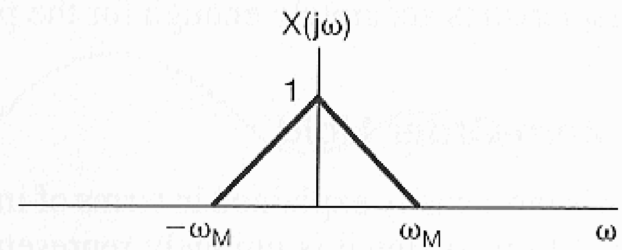
$$G(z) = \frac{0.09969z - 0.07382}{z^2 - 1.724z + 0.7408}$$

$$U(z) = \frac{0.08639}{z - 0.7408} \qquad Y(z) = \frac{0.004528z + 0.004097}{z^2 - 1.724z + 0.7408}$$



- Some Interesting Videos of Sampling Effect
  - Aliasing x1
    - https://www.youtube.com/watch?v=15DC3e4kt-0
  - Aliasing x4
    - https://www.youtube.com/watch?v=z3_TUQp8TaQ
  - 55 Minute Alias
    - http://www.youtube.com/watch?v=xTo3gxsZOWo
  - Airlines Propeller Effect
    - http://www.youtube.com/watch?v=ttvSzoqGQlY
  - Wagon-wheel effect
    - http://www.youtube.com/watch?v=jHS9JGkEOmA&noredirect=1

## ▪ The Sampling Theorem:



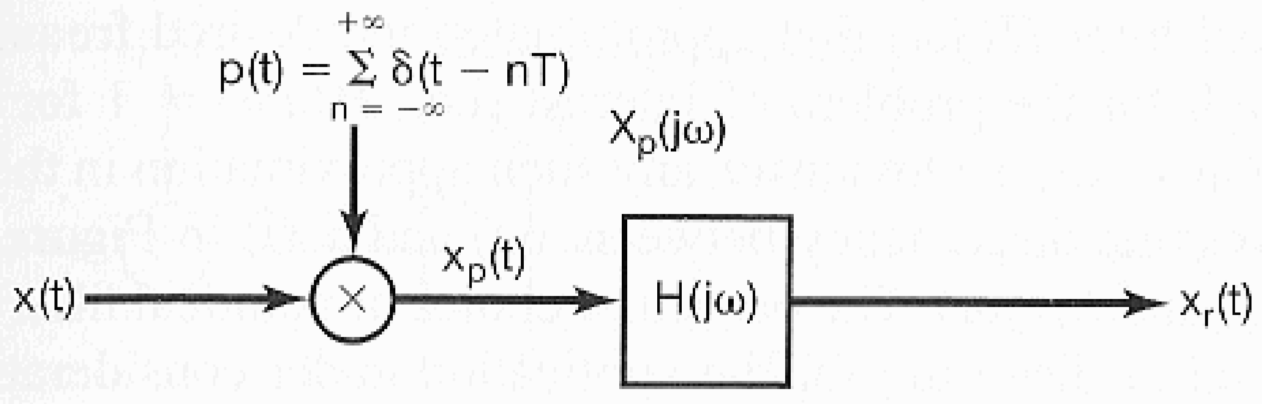
(b)

$x(t)$ : a band-limited signal

with $X(jw) = 0$ for $|w| > w_M$

if $w_s > 2w_M$ where $w_s = \dfrac{2\pi}{T}$

$\Rightarrow$ $x(t)$ is uniquely determined by $x(nT), n = 0, \pm 1, \pm 2, \ldots,$



(a)

$\Rightarrow$ $2w_M$ : Nyquist rate

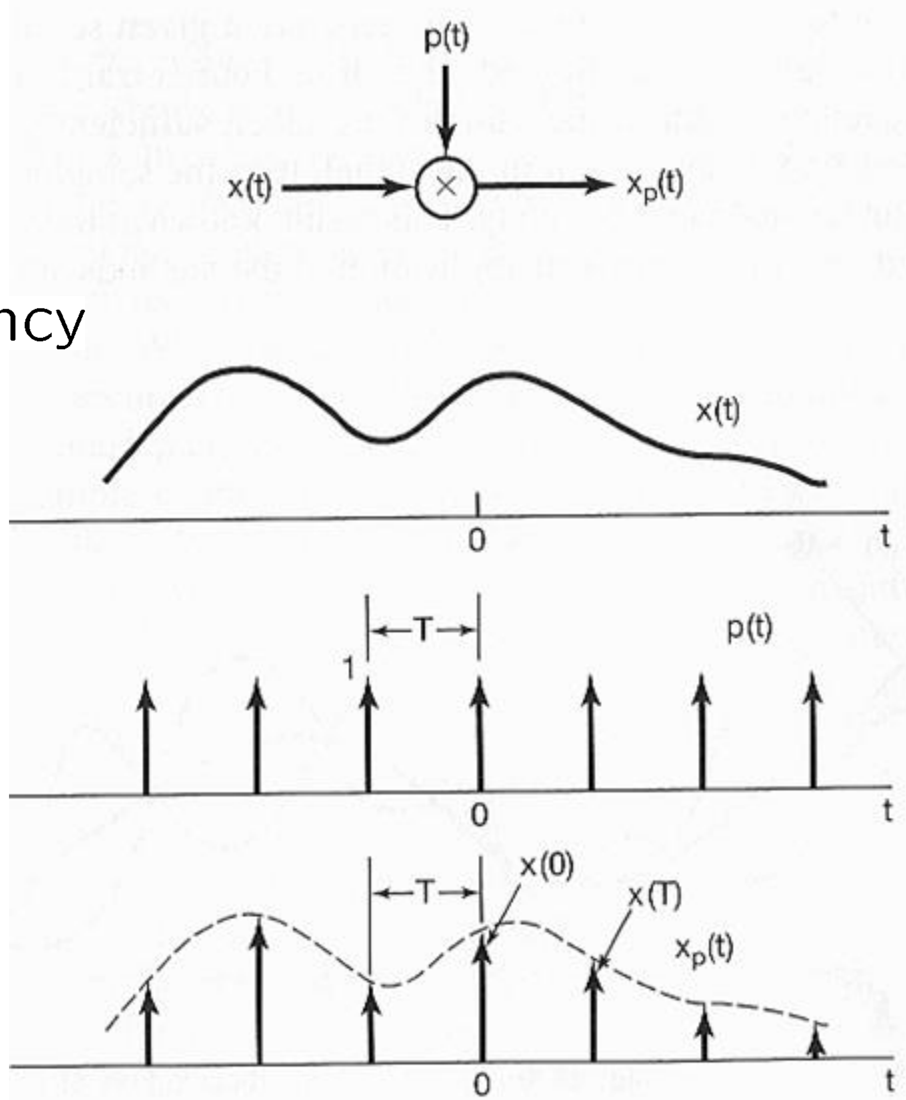
$w_M$ : Nyquist frequency

## Impulse-Train Sampling:

$p(t)$ : sampling function

$T$ : sampling period

$w_s = \frac{2\pi}{T}$ : sampling frequency



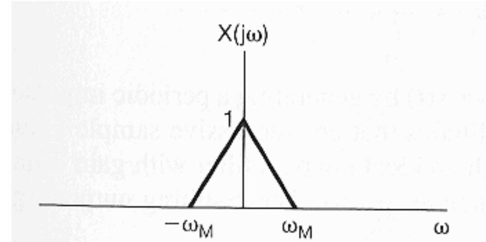

$$\Rightarrow \quad x_p(t) = x(t)\, p(t)$$

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

$$x_p(t) = \sum_{n=-\infty}^{+\infty} x(nT)\delta(t - nT)$$

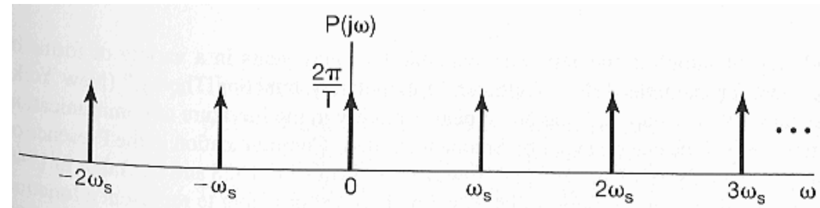
## ▪ Impulse-Train Sampling:

$$x(t) \xleftrightarrow{\mathcal{F}} X(jw)$$



(a)

$$P(jw) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(w - kw_s)$$



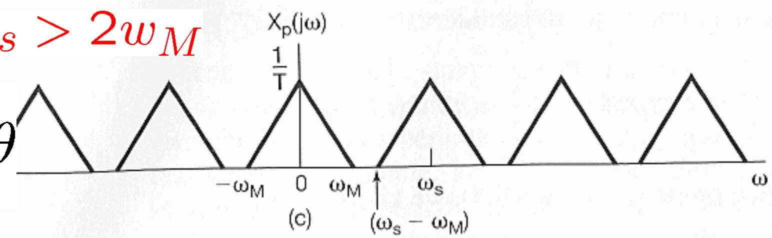
(b)

From multiplication property,

$$X_p(jw) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\theta) P(j(w-\theta)) d\theta$$

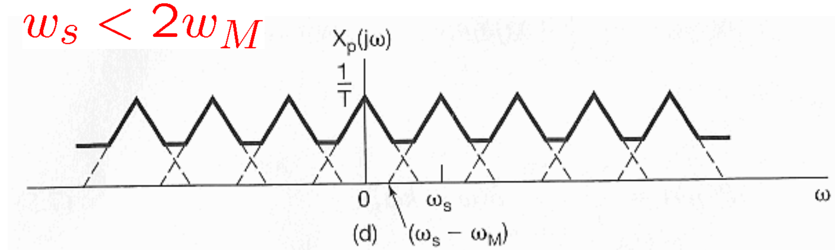
$$= \frac{1}{T} \sum_{k=-\infty}^{+\infty} X(j(w - kw_s))$$

$w_s > 2w_M$



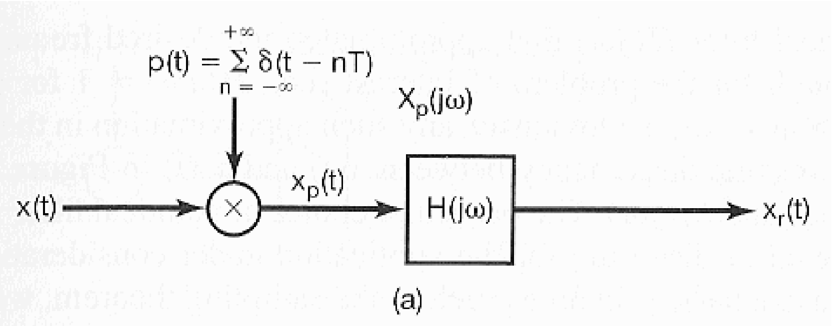
(c)

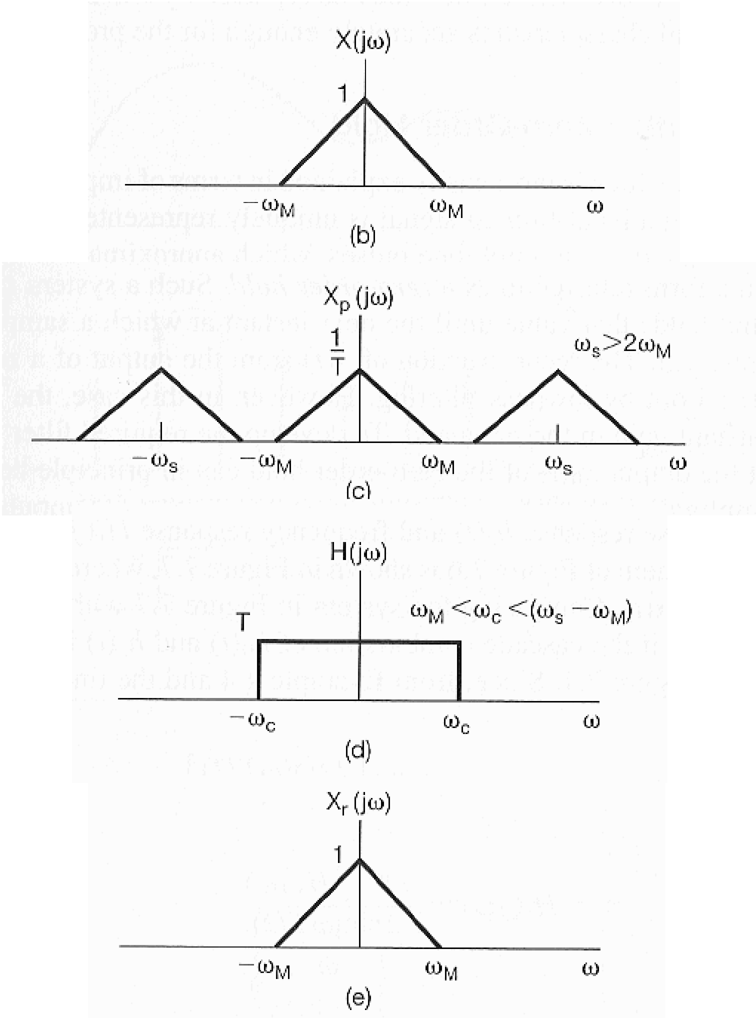
$w_s < 2w_M$



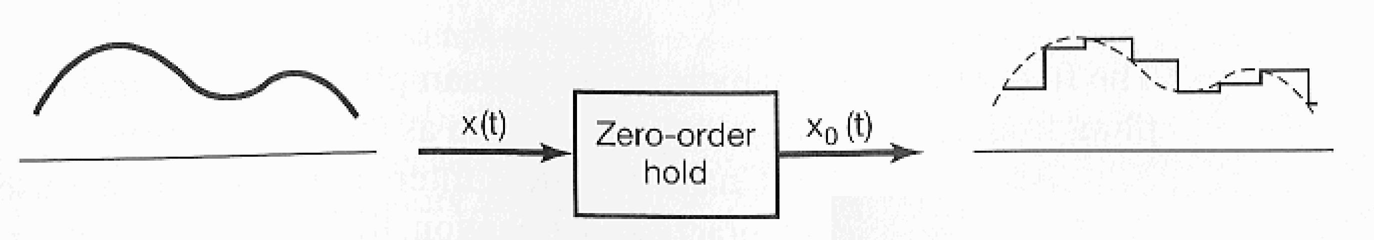
(d)

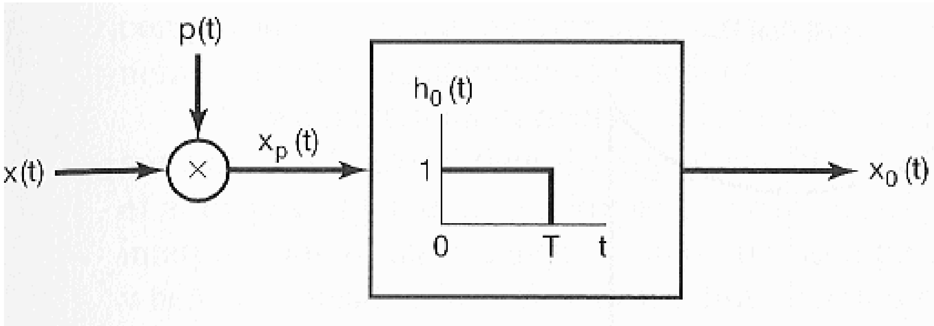
- Exact Recovery by an Ideal Lowpass Filter:

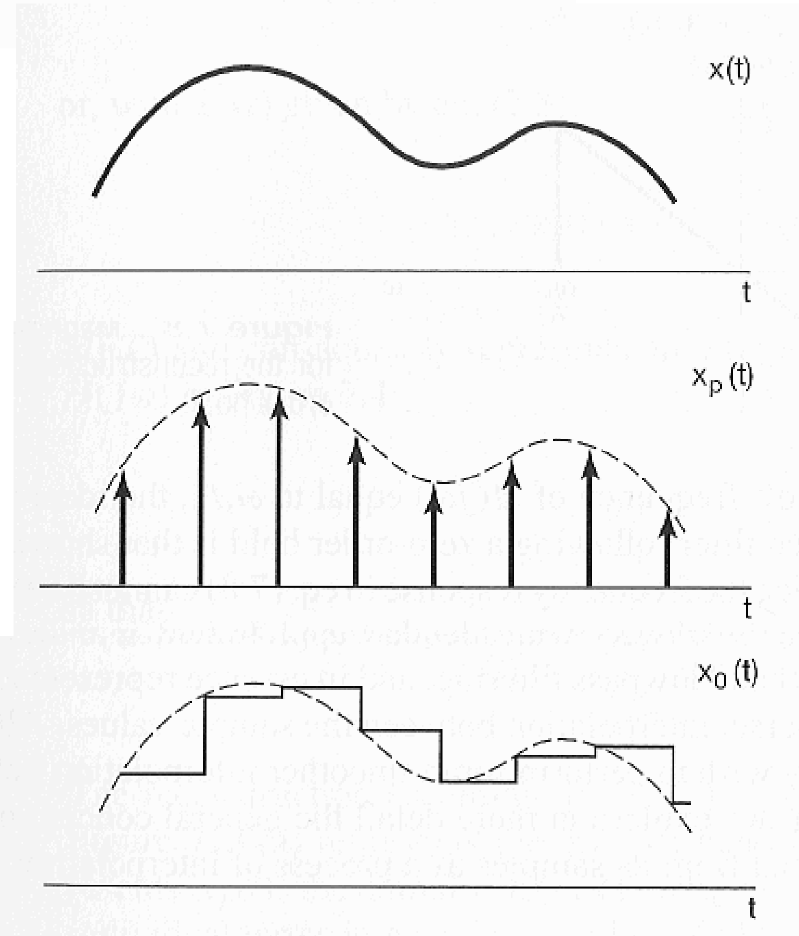


- Sampling with Zero-Order Hold:



# Outline



- Representation of of a CT Signal by Its Samples: The Sampling Theorem
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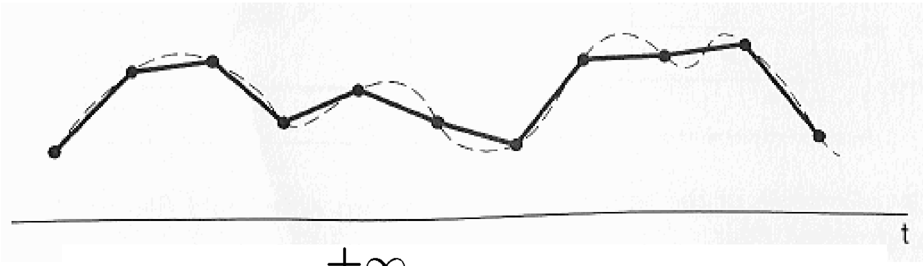
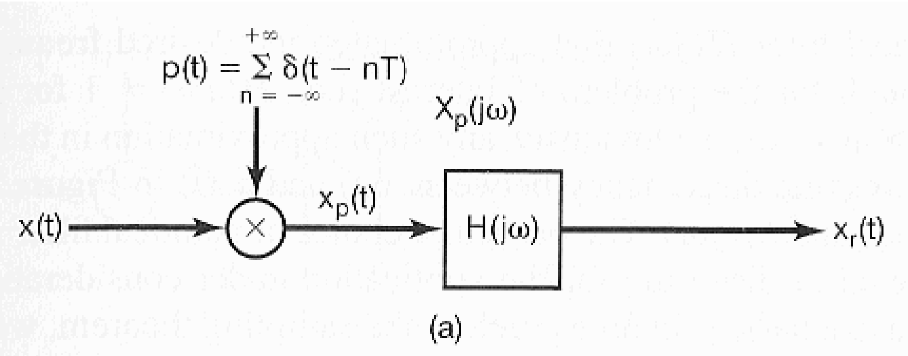
- Exact Interpolation:



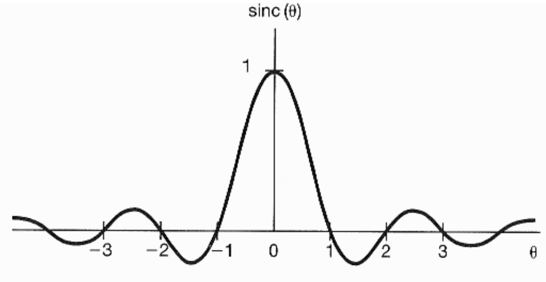




$$x_p(t) = \sum_{n=-\infty}^{+\infty} x(nT)\delta(t - nT)$$

ideal lowpass filter

$$x_r(t) = x_p(t) * h(t)$$

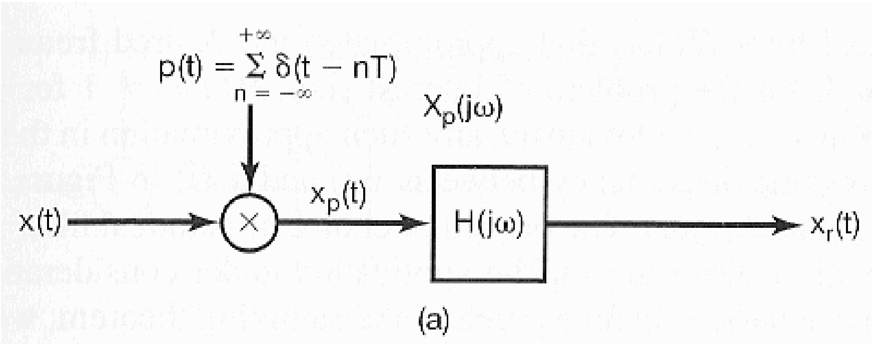
$$h(t) = \frac{w_c T \sin(w_c t)}{\pi w_c t}$$

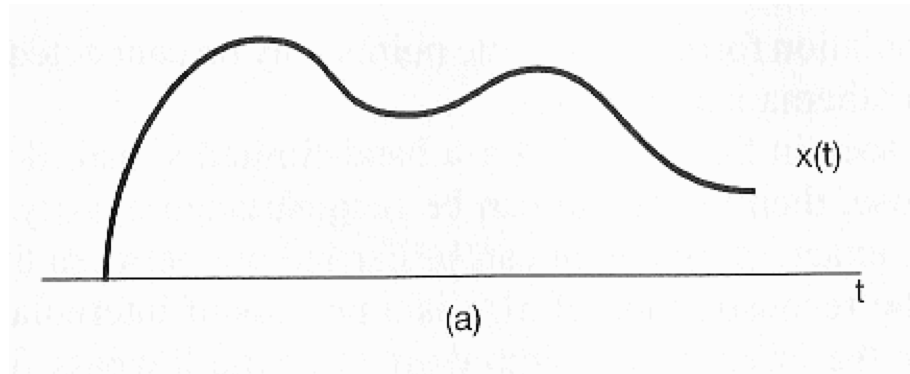



$$x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT)h(t - nT)$$

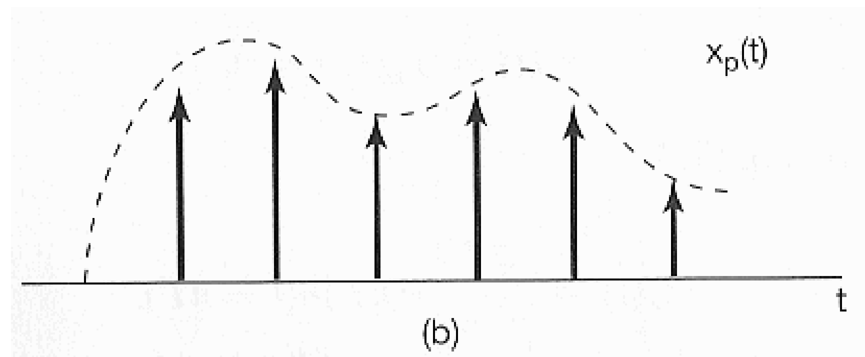
$$x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT)\frac{w_c T}{\pi}\frac{\sin(w_c(t - nT))}{w_c(t - nT)}$$

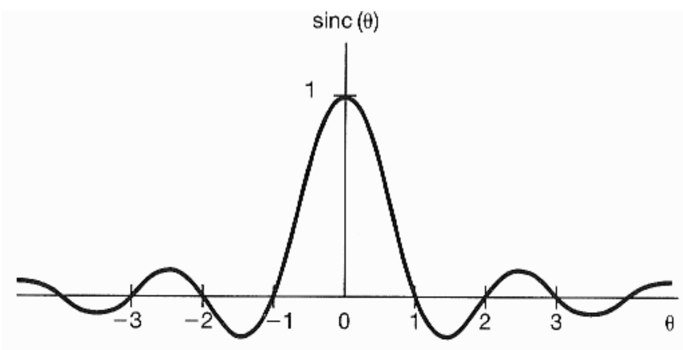
- Exact Interpolation:

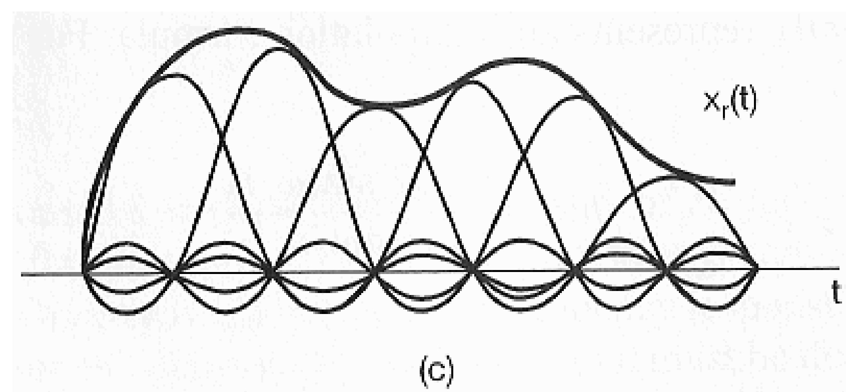



$$x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT) \frac{w_c T}{\pi} \frac{\sin(w_c(t - nT))}{w_c(t - nT)}$$

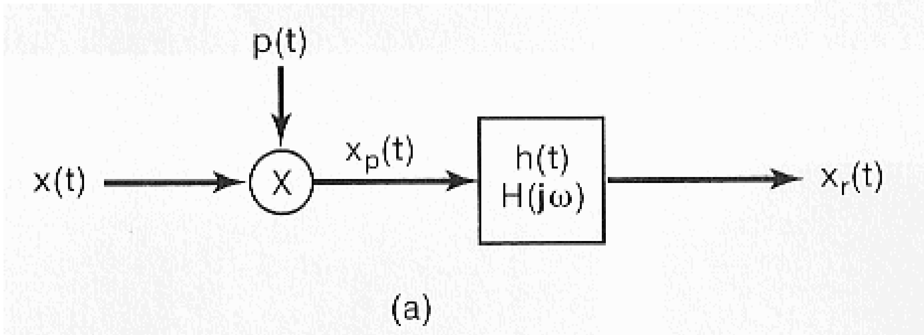


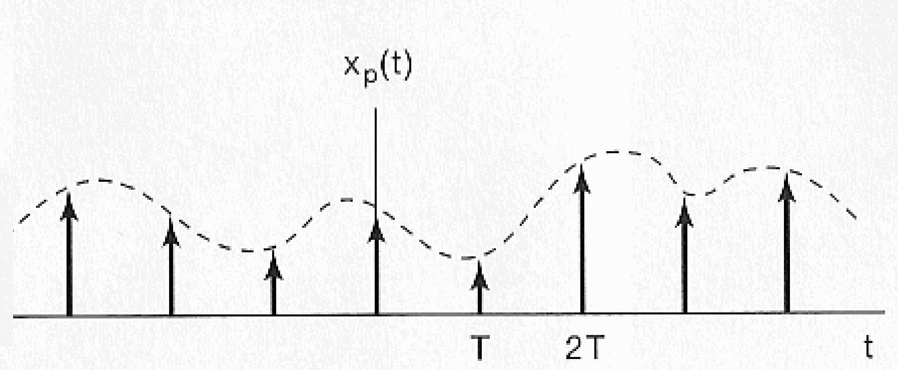



# Reconstruction of a Signal from its Samples Using Interpolation

## ▪ Higher-Order Holds:

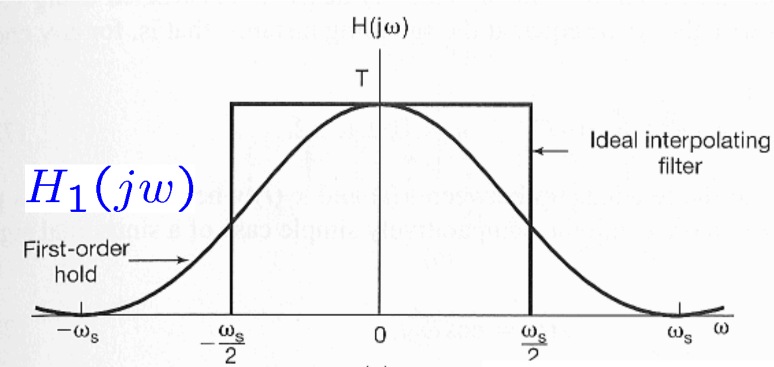



(a)

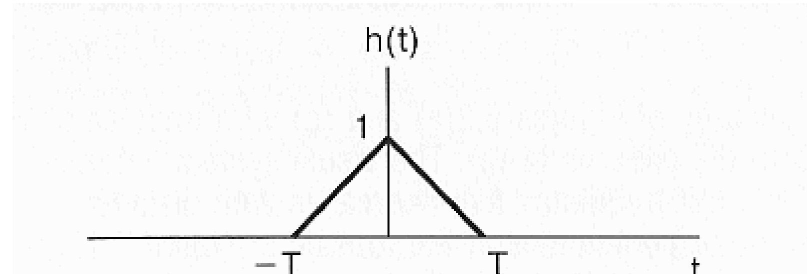



(b)

$H_1(jw)$



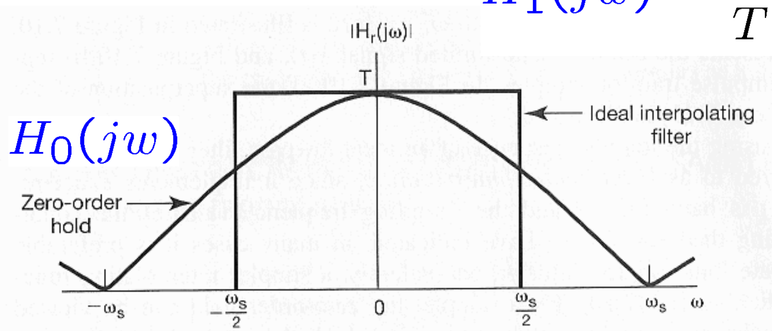

(e)

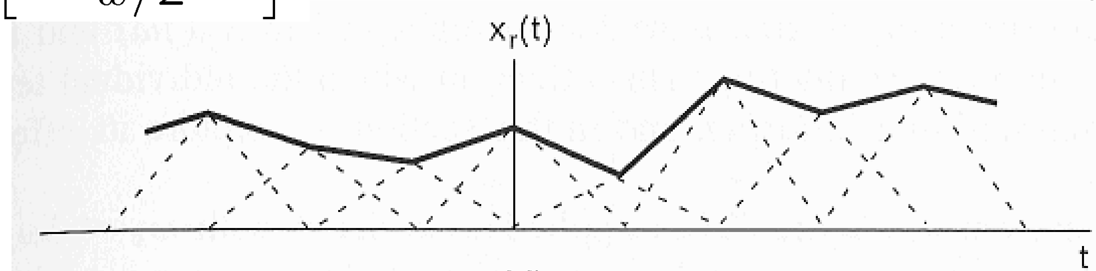



(c)

$$H_1(jw) = \frac{1}{T}\left[\frac{\sin(wT/2)}{w/2}\right]^2$$

$H_0(jw)$




(d)

# Outline



- Representation of of a CT Signal by Its Samples: The Sampling Theorem
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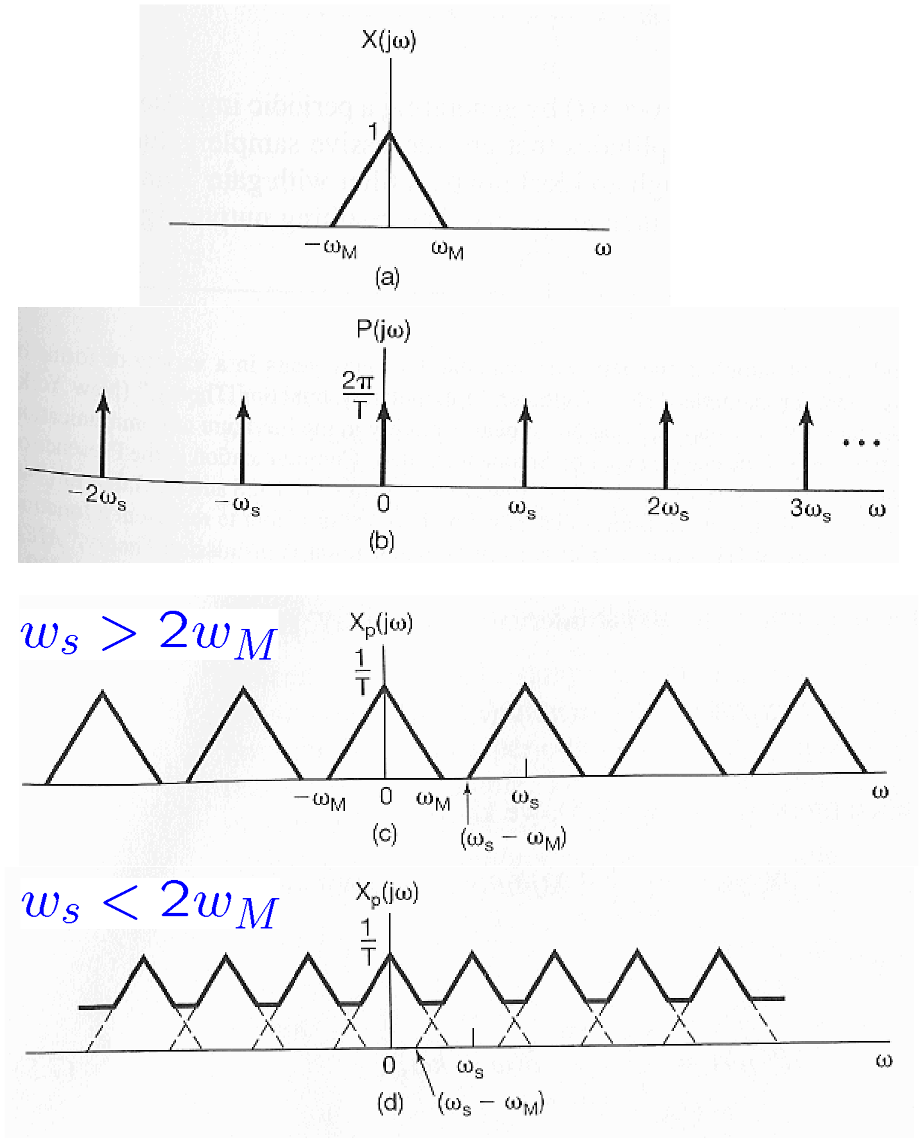
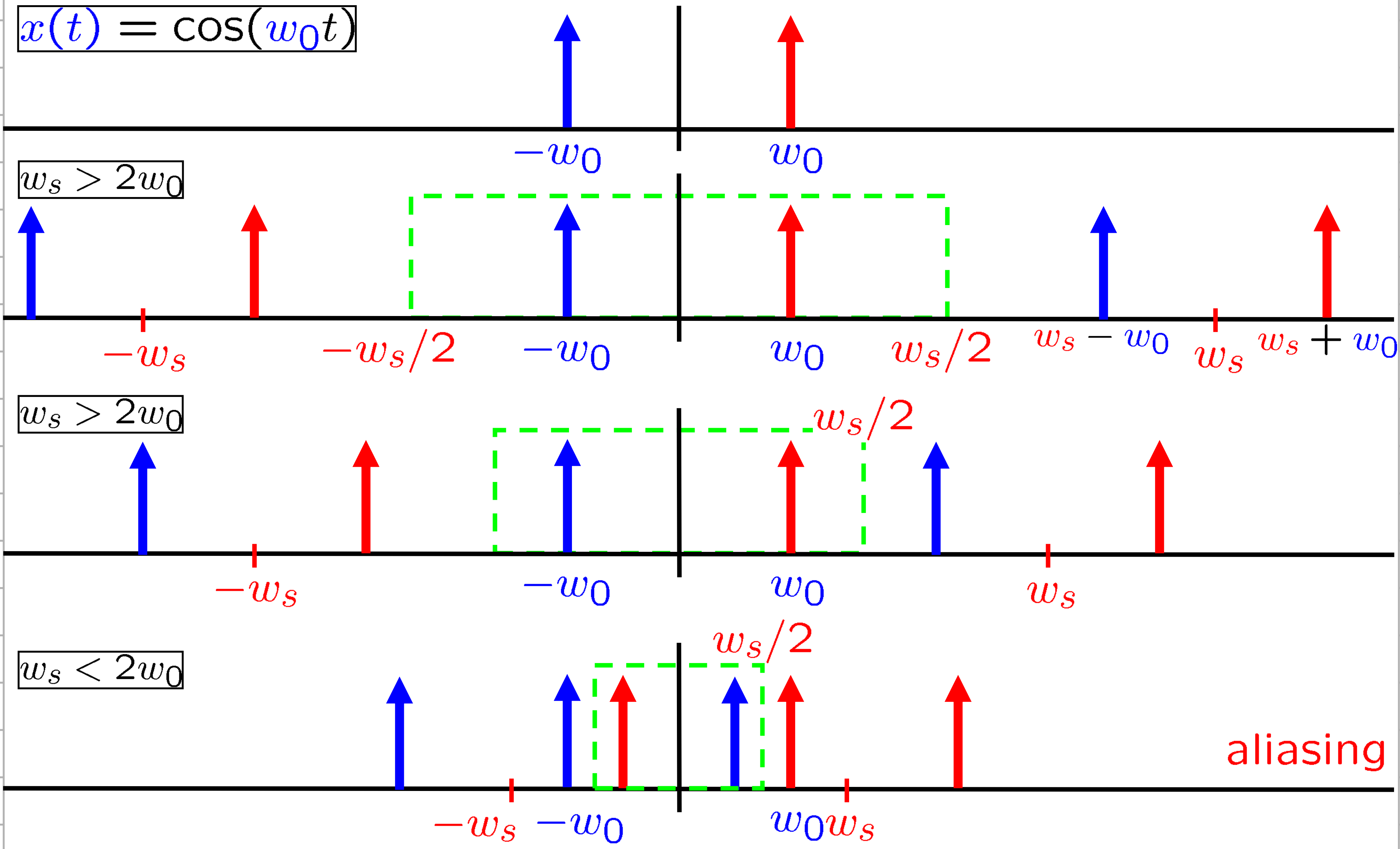
- Overlapping in Frequency-Domain: Aliasing



# Effect of Under-sampling: Aliasing

- Overlapping in Frequency-Domain: Aliasing



# Effect of Under-sampling: Aliasing

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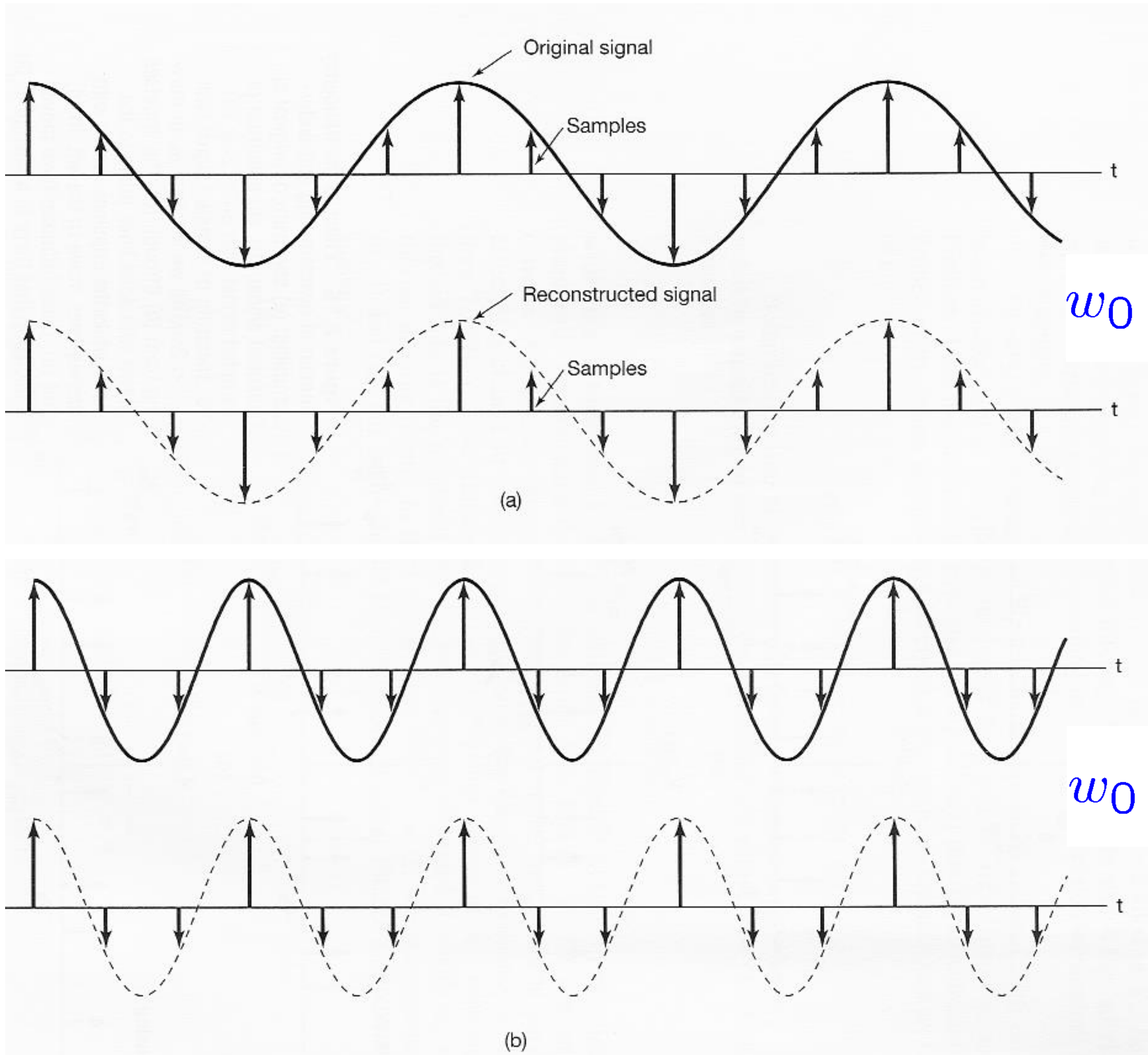
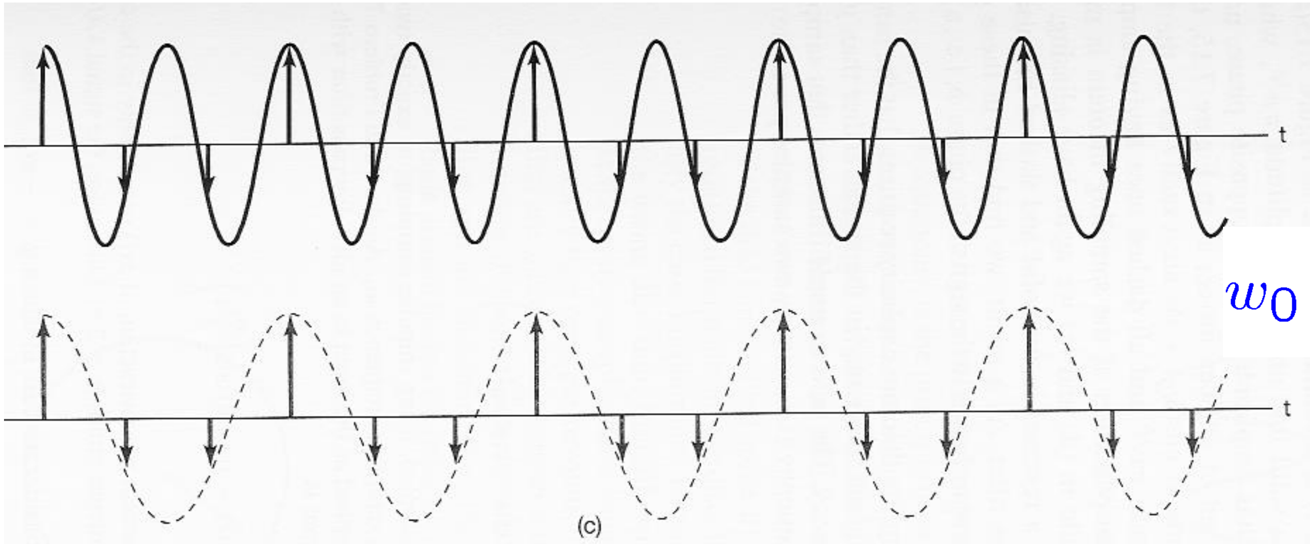



$$w_0 = \frac{w_s}{6}$$

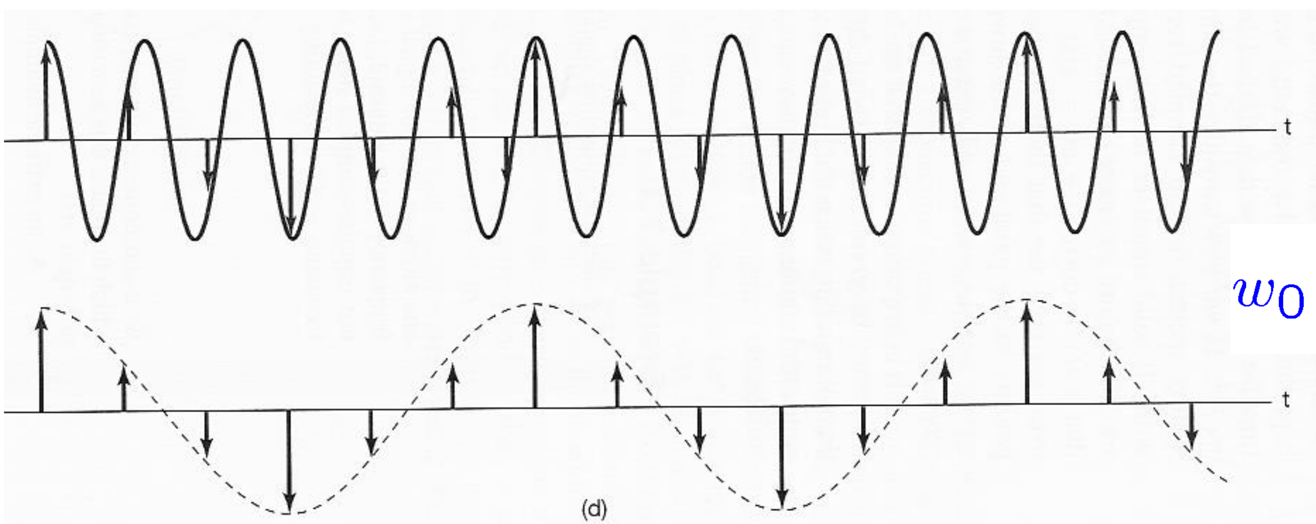
$$w_0 = \frac{2w_s}{6}$$

- Overlapping in Frequency-Domain: Aliasing




(c)

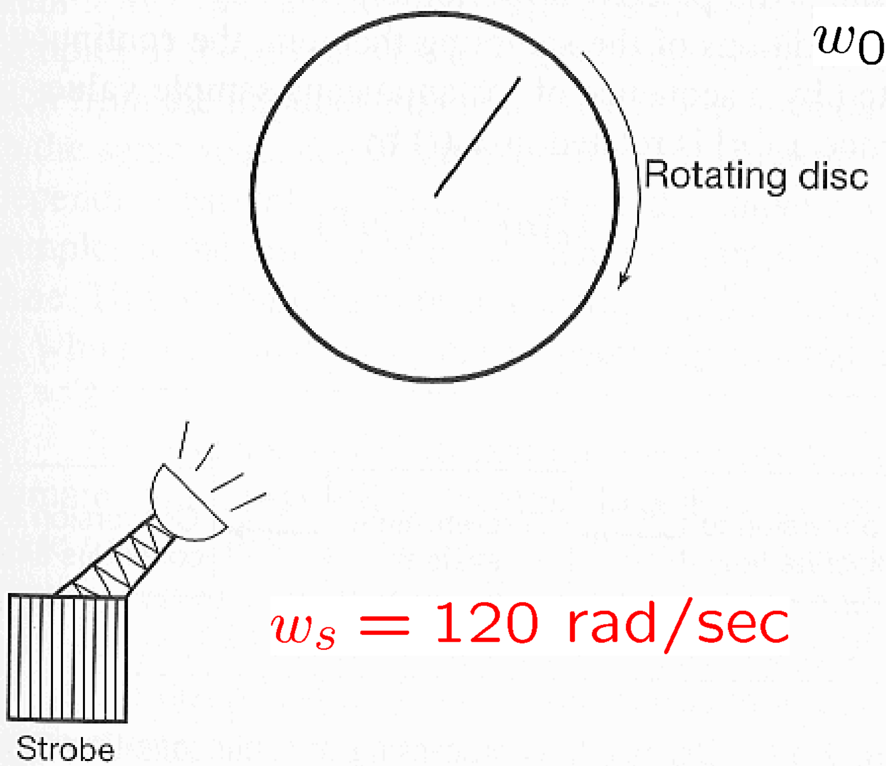
$$w_0 = \frac{4w_s}{6}$$




(d)

$$w_0 = \frac{5w_s}{6}$$

- Strobe Effect:

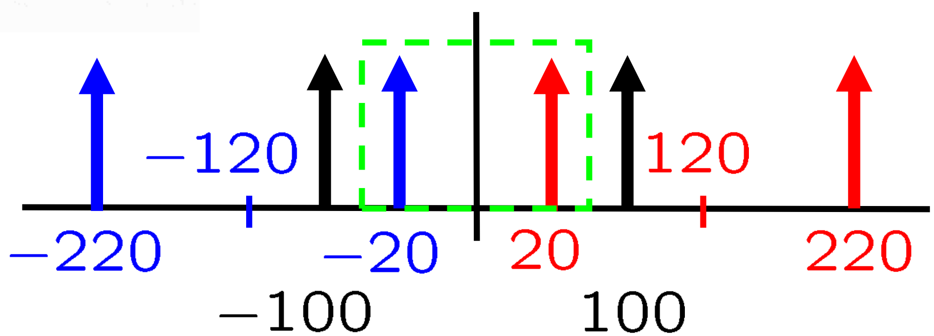



$w_0 = 100$ rad/sec

$w_s = 120$ rad/sec

$$\Rightarrow w = \pm w_s \pm w_0$$

$$= +20, -20$$



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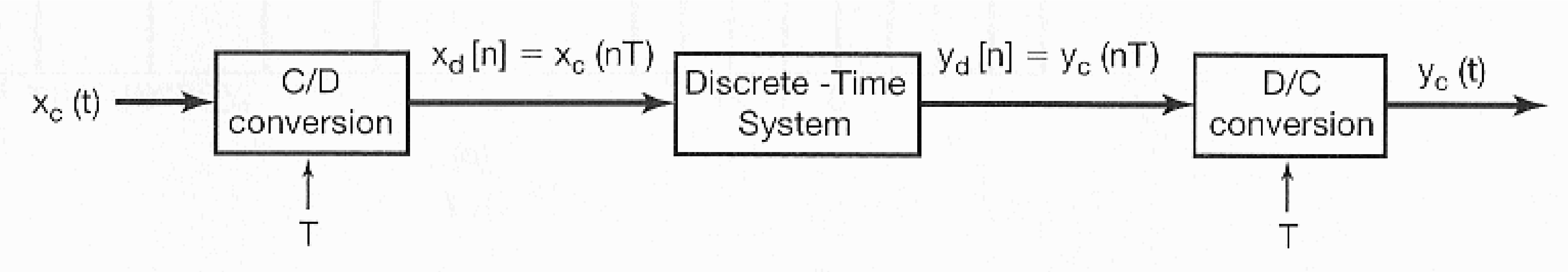


C/D: continous-to-discrete-time conversion

A-to-D: analog-to-digital converter

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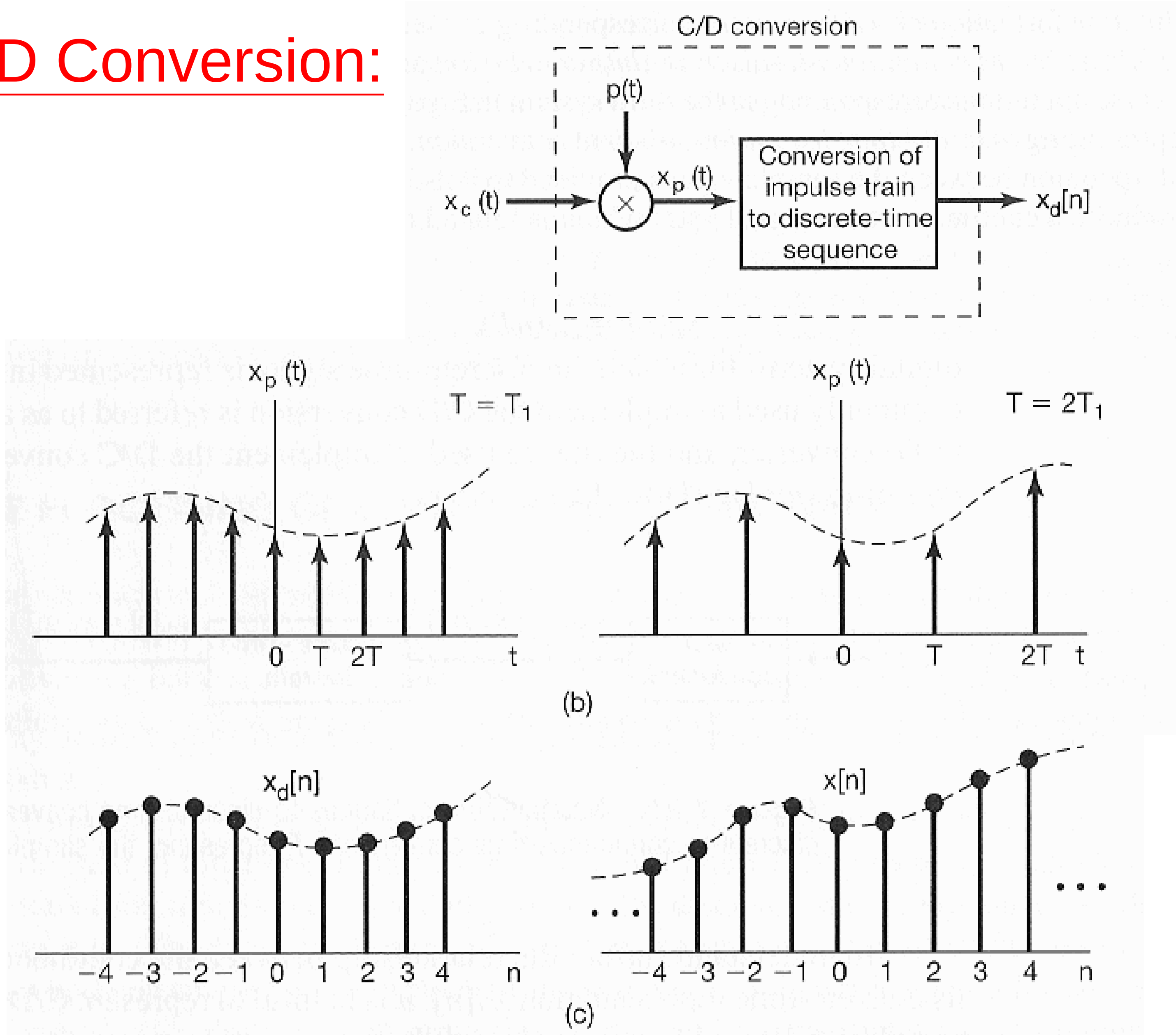
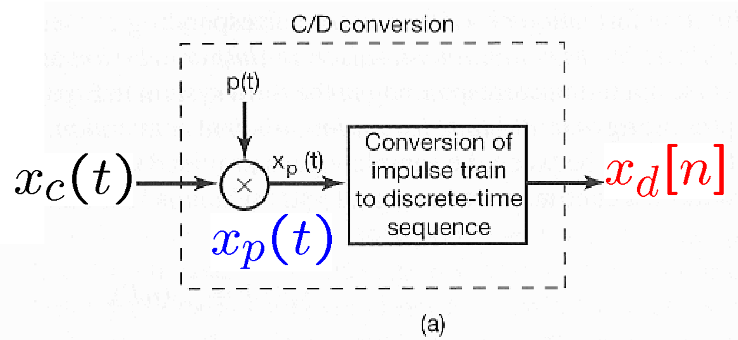
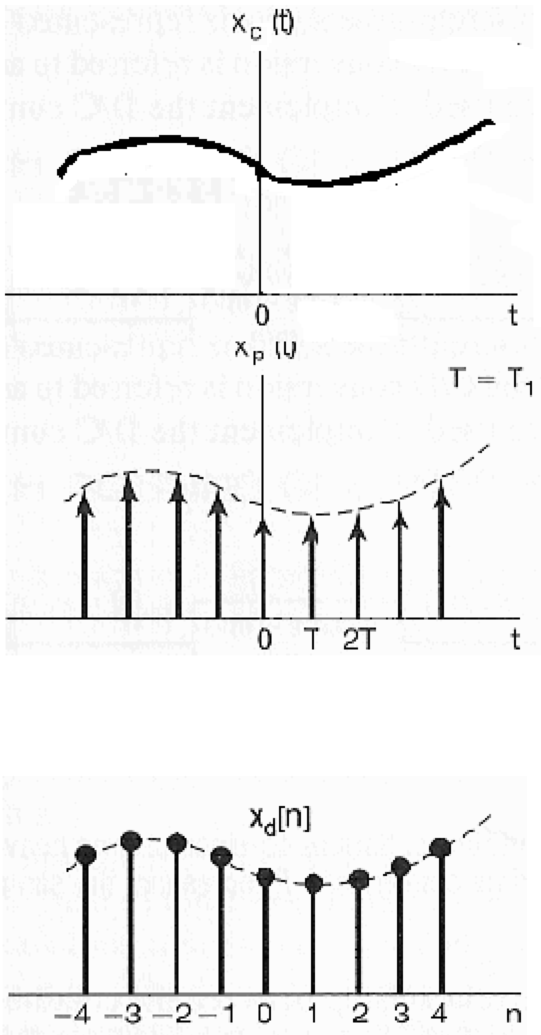
D-to-A: digital-to-analog converter

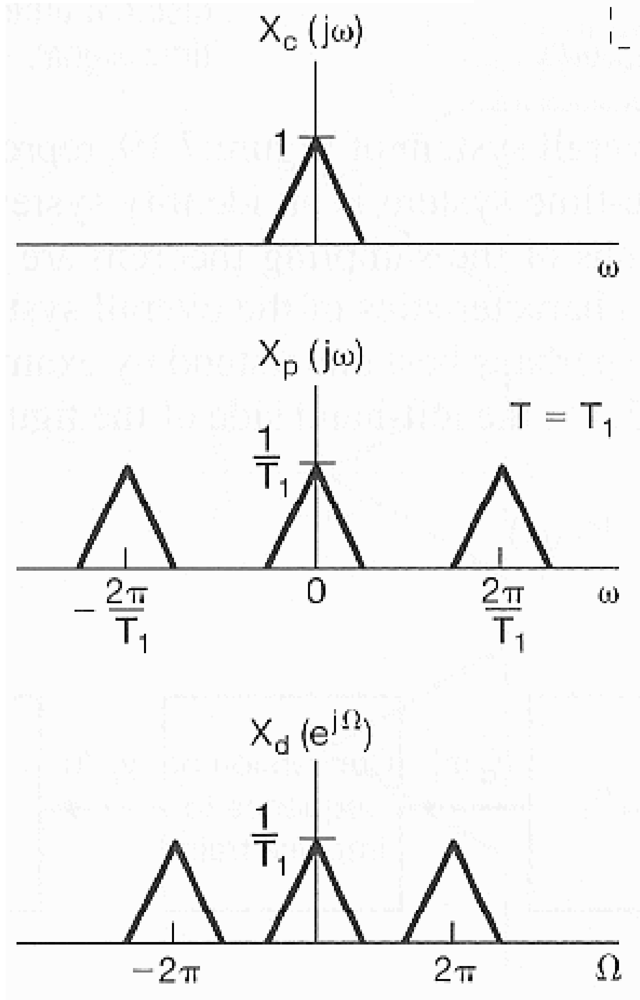
- C/D Conversion:



- C/D Conversion:



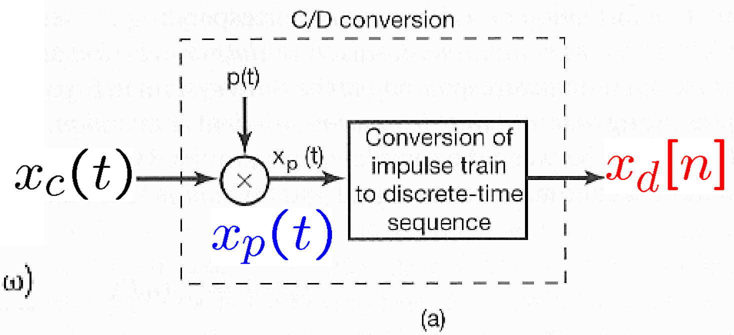



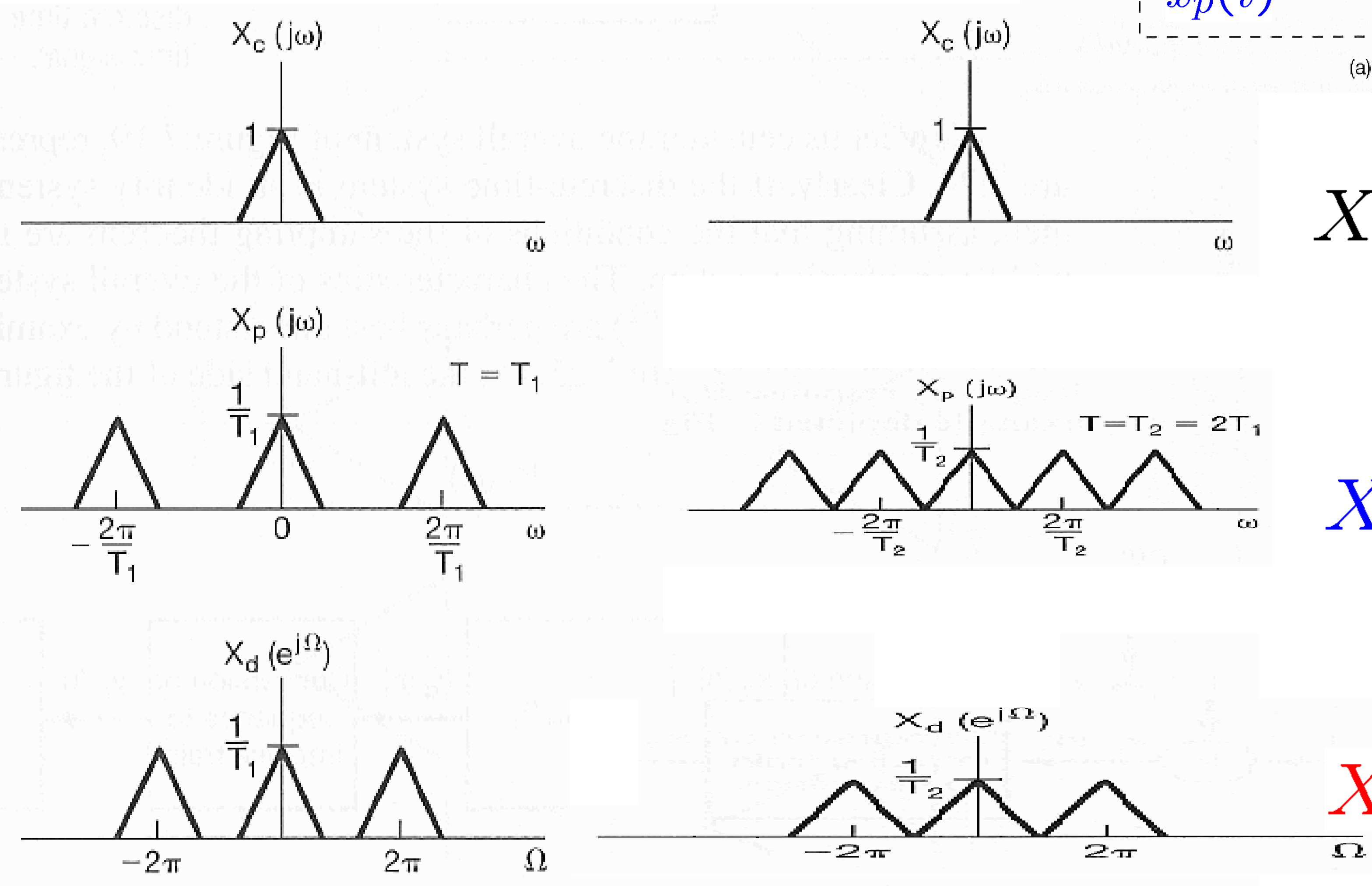

$$X_c(jw)$$

$$X_p(jw)$$

$$X_d(e^{j\Omega})$$

- C/D Conversion:



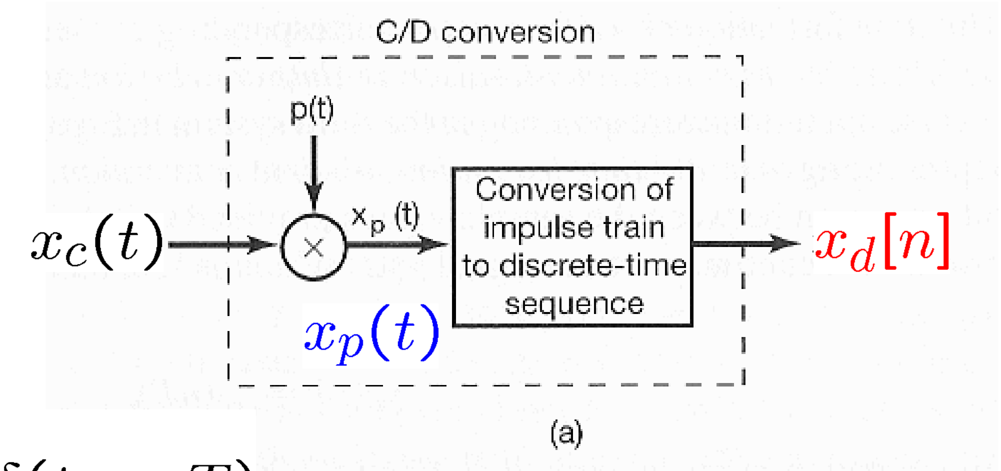




$X_c(jw)$

$X_p(jw)$

$X_d(e^{j\Omega})$

- C/D Conversion:

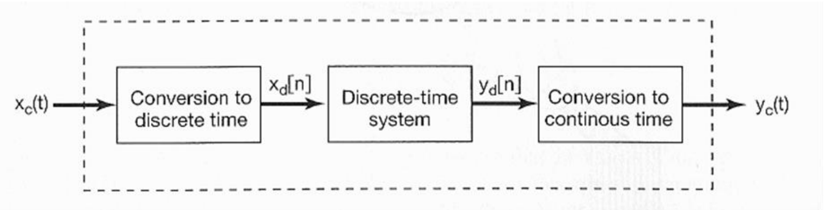



$$x_p(t) = \sum_{n=-\infty}^{+\infty} x_c(nT)\delta(t-nT)$$

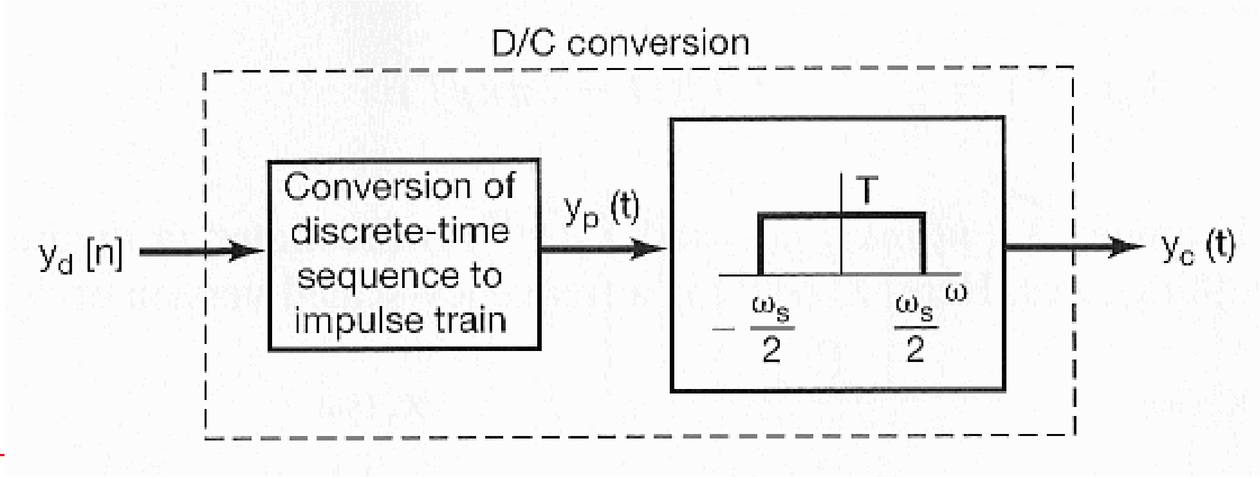
$$X_p(jw) = \sum_{n=-\infty}^{+\infty} x_c(nT)e^{-jwnT} = \frac{1}{T}\sum_{K=-\infty}^{+\infty} X_c(j(w-kw_s))$$

$$X_d(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} x_d[n]e^{-j\Omega n} = \sum_{n=-\infty}^{+\infty} x_c(nT)e^{-j\Omega n}$$

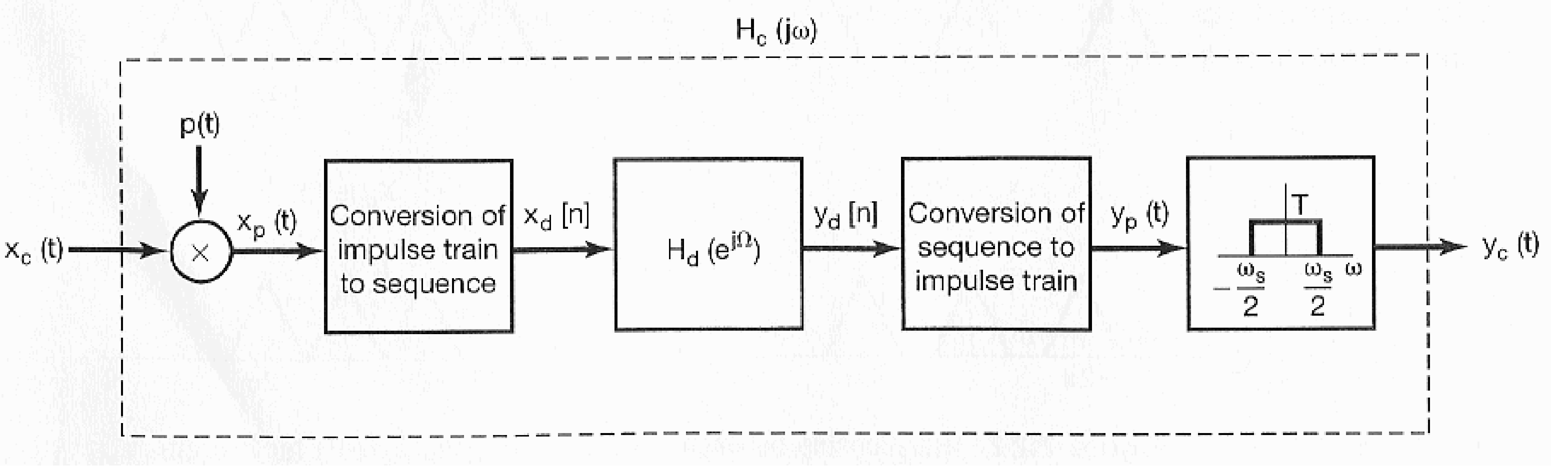
$$\Rightarrow X_d(e^{j\Omega}) = X_p\left(j\frac{\Omega}{T}\right) = \frac{1}{T}\sum_{K=-\infty}^{+\infty} X_c\left(j\left(\frac{\Omega}{T} - k\frac{2\pi}{T}\right)\right)$$

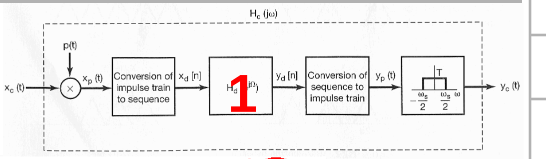



- D/C Conversion:




- Overall





- Frequency-Domain Illustration:

$X_c(jw)$

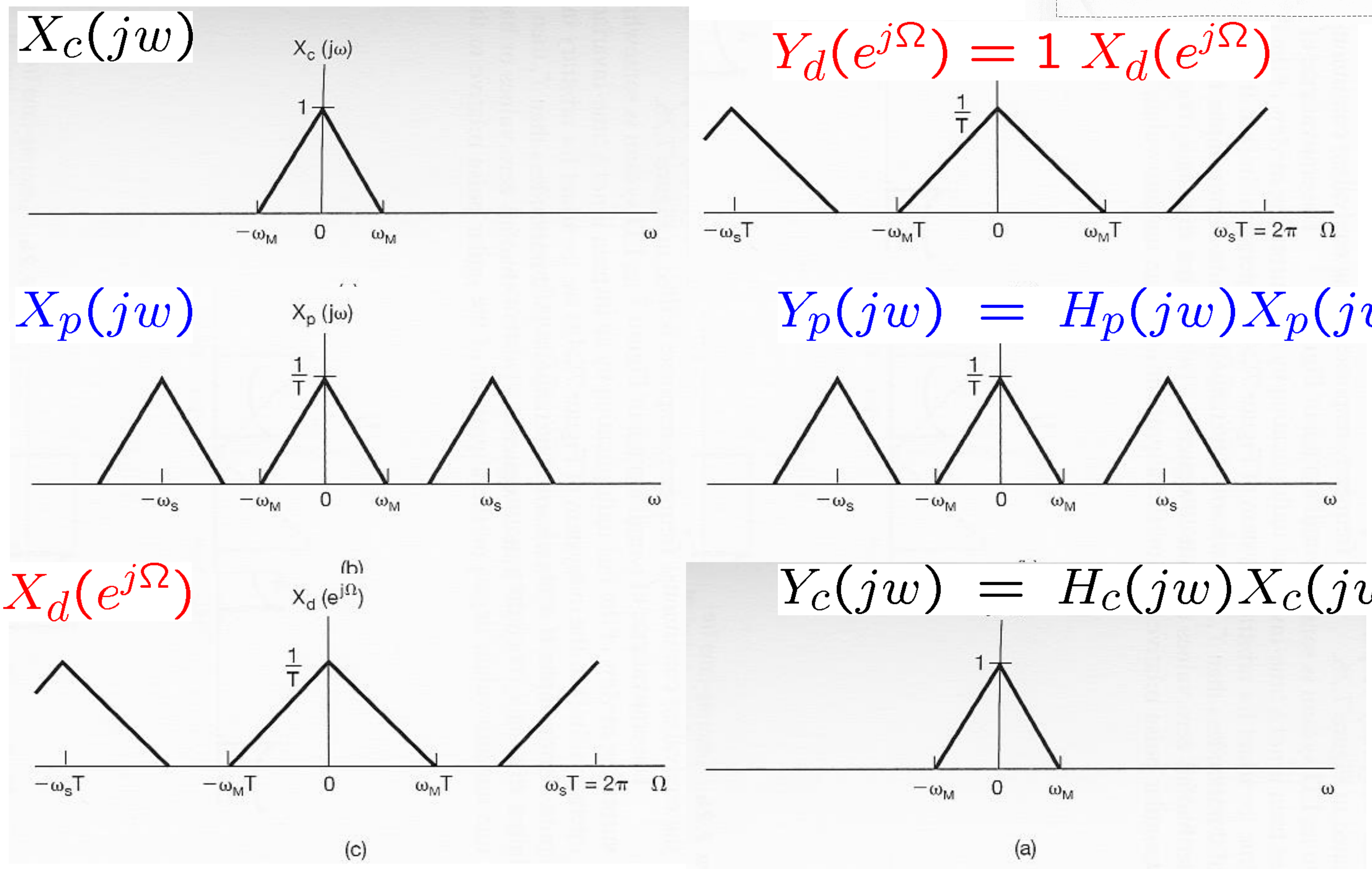
$X_p(jw)$

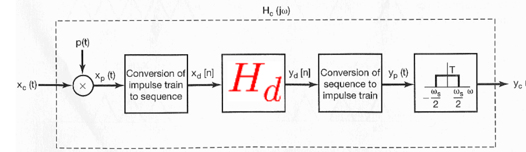
$X_d(e^{j\Omega})$

$Y_d(e^{j\Omega}) = 1\ X_d(e^{j\Omega})$

$Y_p(jw) = H_p(jw) X_p(jw)$

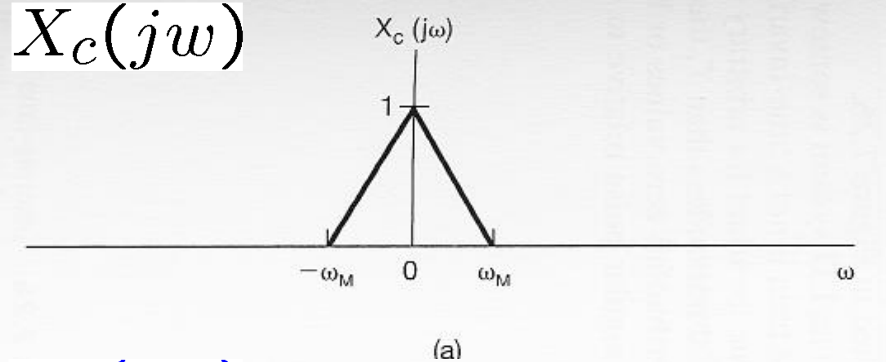
$Y_c(jw) = H_c(jw) X_c(jw)$



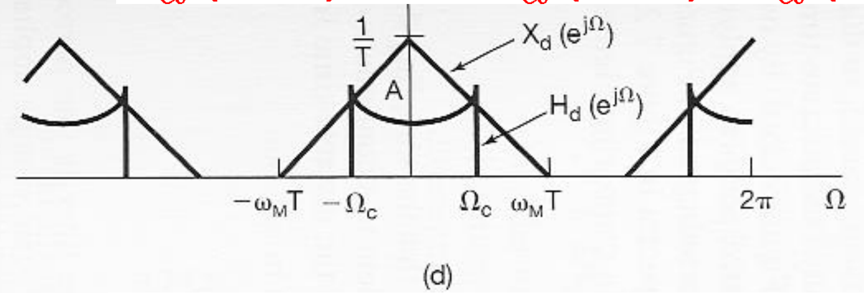



## ▪ Frequency-Domain Illustration:

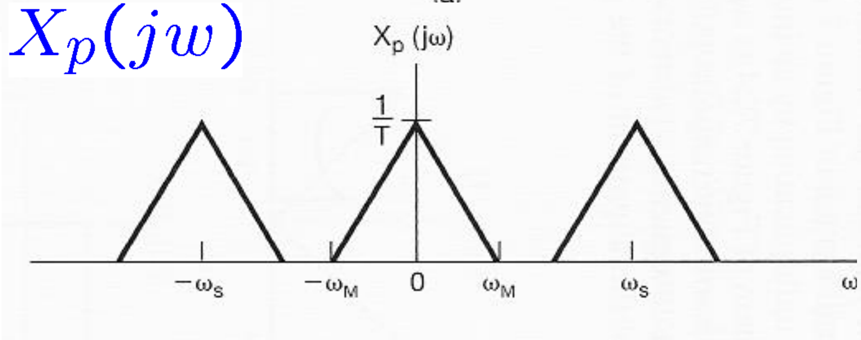
$X_c(jw)$



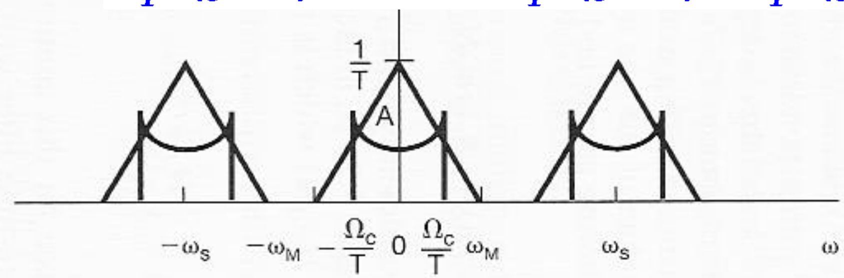

$Y_d(e^{j\Omega}) = H_d(e^{j\Omega})X_d(e^{j\Omega})$



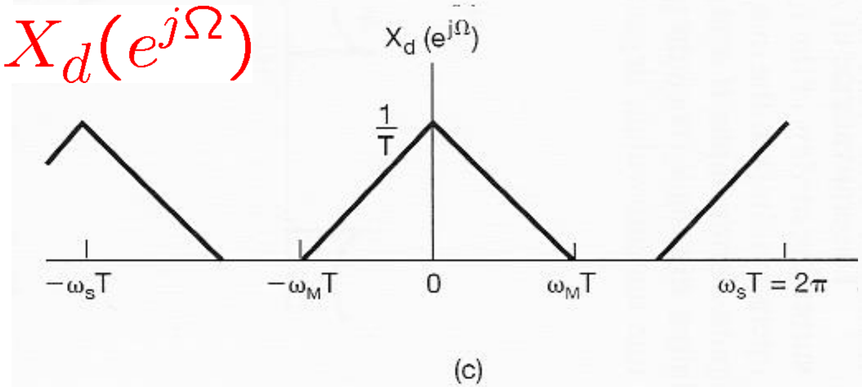

$X_p(jw)$



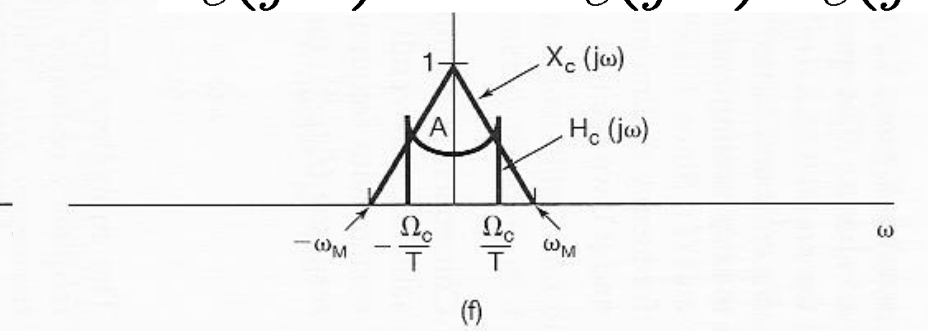

$Y_p(jw) = H_p(jw)X_p(jw)$



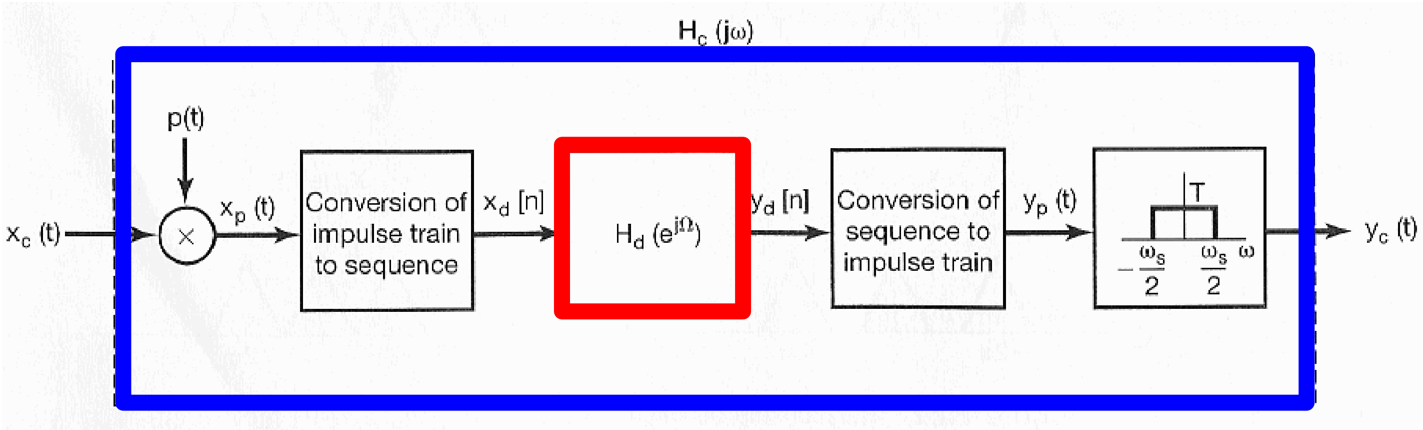

$X_d(e^{j\Omega})$




$Y_c(jw) = H_c(jw)X_c(jw)$

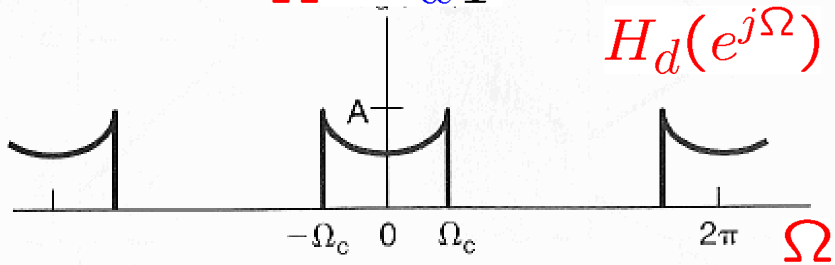


## CT & DT Frequency Responses:

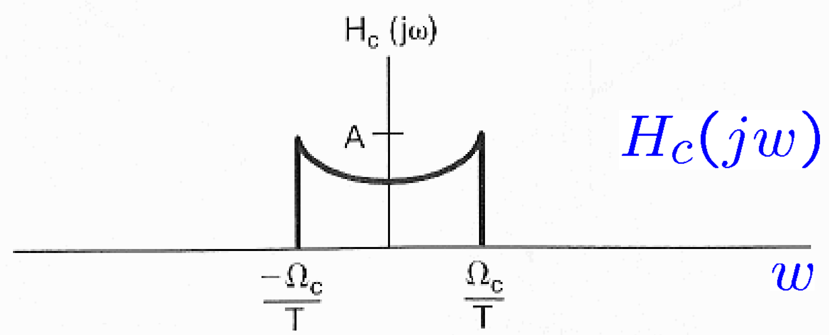



$\Omega = wT$

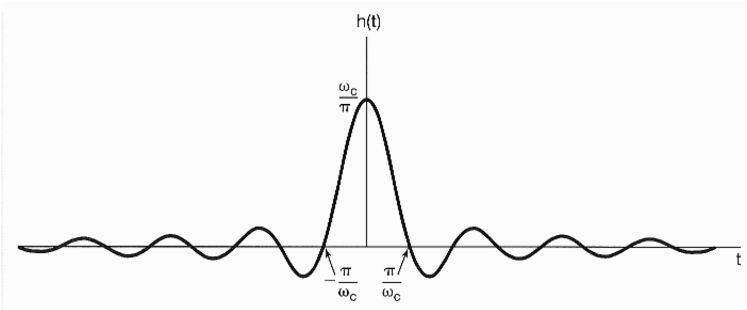
$$Y_c(jw) = X_c(jw) H_d(e^{jwT})$$

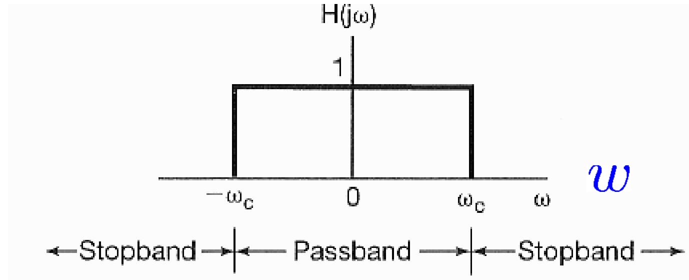



$$H_c(jw) = \begin{cases} H_d(e^{jwT}), & |w| < w_s/2 \\ 0, & |w| > w_s/2 \end{cases}$$



# In Summary



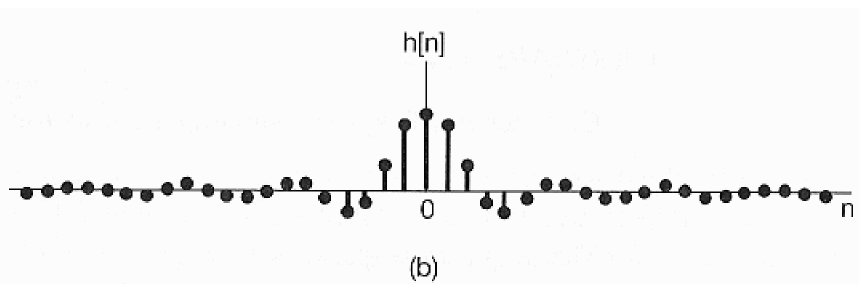

$w$

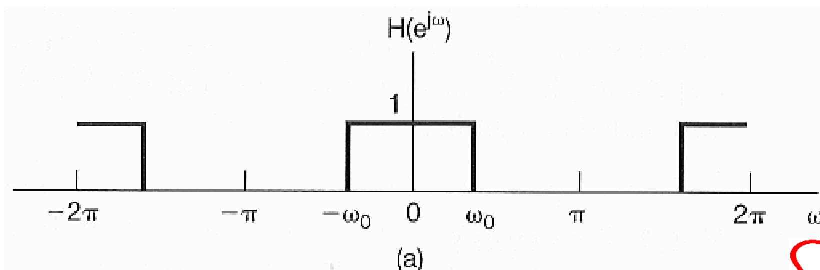
$$h(t) \xleftrightarrow{\mathcal{C.T.F.T.}} H(jw)$$

$$w_s = \frac{2\pi}{T}$$

$$\Omega = wT$$

$$h[n] \xleftrightarrow{\mathcal{D.T.F.T.}} H(e^{j\Omega})$$



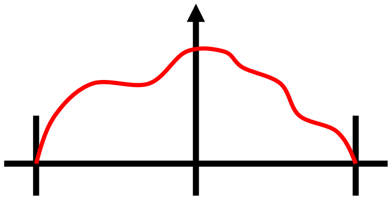




$\Omega$

## ▪ The Sampling Theorem:

- If the sampling instants are sufficiently close, very little is lost by sampling a CT signal
- If the sampling points are too far apart, much of the information about a signal can be lost
- So, when a CT signal can be uniquely given by its sampled version?

## Theorem 7.1: (Shannon, 1949)



- $f(t)$: a continuous-time signal
- $F(w)$: the Fourier transform of $f(t)$

$\rightarrow F(w) = 0$ outside $(-w_0, w_0)$

- $w_s$: sampling frequency

$\Rightarrow$ If $w_s > 2w_0$

Then $f(t)$ can be computed by:

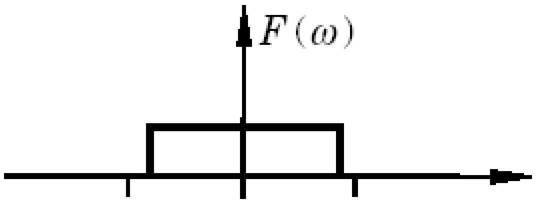
$$f(t) = \sum_{k=-\infty}^{\infty} f(kh) \frac{\sin(w_s(t-kh)/2)}{w_s(t-kh)/2}$$

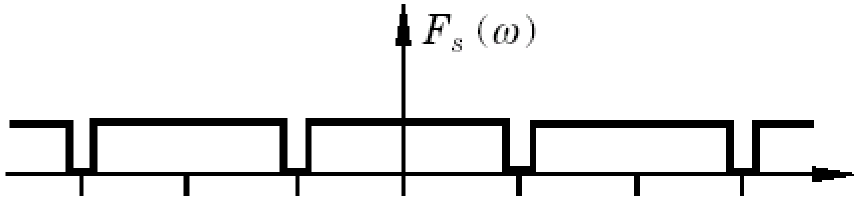
$\text{sinc}\frac{w_s(t-kh)}{2}$

- Note that:
  - $w_N = w_s/2$: Nyquist freqency
  - Reconstruction of signlas:

$$F(w) = 0 \text{ when } w > w_N$$

## Reconstruction:




- $$F(w) = \int_{-\infty}^{\infty} e^{-iwt} f(t) dt$$

- $$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iwt} F(w) dw$$

- $$F_s(w) = \frac{1}{h} \sum_{k=-\infty}^{\infty} F(w + kw_s)$$
$$= \sum_{k=-\infty}^{\infty} C_k e^{-ikhw}$$
$$= \sum_{k=-\infty}^{\infty} f(kh) e^{-ikhw}$$

$$C_k = \frac{1}{w_s} \int_0^{w_s} e^{ikhw} F_s(w) dw$$

- $$F(w) = \begin{cases} hF_s(w) & |w| \leq \frac{w_s}{2} \\ 0 & |w| > \frac{w_s}{2} \end{cases}$$

## Shannon Reconstruction:

- For periodic sampling of band-limited signals

$$f(t) = \sum_{k=-\infty}^{\infty} f(kh) \frac{\sin(w_s(t-kh)/2)}{w_s(t-kh)/2}$$

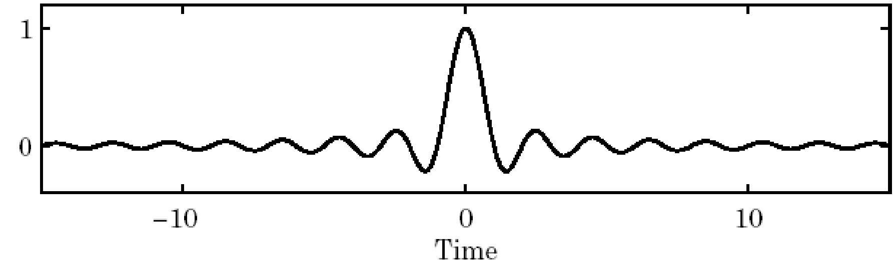
- However, it is NOT a caucal operator

# Shannon Reconstruction:

- Let's look at the impulse response:

$h = 1$

$$h(t) = \frac{\sin(w_s t/2)}{w_s t/2}$$

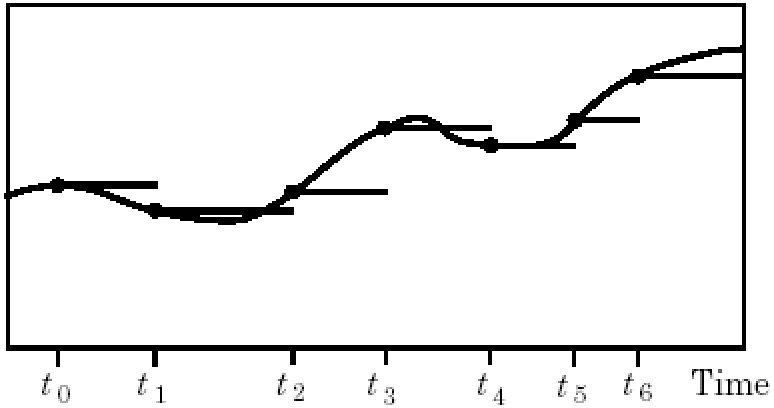


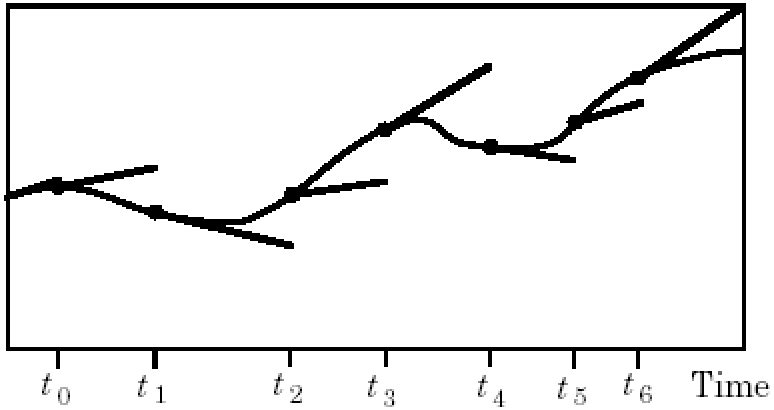

- The weights are 10% after 3 samples
  $< 5\%$ after 6 samples

- This construction has a delay
  $\Rightarrow$ Not good for control

- Only applied to periodic sampling

## ▪ Zero-Order Hold (ZOH) & First-Order Hold (FOH)



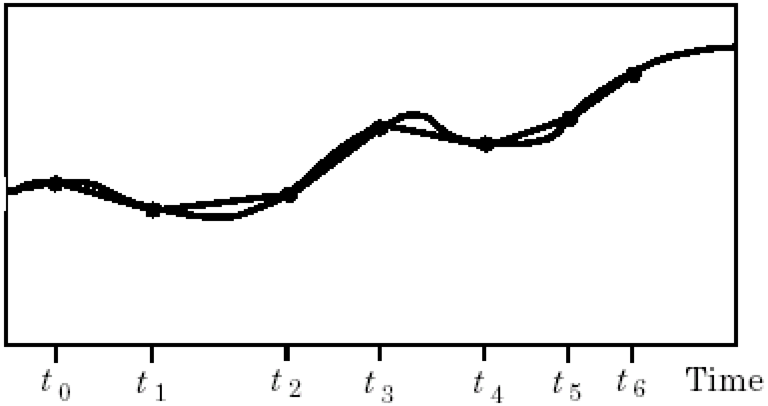

- They are caucal operators

## ▪ Predictive FOH:

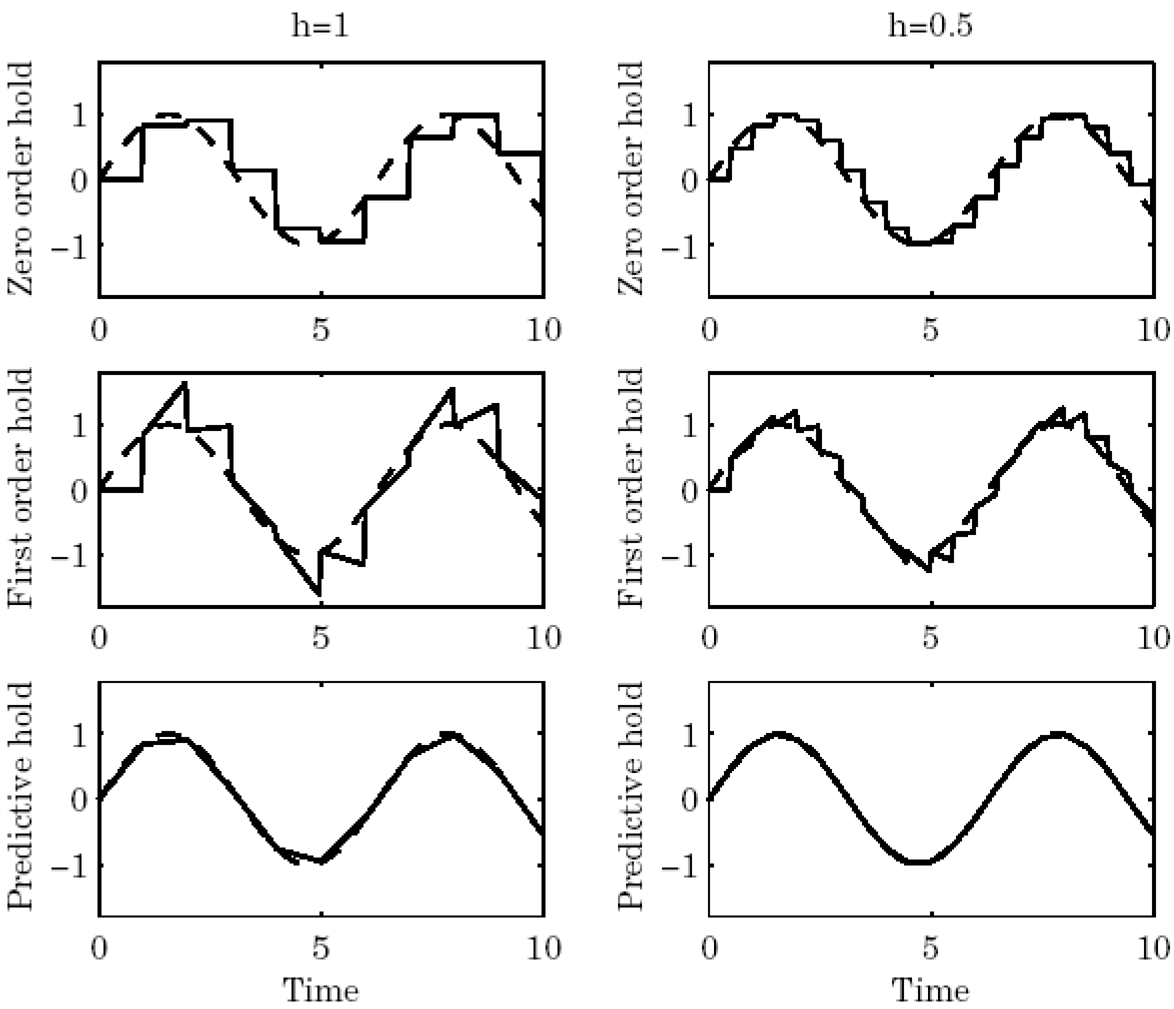
- It is NOT caucal

  But, can be replaced

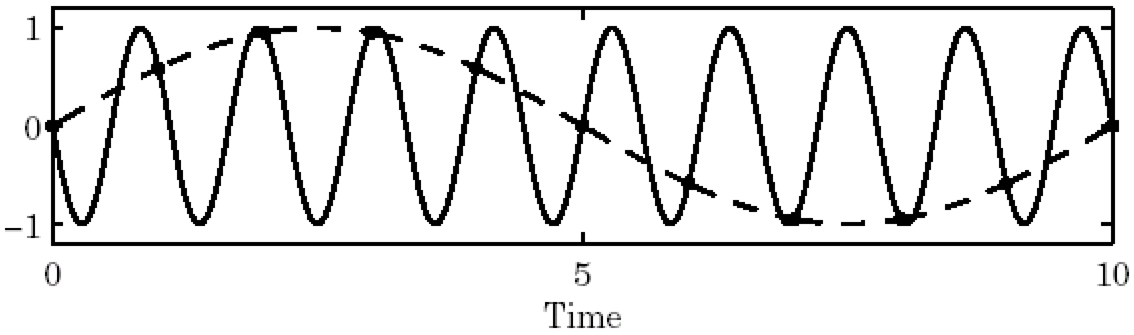
  by model prediction



- Sinusoidal signal with h = 1 and h = 0.5

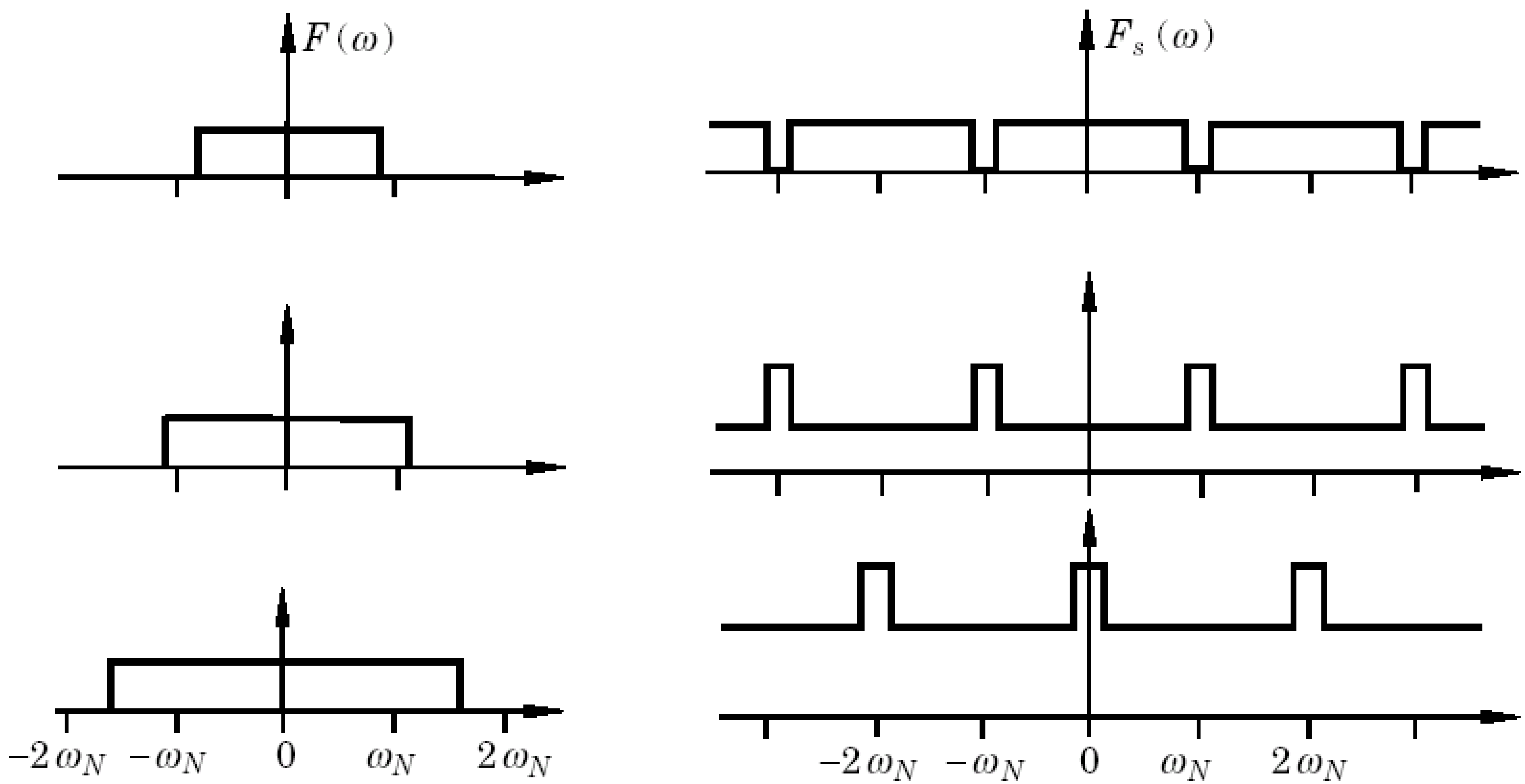


- Aliasing:
  - Two signals with frequency, 0.1 Hz and 0.9 Hz
  - They have the same values at all sampling instants

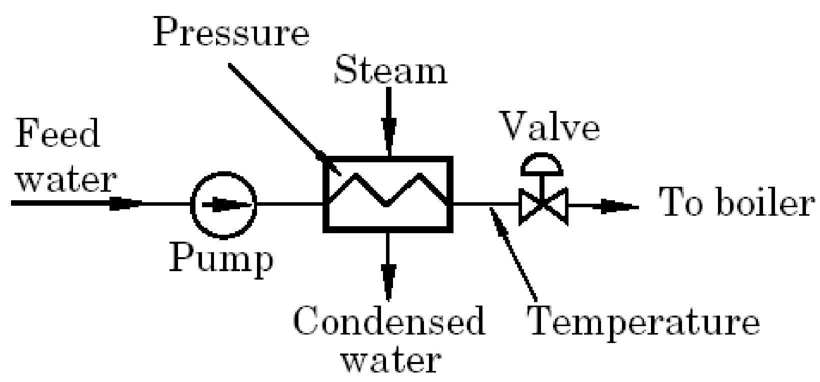


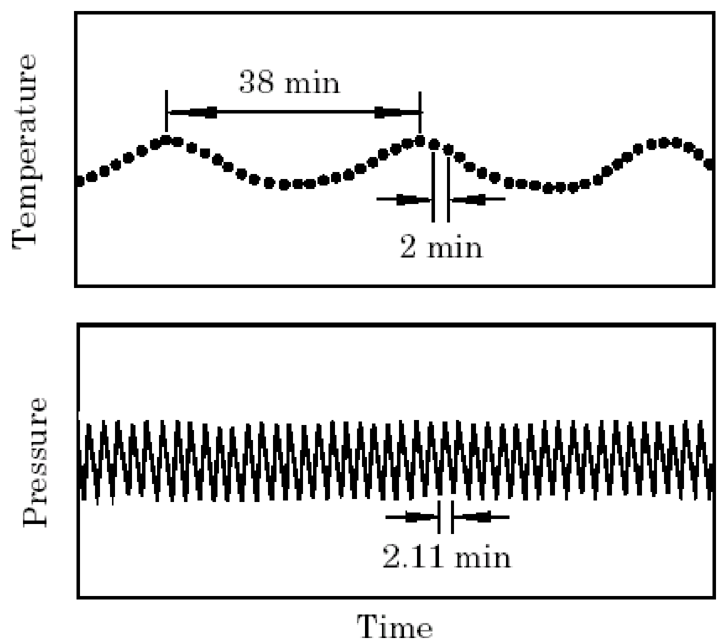
- Fourier transform of sampled signal:

  - $F(w) = \int_{-\infty}^{\infty} e^{-iwt} f(t) dt$
  - $F_s(w) = \sum_{k=-\infty}^{\infty} f(kh) e^{-ikhw}$



- Example 7.1: Feed-water heater in a ship boiler

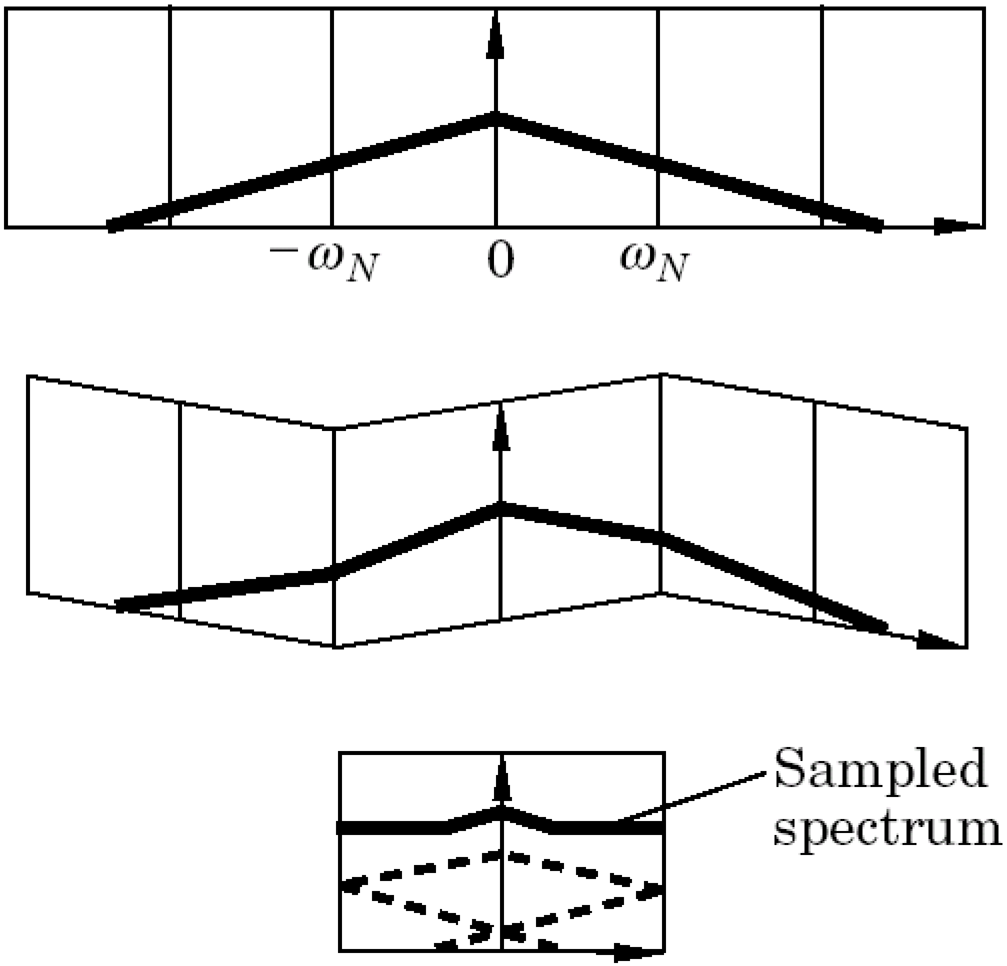


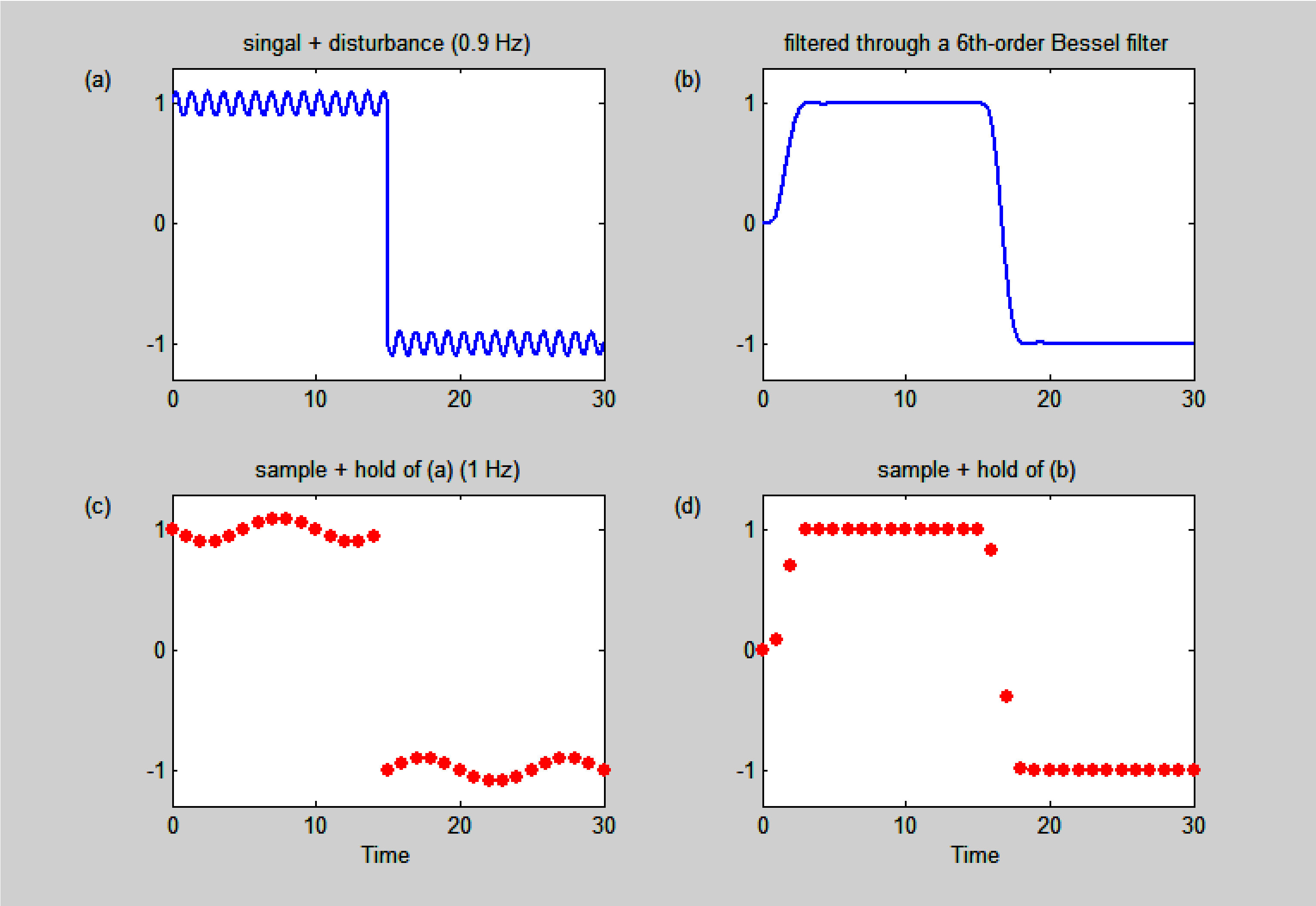

- $w_s = \dfrac{2\pi}{2} = 3.142$ rad/min
- $w_0 = \dfrac{2\pi}{2.11} = 2.978$ rad/min
- $w_s - w_0 = 0.1638$ rad/min

  $\Rightarrow \ T_s = 38$ min

- Frequency Folding

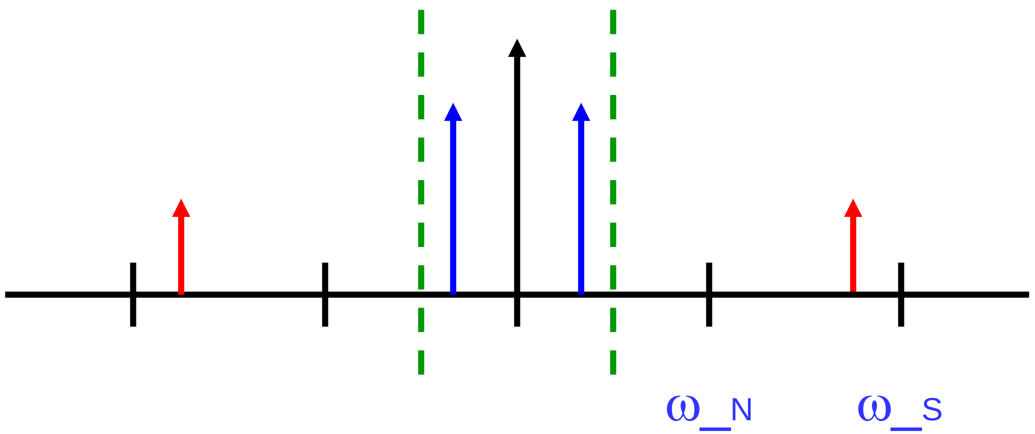


- Pre-Sampling Filter in Example 7.2:



## Pre-Sampling Filter in Example 7.2:

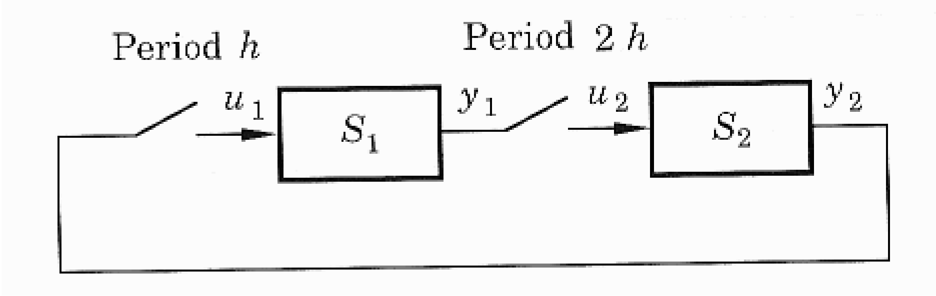
- With a sinusoidal perturbation (0.9Hz)
- Sampling frequency = 1 Hz



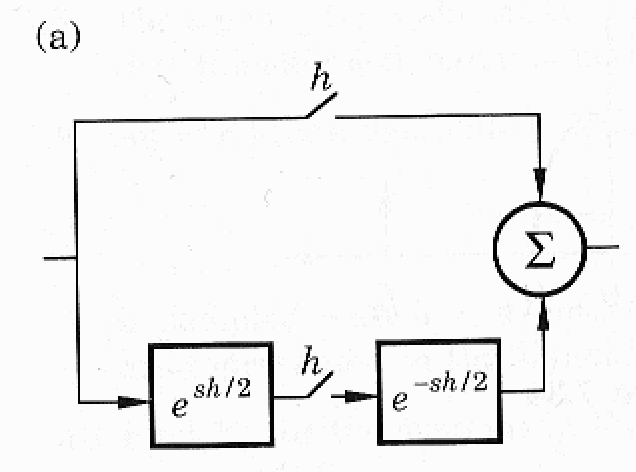
- Post-Sampling Filter:

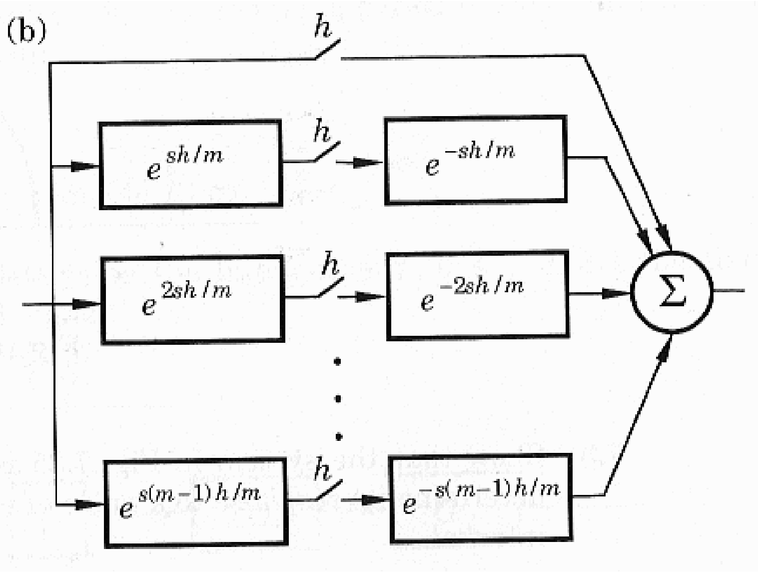
  - Because signal from D/A is piecewise constant
    - May excite some oscillatory modes
    - So, use higher-order hold!
      such as piecewise linear signal

- Multi-Rate System:




- Switch Decomposition:



# Further Readings: Multi-Rate Sampling

- Multi-Rate Systems

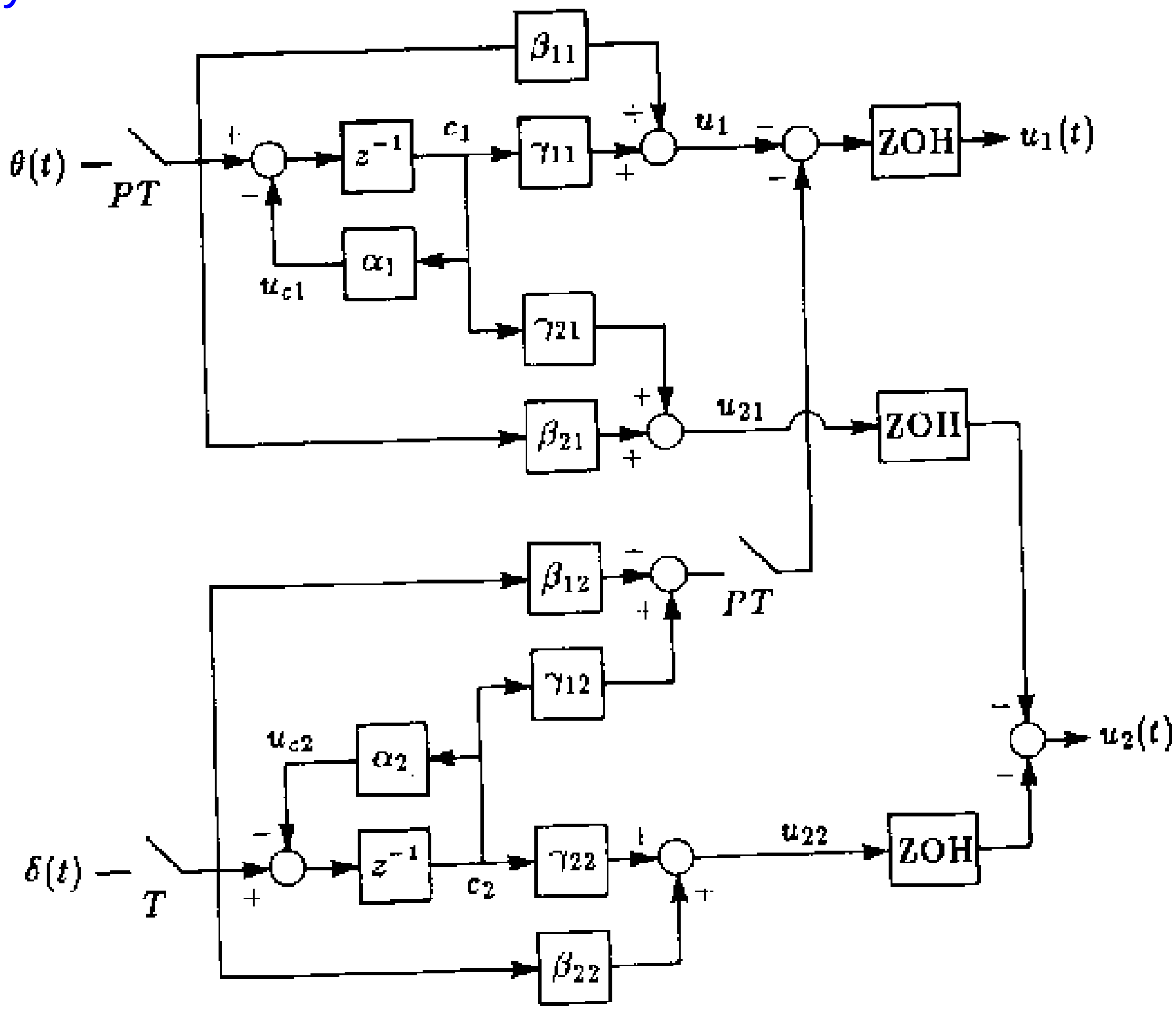



Fig. 7. TLA compensator structure.

M.C. Berg, N. Amit, J.D. Powell, "Multirate digital control system design", IEEE-TAC 33(12): 1139-1150, Dec 1988