

Spring 2021

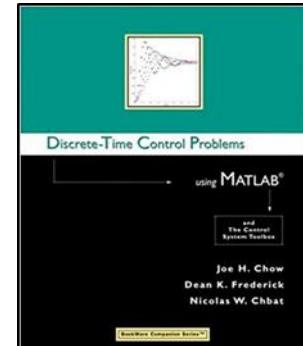
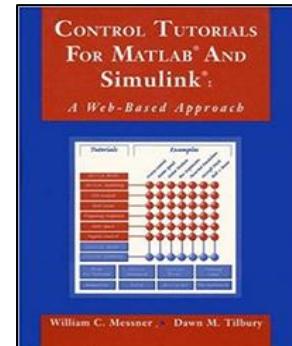
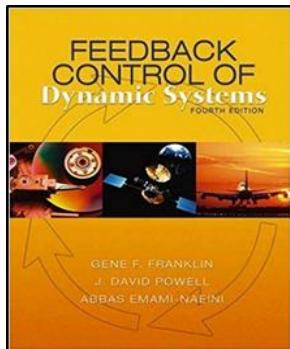
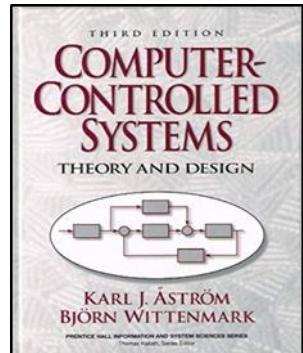
數位控制系統 Digital Control Systems

DCS-13
z Transform

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- Continuous-time systems: \Rightarrow Laplace transform
 - Discrete-time systems: \Rightarrow z transform
 - z transform maps a semi-infinite time sequence into a function of a complex variable
 - Range of z-transform and operator calculus
 - operator calculus: $\Rightarrow \{f(k) : k = \dots, -1, 0, 1, \dots\}$
 - z transform: $\Rightarrow \{f(k) : k = 0, 1, 2, \dots\}$
- \Rightarrow also take the initial values into consideration

- Definition: z-transform
- Consider the discrete-time signal

$$\{f(k) : k = 0, 1, 2, \dots\}$$

- The z transform of $f(k)$ is defined as:

$$\mathcal{Z}\{f(k)\} = F(z) = \sum_{k=0}^{\infty} f(k)z^{-k}$$

where z is a complex variable.

- The inverse z-transform is given by

$$f(k) = \frac{1}{2\pi i} \oint F(z)z^{k-1} dz$$

where the contour of integration
encloses the singularities of $F(z)$.

The z-Transform: Example

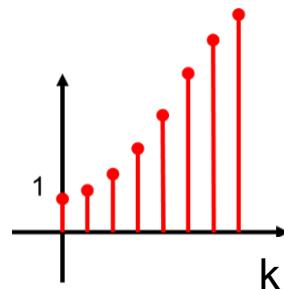
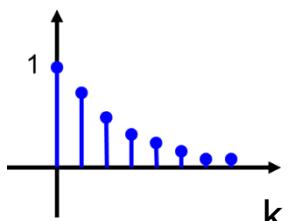
- Example: z Transform of a function

$$f[k] = a^k u[k]$$

$$\Rightarrow F(z) = \sum_{k=0}^{\infty} a^k u[k] z^{-k}$$

$$= \sum_{k=0}^{\infty} (az^{-1})^k$$

$$= \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \quad |z| > |a|$$



$$\mathcal{Z}\{f(k)\} = F(z)$$

$$= \sum_{k=0}^{\infty} f(k) z^{-k}$$

- Example: Transform of a ramp

$$\mathcal{Z}\{f(k)\} = F(z)$$

- Consider a ramp signal:

$$y(k) = kh \text{ for } k \geq 0$$

$$= \sum_{k=0}^{\infty} f(k)z^{-k}$$

$$\Rightarrow Y(z) = y(0)z^0 + y(1)z^{-1} + y(2)z^{-2} + \dots$$

$$= 0 + hz^{-1} + 2hz^{-2} + \dots$$

$$= h(z^{-1} + 2z^{-2} + \dots)$$

$$= \frac{hz}{(z - 1)^2}$$

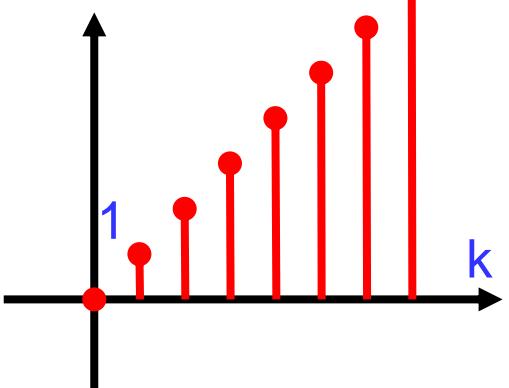


Table 2.3 Some time functions and corresponding Laplace and *z*-transforms. Warning: Use the table only as prescribed!

f	$\mathcal{L}f$	Zf
$\delta(k)$ (pulse)	—	1
$1 \quad k \geq 0$ (step)	$\frac{1}{s}$	$\frac{z}{z - 1}$
kh	$\frac{1}{s^2}$	$\frac{hz}{(z - 1)^2}$
$\frac{1}{2} (kh)^2$	$\frac{1}{s^3}$	$\frac{h^2 z(z + 1)}{2(z - 1)^3}$
$e^{-kh/T}$	$\frac{T}{1 + sT}$	$\frac{z}{z - e^{-h/T}}$
$1 - e^{-kh/T}$	$\frac{1}{s(1 + sT)}$	$\frac{z(1 - e^{-h/T})}{(z - 1)(z - e^{-h/T})}$
$\sin \omega kh$	$\frac{\omega}{s^2 + \omega^2}$	$\frac{z \sin \omega h}{z^2 - 2z \cos \omega h + 1}$

1. Definition:

$$F(z) = \sum_{k=0}^{\infty} f(k)z^{-k}$$

2. Inversion:

$$f(k) = \frac{1}{2\pi i} \oint F(z)z^{k-1} dz$$

3. Linearity:

$$\mathcal{Z}\{af + bg\} = \mathcal{Z}\{af\} + \mathcal{Z}\{bg\}$$

4. Time shift:

$$\mathcal{Z}\{q^{-n}f[k]\} = z^{-n}F(z)$$

$$\mathcal{Z}\{f[k-n]\} = z^{-n}F(z)$$

$$\mathcal{Z}\{q^n f[k]\} = z^n(F(z) - F_1)$$

$$\mathcal{Z}\{f[k+n]\} = z^n(F(z) - F_1)$$

where $F_1(z) = \sum_{j=0}^{n-1} f(j)z^{-j}$

5 Initial-value theorem:

$$f(0) = \lim_{z \rightarrow \infty} F(z)$$

6 Final-value theorem:

If $(1 - z^{-1})F(z)$ does not have any poles on or outside the unit circle:

$$\lim_{k \rightarrow \infty} f(k) = \lim_{z \rightarrow 1} (1 - z^{-1})F(z)$$

7 Convolution:

$$\mathcal{Z}\{f * g\} = (\mathcal{Z}f)(\mathcal{Z}g)$$

$$f * g = \sum_{n=0}^k f(n)g(k-n)$$

- Example – Use z-Transform to find system response

- Consider the difference equation:

$$y[k] - \frac{1}{2}y[k-1] = u[k] + \frac{1}{3}u[k-1]$$

$$\Rightarrow Y(z) - \frac{1}{2}z^{-1}Y(z) = U(z) + \frac{1}{3}z^{-1}U(z)$$

$$\Rightarrow G(z) = \frac{Y(z)}{U(z)} = \frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}} = \frac{z + \frac{1}{3}}{z - \frac{1}{2}}$$

Time shift:

$$\mathcal{Z}\{f[k-n]\} = z^{-n}F(z)$$

$$\mathcal{Z}\{f[k+n]\} = z^n(F(z) - F_1)$$

where $F_1(z) = \sum_{j=0}^{n-1} f(j)z^{-j}$

The z-Transform: From State Space to Pulse Transfer Function

$$y(k) + a_1 y(k-1) + \cdots + a_n y(k-n)$$

$$= b_0 u(k) + b_1 u(k-1) + \cdots + b_n u(k-n)$$

$$y(k+n) + a_1 y(k+n-1) + \cdots + a_n y(k)$$

$$= b_0 u(k+n) + b_1 u(k+n-1) + \cdots + b_n u(k)$$

- Take **z-transform** of both sides:

$$\underbrace{[z^n + a_1 z^{n-1} + \cdots + a_n]}_{A(z)} Y(z) = \underbrace{[b_0 z^n + b_1 z^{n-1} + b_2 z^{n-2} + \cdots + b_n]}_{B(z)} U(z)$$

$\Rightarrow \frac{B(z)}{A(z)}$ is the **pulse transfer function**: $u \rightarrow y$

- The corresponding state-space form:

$$x(k+1) = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & \cdots & 0 \\ \ddots & & & \vdots \\ & & 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u(k)$$

$$y(k) = [b_1 \ b_2 \ \cdots \ b_n] x(k)$$

$$\begin{cases} x(k+1) = \mathbf{F}x(k) + \mathbf{H}u(k) \\ y(k) = \mathbf{C}x(k) + \mathbf{D}u(k) \end{cases}$$

$$\begin{cases} z(X(z) - x(0)) = \mathbf{F}X(z) + \mathbf{H}U(z) \\ Y(z) = \mathbf{C}X(z) + \mathbf{D}U(z) \end{cases}$$

$$\Rightarrow zX(z) - \mathbf{F}X(z) = zx(0) + \mathbf{H}U(z)$$

$$\Rightarrow (z\mathbf{I} - \mathbf{F})X(z) = zx(0) + \mathbf{H}U(z)$$

$$\Rightarrow X(z) = (z\mathbf{I} - \mathbf{F})^{-1}[zx(0) + \mathbf{H}U(z)]$$

$$\Rightarrow Y(z) = \mathbf{C}(z\mathbf{I} - \mathbf{F})^{-1}[zx(0) + \mathbf{H}U(z)] + \mathbf{D}U(z)$$

$$\begin{aligned} \Rightarrow Y(z) &= \mathbf{C}(z\mathbf{I} - \mathbf{F})^{-1}zx(0) \\ &\quad + (\mathbf{C}(z\mathbf{I} - \mathbf{F})^{-1}\mathbf{H} + \mathbf{D})U(z) \end{aligned}$$

$$\Rightarrow G(z) = \mathbf{C}(z\mathbf{I} - \mathbf{F})^{-1}\mathbf{H} + \mathbf{D}$$

The z-Transform: From State Space to Pulse Transfer Function

$$\begin{cases} x(k+1) = \mathbf{F}x(k) + \mathbf{H}u(k) \\ y(k) = \mathbf{C}x(k) + \mathbf{D}u(k) \end{cases}$$

$$\Rightarrow G(z) = \mathbf{C}(z\mathbf{I} - \mathbf{F})^{-1}\mathbf{H} + \mathbf{D}$$

$$= [b_1 \ b_2 \ \dots \ b_n] (z\mathbf{I} - \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & \cdots & 0 \\ \ddots & & & \vdots \\ & & 1 & 0 \end{bmatrix})^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= [b_1 \ b_2 \ \dots \ b_n] \begin{bmatrix} z + a_1 & a_2 & \cdots & a_n \\ -1 & z & \cdots & 0 \\ \ddots & & & \vdots \\ & & -1 & z \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= \frac{1}{\Delta(z)} \text{Adj } (z\mathbf{I} - \mathbf{F})$$

$$= \frac{(b_0 z^n b_1 z^{n-1} + b_2 z^{n-2} + \dots + b_n)}{(z^n + a_1 z^{n-1} + \dots + a_n)} = \frac{B(z)}{A(z)}$$

- Theorem 2.5:

- The pulse response $g(k)$ and the pulse-transfer function $G(z)$ are a z -transform pair, that is, $\mathcal{Z}\{g(k)\} = G(z)$



$$U = \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(N-1) \end{bmatrix} Y = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N-1) \end{bmatrix}$$

- The Pulse Response for the D.T. system:

$$g(k) = \begin{cases} 0 & k < 0 \\ D & k = 0 \\ C\mathbf{F}^{k-1}\mathbf{H} & k \geq 1 \end{cases}$$

- Pulse Transfer Function

$$G(z) = C(zI - F)^{-1}H + D$$

- Method 1:

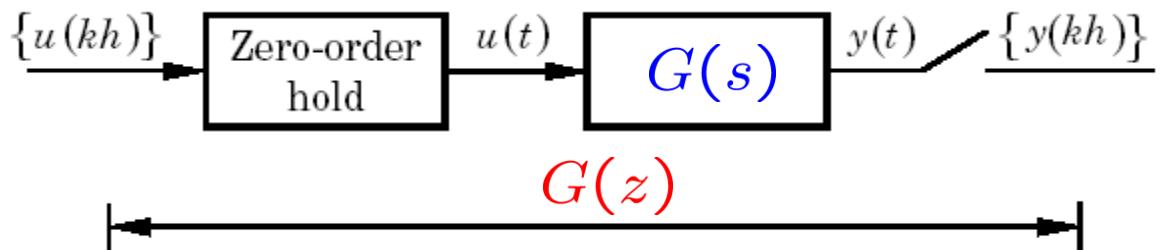
- Make state-space realization
- Calculate \mathbf{F} and \mathbf{H}
- Find $G(z)$

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\ y(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t) \end{cases}$$

$$\Rightarrow \begin{cases} x(k+1) = \mathbf{F}x(k) + \mathbf{H}u(k) \\ y(k) = \mathbf{C}x(k) + \mathbf{D}u(k) \end{cases}$$

$$\Rightarrow G(z) = \mathbf{C}(z\mathbf{I} - \mathbf{F})^{-1}\mathbf{H} + \mathbf{D}$$

• Method 2:



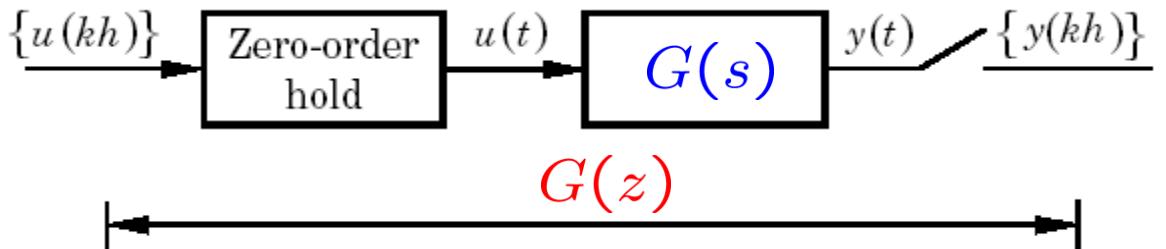
- Step response of $G(s)$
 - z -transform of $\frac{G(s)}{s}$
 - Divide by the z -transform of step function

$$\Rightarrow Y(s) = \frac{G(s)}{s}$$

$$\Rightarrow \hat{Y}(z) = \mathcal{Z}\{\mathcal{L}^{-1}Y(s)\}$$

$$\Rightarrow G(z) = (1 - z^{-1})\hat{Y}(z)$$

- Method 2:



$$\Rightarrow G(z) = \frac{z-1}{z} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{sh}}{z - e^{sh}} \frac{G(s)}{s} ds$$

- If $G(s)$ goes to zero at least as fast as $|s|^{-1}$ for large s
- and $G(s)$ has distinct poles, none are at the origin

$$\Rightarrow G(z) = \sum_{s=s_i} \frac{1}{z - e^{sh}} \text{Res} \left\{ \frac{e^{sh} - 1}{s} \right\} G(s)$$

where s_i are the poles of $G(s)$
and Res denotes the residue.

- Example (Pole-Zero Cancellation):

- Consider the difference equation:

$$y[k+1] + ay[k] = u[k+1] + au[k]$$

- By using z-transform, Its pulse-transfer function:

$$G(z) = \frac{z + a}{z + a} = 1 \Rightarrow y[k] = u[k]$$

- But, the solution of the difference equation is:

$$y[k] = (-a)^k y[0] + u[k], \quad k \geq 1$$

- That is, by using the shift-operator calculus:

$$(q + a)y[k] = (q + a)u[k]$$

- Different conclusions for:

$$\begin{cases} |a| < 0 & \text{stable mode} \\ |a| \geq 0 & \text{unstable mode} \end{cases}$$

- Use modified z -transform to study the intersampling behavior
- Definition: The Modified z -Transform

$$\tilde{F}(z, m) = \sum_{k=0}^{\infty} z^{-k} f(kh - h + mh)$$

$$0 \leq m \leq 1$$

- The inverse transform:

$$f(kh - h + mh) = \frac{1}{2\pi i} \int_{\Gamma} \tilde{F}(z, m) z^{n-1} dz$$

where the contour Γ encloses the singularities of integrand.

- Poles and zeros

$$Y(z) = \underbrace{[\mathbf{C}(\cancel{z}\mathbf{I} - \mathbf{F})^{-1}\mathbf{H} + \mathbf{D}]}_{G(z)} U(z)$$

- Poles:
 - The points $p \in C$ where $G(p) = \infty$ are the poles of $G(z)$
 - They are eigenvalues of \mathbf{F} and determine stability
- Zeros:
 - The points $p \in C$ where $G(p) = 0$ are the zeros of $G(z)$

- Interpretation of poles and zeros
 - Poles:
 - A **pole** $z = a$ is associated with the time function $z(k) = a^k$
 - A **pole** $z = a$ is an eigenvalue of \mathbf{F}
 - Zeros:
 - A **zero** $z = b$ implies that the transmission of $u(k) = b^k$ is **blocked** by the system
 - A **zero** is related to how **inputs** and **outputs** are coupled to the **states**

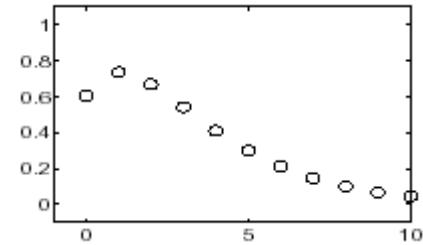
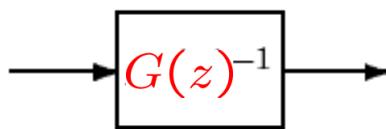
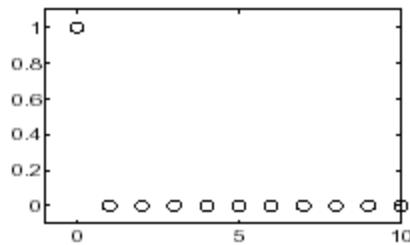
- Poles determine stability
 - All poles of $G(z) = \mathbf{C}(z\mathbf{I} - \mathbf{F})^{-1}\mathbf{H} + \mathbf{D}$ are eigenvalues of \mathbf{F}
 - The matrix \mathbf{F} can always be written as:

$$\mathbf{F} = U \begin{bmatrix} \lambda_1 & & * \\ & \dots & \\ 0 & & \lambda_n \end{bmatrix} U^{-1} \quad \mathbf{F}^k = U \begin{bmatrix} \lambda_1^k & & * \\ & \dots & \\ 0 & & \lambda_n^k \end{bmatrix} U^{-1}$$

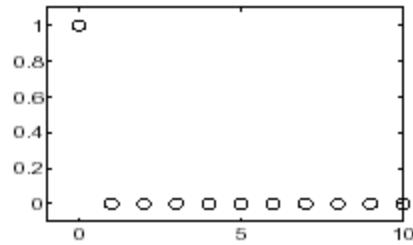
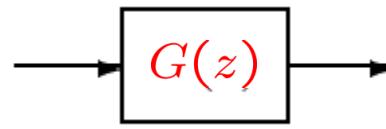
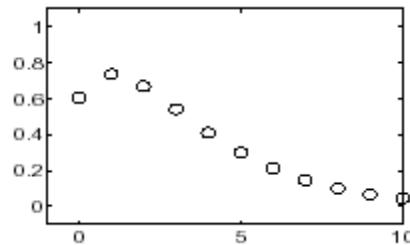
- The diagonal elements λ_k are the eig(\mathbf{F})
- \mathbf{F}^k decays exponentially iff $|\lambda_k| < 1$, $\forall k$

- Zeros: blocking of signals

- Assume that $G(z) = \frac{B(z)}{A(z)}$
and $G(z)^{-1}$ has pole at $z = b$, $b < 1$



- The pulse response of $G(z)^{-1}$ decays as b^k



- $u(k) = b^k$ to $G(z) \Rightarrow$ zero output

- Non-minimum phase
 - Unstable zeros are sometimes called non-minimum phase zeros
 - Because:
 $\frac{s-3}{(s+1)(s+2)}$ and $\frac{s+3}{(s+1)(s+2)}$ have Bode diagrams with equal amplitude:
 $|iw - 3| = \sqrt{w^2 + 3^2} = |iw + 3|$
 - But the first has more phase lag
 - This is similar to a time delay and make control harder

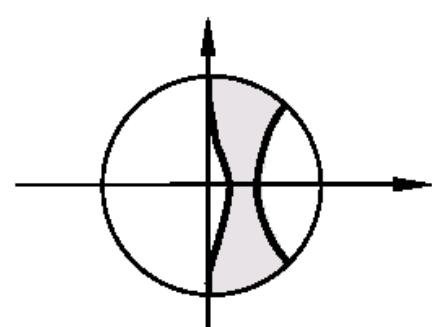
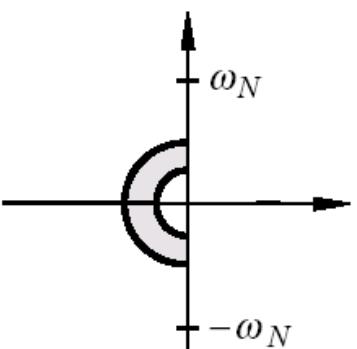
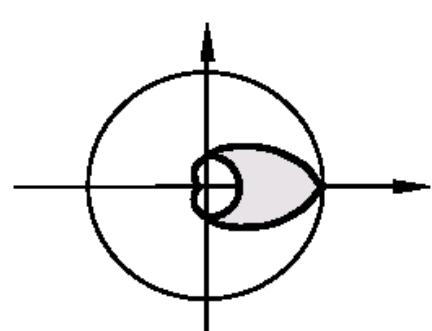
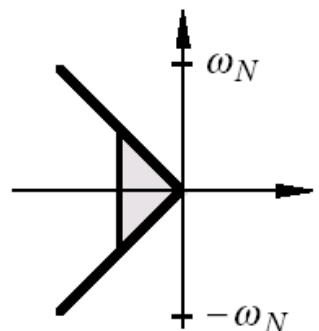
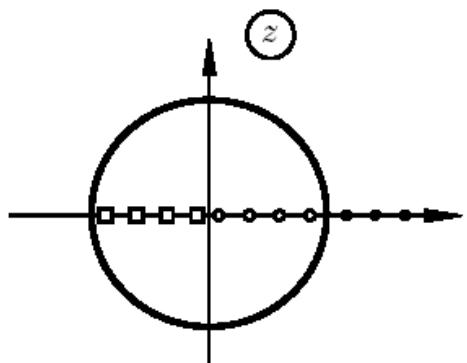
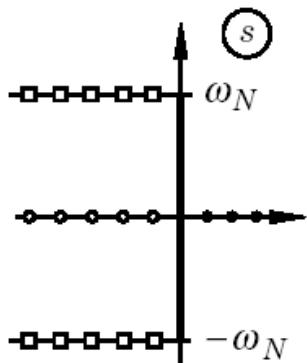
- Non-minimum phase zeros make control harder
 - To get a pulse as output, one should use an input which is the impulse response of G^{-1}
 - If there are non-minimum phase zeros, the input needs to grow exponentially
 - For a stable system with one unstable zero, the step response will initially have different sign than the stationary value.
⇒ This make control harder

- Mapping between s & z

- Because $\mathbf{F} = \exp(\mathbf{A}h)$

$$\Rightarrow \lambda_i(\mathbf{F}) = e^{\lambda_i(\mathbf{A})h}$$

$$\Rightarrow z = e^{sh}$$



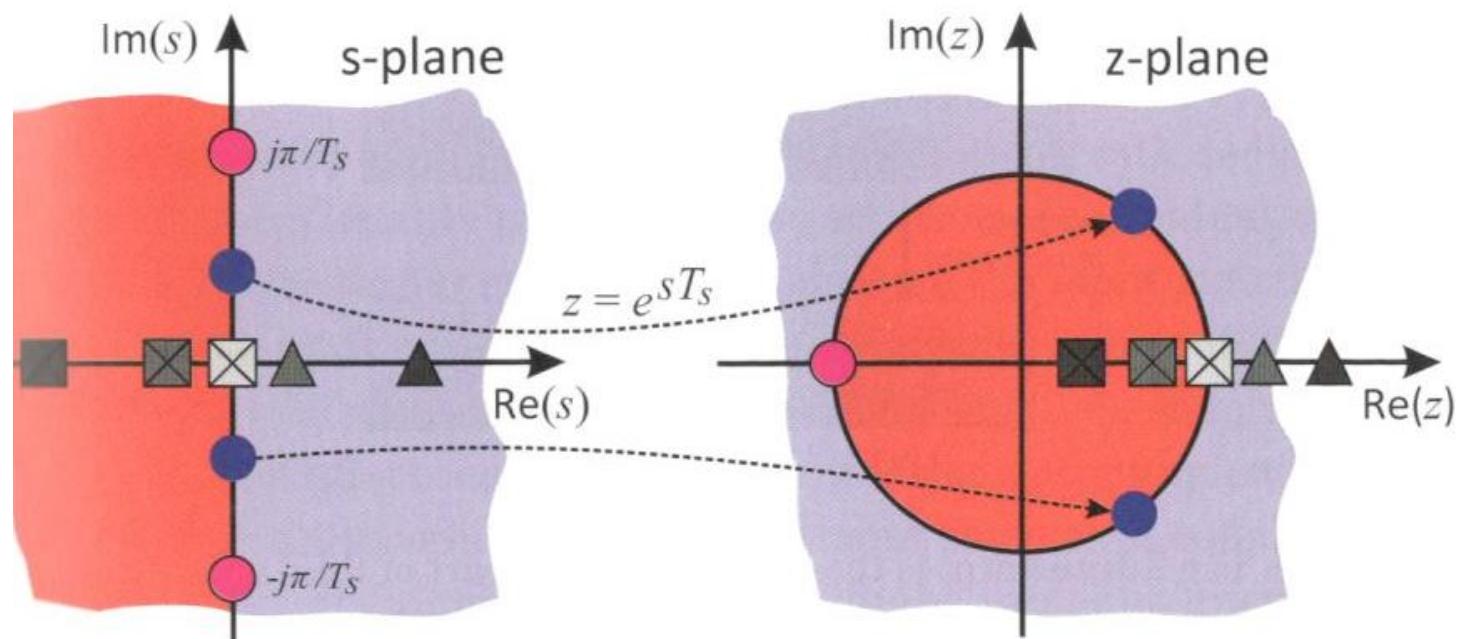
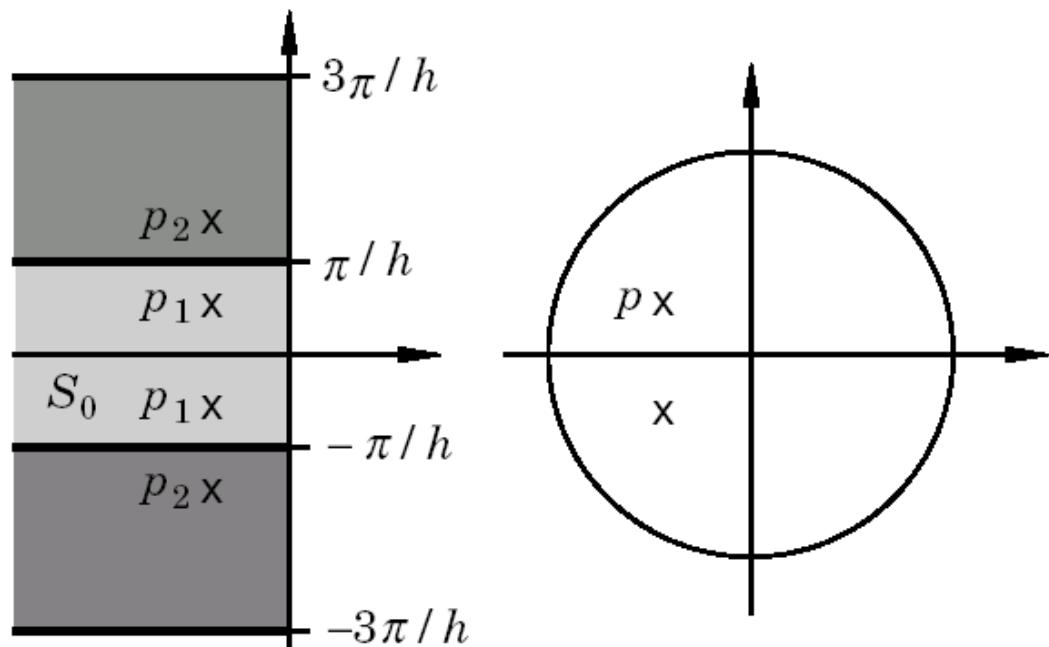


Figure 4.42: Corresponding points and areas in *s*- and *z*-plane.

- Alias Problem

$z = e^{sh}$ \Rightarrow Several points in the *s*-plane
map into the same point in the *z*-plane
 \Rightarrow The map is not bijective



- Sampling of a second-order system

$$\frac{w_0^2}{s^2 + 2\zeta w_0 s + w_0^2}$$

- Poles of the D.T. system:

$$z^2 + a_1 z + a_2 = 0$$

where

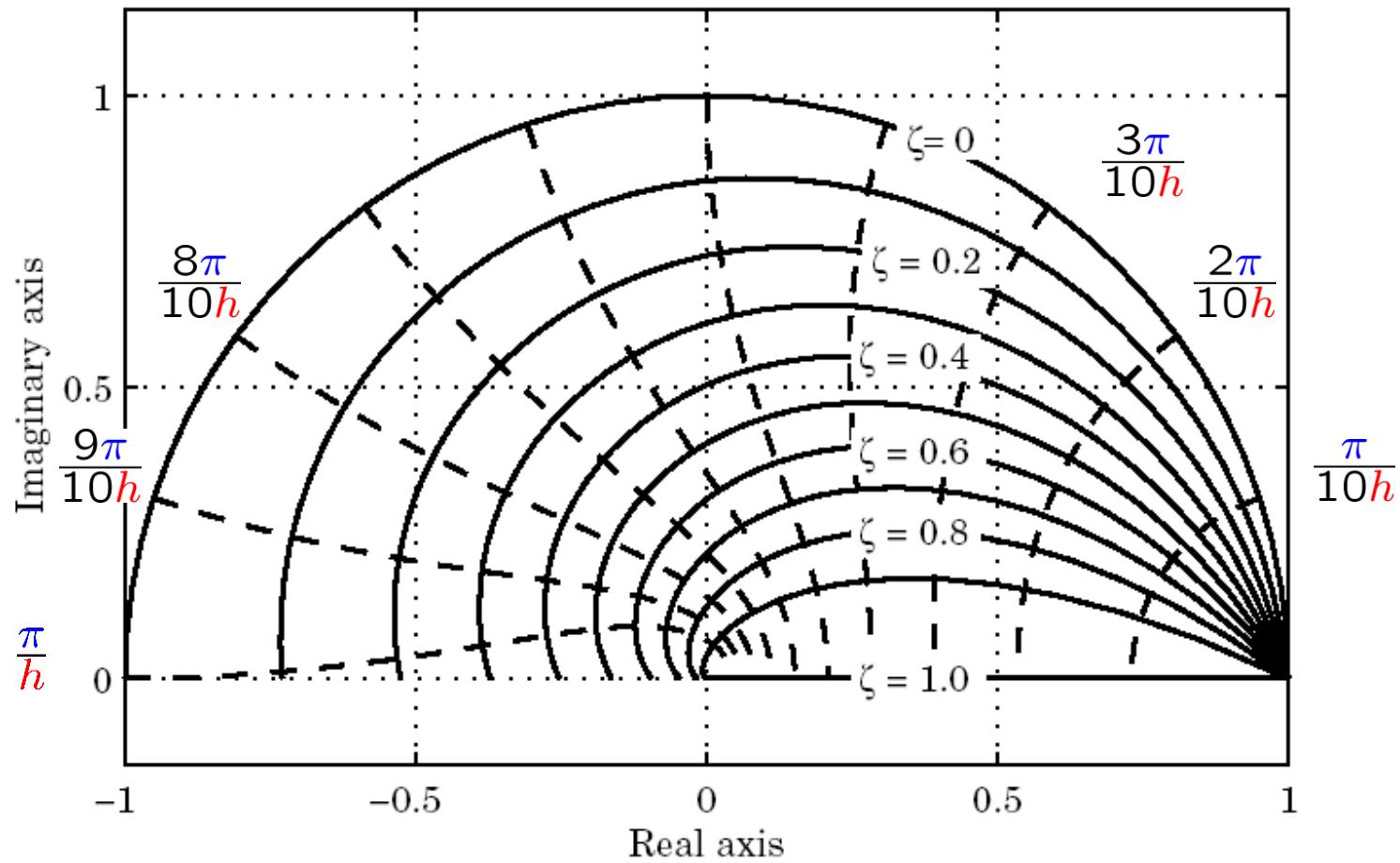
$$\begin{aligned}a_1 &= -2e^{-\zeta w_0 h} \cos \left(\sqrt{1 - \zeta^2} w_0 h \right) \\a_2 &= e^{-2\zeta w_0 h}\end{aligned}$$

$$z^2 + a_1 z + a_2 = 0$$

- Sampling of a second-order system

$$a_1 = -2e^{-\zeta w_0 h} \cos(\sqrt{1-\zeta^2} w_0 h)$$

$$a_2 = e^{-2\zeta w_0 h}$$



$w_0 h$: - - -

- Transformation of zeros

- More difficult than poles
- In general,
more sampled zeros than continuous zeros
- For short sampling periods $z_i \approx e^{s_i h}$
- For large s then $G(s) \approx e^{-d}$
where $d = \deg A(s) - \deg B(s)$
- The $r = d - 1$ zeros
introduced by the sampling
go to the zeros of the polynomials z_d

d	Z_d	
1	1	$d = \#(p) - \#(z)$
2	$z + 1$	
3	$z^2 + 4z + 1$	
4	$z^3 + 11z^2 + 11z + 1$	
5	$z^4 + 26z^3 + 66z^2 + 26z + 1$	

■ Example:

- Consider the CT transfer function:

$$\frac{2}{(s+1)(s+2)}$$

- The zero of the pulse-transfer function:

$$z = \frac{(1 - e^{-2h})e^{-h} - 2(1 - e^{-h})e^{-2h}}{2(1 - e^{-h}) - (1 - e^{-2h})}$$

- When h is small:

$$z \approx -1 + 3h$$

- When h approaches 0:

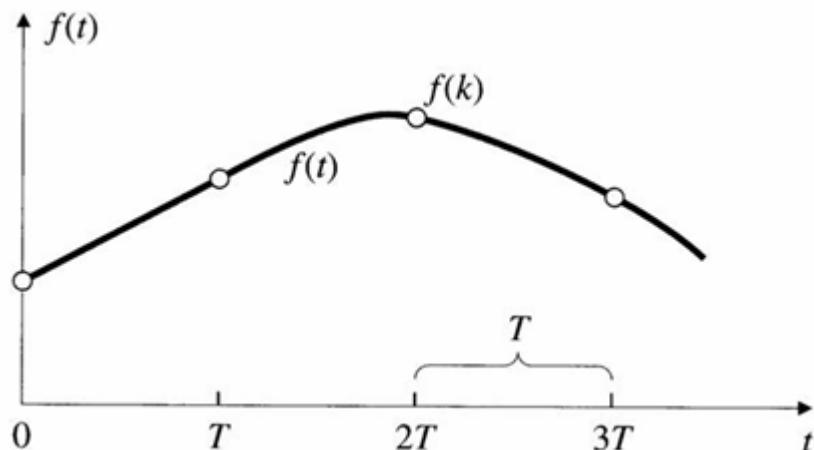
$$z \rightarrow -1$$

- When h increases:

$$z \rightarrow 0$$

$$H(q) = \frac{b_1 q^{n-1} + b_2 q^{n-2} + \cdots + b_n}{q^n + a_1 q^{n-1} + \cdots + a_n}$$

$$\begin{aligned} b_1 &= \frac{b(1 - e^{-ah}) - a(1 - e^{-bh})}{b - a} \\ \frac{ab}{(s+a)(s+b)} &= \frac{a(1 - e^{-bh})e^{-ah} - b(1 - e^{-ah})e^{-bh}}{b - a} \\ a \neq b & \\ a_1 &= -(e^{-ah} + e^{-bh}) \\ a_2 &= e^{-(a+b)h} \end{aligned}$$



- Laplace transform:

$$\mathcal{L} \left\{ f(t) \right\} = F(s) \triangleq \int_0^{\infty} f(t) e^{-st} dt$$

$$\mathcal{L} \left\{ \dot{f}(t) \right\} = s F(s)$$

- z-Transform:

$$\mathcal{Z} \left\{ f[k] \right\} = F(z) \triangleq \sum_{k=0}^{\infty} f[k] z^{-k}$$

$$\mathcal{Z} \left\{ f[k-1] \right\} = z^{-1} F(z)$$

- Analysis of discrete systems by the z-transform:

$$y[k] + a_1 y[k-1] + a_2 y[k-2]$$

$$= b_0 u[k] + b_1 u[k-1] + b_2 u[k-2]$$

$$\Rightarrow Y(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z)$$

$$= b_0 U(z) + b_1 z^{-1} U(z) + b_2 z^{-2} U(z)$$

$$\Rightarrow \frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} = \frac{b_0 z^2 + b_1 z + b_2}{z^2 + a_1 z + a_2}$$

■ The z-transform inversion:

- Long division

$$\frac{Y(z)}{U(z)} = \frac{1}{1 - a z^{-1}}$$

- For a unit pulse input: $u[0] = 1$, & $u[k] = 0, k \neq 0$

$$\Rightarrow U(z) = 1$$

$$\Rightarrow Y(z) = \frac{1}{1 - a z^{-1}}$$

$$= 1 + az^{-1} + a^2 z^{-2} + a^3 z^{-3} + \dots$$

$$= y[0] + y[1]z^{-1} + y[2]z^{-2} + y[3]z^{-3} + \dots$$

- The z-transform inversion:
 - The z-transform table

TABLE 10.2 SOME COMMON z -TRANSFORM PAIRS

Signal	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1 - z^{-1}}$	$ z > 1$
3. $-u[-n - 1]$	$\frac{1}{1 - z^{-1}}$	$ z < 1$
4. $\delta[n - m]$	z^{-m}	All z , except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $\alpha^n u[n]$	$\frac{1}{1 - \alpha z^{-1}}$	$ z > \alpha $
6. $-\alpha^n u[-n - 1]$	$\frac{1}{1 - \alpha z^{-1}}$	$ z < \alpha $
7. $n\alpha^n u[n]$	$\frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$	$ z > \alpha $
8. $-n\alpha^n u[-n - 1]$	$\frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$	$ z < \alpha $
9. $[\cos \omega_0 n]u[n]$	$\frac{1 - [\cos \omega_0]z^{-1}}{1 - [2 \cos \omega_0]z^{-1} + z^{-2}}$	$ z > 1$
10. $[\sin \omega_0 n]u[n]$	$\frac{[\sin \omega_0]z^{-1}}{1 - [2 \cos \omega_0]z^{-1} + z^{-2}}$	$ z > 1$
11. $[r^n \cos \omega_0 n]u[n]$	$\frac{1 - [r \cos \omega_0]z^{-1}}{1 - [2r \cos \omega_0]z^{-1} + r^2 z^{-2}}$	$ z > r$
12. $[r^n \sin \omega_0 n]u[n]$	$\frac{[r \sin \omega_0]z^{-1}}{1 - [2r \cos \omega_0]z^{-1} + r^2 z^{-2}}$	$ z > r$

$F(s)$ is the Laplace transform of $f(t)$, and $F(z)$ is the z -transform of $f(kT)$. Note: $f(t) = 0$ for $t = 0$.

No.	$F(s)$	$f(kT)$	$F(z)$
1		$1, k = 0; 0, k \neq 0$	1
2		$1, k = k_o; 0, k \neq k_o$	z^{-k_o}
3	$\frac{1}{s}$	$1(kT)$	$\frac{z}{z - 1}$
4	$\frac{1}{s^2}$	kT	$\frac{Tz}{(z - 1)^2}$
5	$\frac{1}{s^3}$	$\frac{1}{2!}(kT)^2$	$\frac{T^2}{2} \left[\frac{z(z + 1)}{(z - 1)^3} \right]$
6	$\frac{1}{s^4}$	$\frac{1}{3!}(kT)^3$	$\frac{T^3}{6} \left[\frac{z(z^2 + 4z + 1)}{(z - 1)^4} \right]$
7	$\frac{1}{s^m}$	$\lim_{a \rightarrow 0} \frac{(-1)^{m-1}}{(m-1)!} \left(\frac{\partial^{m-1}}{\partial a^{m-1}} e^{-akT} \right)$	$\lim_{a \rightarrow 0} \frac{(-1)^{m-1}}{(m-1)!} \left(\frac{\partial^{m-1}}{\partial a^{m-1}} \frac{z}{z - e^{-aT}} \right)$
8	$\frac{1}{s+a}$	e^{-akT}	$\frac{z}{z - e^{-aT}}$
9	$\frac{1}{(s+a)^2}$	$kT e^{-akT}$	$\frac{Tze^{-aT}}{(z - e^{-aT})^2}$
10	$\frac{1}{(s+a)^3}$	$\frac{1}{2}(kT)^2 e^{-akT}$	$\frac{T^2}{2} e^{-aT} z \frac{(z + e^{-aT})}{(z - e^{-aT})^3}$
11	$\frac{1}{(s+a)^m}$	$\frac{(-1)^{m-1}}{(m-1)!} \left(\frac{\partial^{m-1}}{\partial a^{m-1}} e^{-akT} \right)$	$\frac{(-1)^{m-1}}{(m-1)!} \left(\frac{\partial^{m-1}}{\partial a^{m-1}} \frac{z}{z - e^{-aT}} \right)$

In Summary

12	$\frac{a}{s(s+a)}$	$1 - e^{-akT}$	$\frac{z(1 - e^{-aT})}{(z-1)(z - e^{-aT})}$
13	$\frac{a}{s^2(s+a)}$	$\frac{1}{a}(akT - 1 + e^{-akT})$	$\frac{z[(aT - 1 + e^{-aT})z + (1 - e^{-aT} - aTe^{-aT})]}{a(z-1)^2(z - e^{-aT})}$
14	$\frac{b-a}{(s+a)(s+b)}$	$e^{-akT} - e^{-bkT}$	$\frac{(e^{-aT} - e^{-bT})z}{(z - e^{-aT})(z - e^{-bT})}$
15	$\frac{s}{(s+a)^2}$	$(1 - akT)e^{-akT}$	$\frac{z[z - e^{-aT}(1 + aT)]}{(z - e^{-aT})^2}$
16	$\frac{a^2}{s(s+a)^2}$	$1 - e^{-akT}(1 + akT)$	$\frac{z[z(1 - e^{-aT} - aTe^{-aT}) + e^{-2aT} - e^{-aT} + aTe^{-aT}]}{(z-1)(z - e^{-aT})^2}$
17	$\frac{(b-a)s}{(s+a)(s+b)}$	$be^{-bkT} - ae^{-akT}$	$\frac{z[z(b-a) - (be^{-aT} - ae^{-bT})]}{(z - e^{-aT})(z - e^{-bT})}$
18	$\frac{a}{s^2 + a^2}$	$\sin akT$	$\frac{z \sin aT}{z^2 - (2 \cos aT)z + 1}$
19	$\frac{s}{s^2 + a^2}$	$\cos akT$	$\frac{z(z - \cos aT)}{z^2 - (2 \cos aT)z + 1}$
20	$\frac{s+a}{(s+a)^2 + b^2}$	$e^{-akT} \cos bkT$	$\frac{z(z - e^{-aT} \cos bT)}{z^2 - 2e^{-aT}(\cos bT)z + e^{-2aT}}$
21	$\frac{b}{(s+a)^2 + b^2}$	$e^{-akT} \sin bkT$	$\frac{ze^{-aT} \sin bT}{z^2 - 2e^{-aT}(\cos bT)z + e^{-2aT}}$
22	$\frac{a^2 + b^2}{s[(s+a)^2 + b^2]}$	$1 - e^{-akT} \left(\cos bkT + \frac{a}{b} \sin bkT \right)$	$\frac{z(Az + B)}{(z-1)[z^2 - 2e^{-aT}(\cos bT)z + e^{-2aT}]}$

$$A = 1 - e^{-aT} \cos bT - \frac{a}{b} e^{-aT} \sin bT$$

$$B = e^{-2aT} + \frac{a}{b} e^{-aT} \sin bT - e^{-aT} \cos bT$$

■ Properties of the z-transform:

1. Definition:

$$F(z) = \sum_{k=0}^{\infty} f[k]z^{-k}$$

2. Inversion:

$$f[k] = \frac{1}{2\pi i} \oint F(z)z^{k-1}dz$$

3. Linearity:

$$\mathcal{Z}\{af + bg\} = a\mathcal{Z}\{f\} + b\mathcal{Z}\{g\}$$

4. Time shift:

$$\mathcal{Z}\{q^{-n}f\} = z^{-n}F$$

$$\mathcal{Z}\{q^n f\} = z^n(F - F_1)$$

where $F_1(z) = \sum_{j=0}^{n-1} f[j]z^{-j}$

5 Initial-value theorem:

$$f[0] = \lim_{z \rightarrow \infty} F(z)$$

6 Final-value theorem:

If $(1 - z^{-1})F(z)$ does not have any poles on or outside the unit circle:

$$\lim_{k \rightarrow \infty} f[k] = \lim_{z \rightarrow 1} (1 - z^{-1})F(z)$$

7 Convolution:

$$\mathcal{Z}\{f * g\} = (\mathcal{Z}f)(\mathcal{Z}g)$$

$$f * g = \sum_{n=0}^k f[n]g[k-n]$$

Properties of the z-transform:

Section	Property	Signal	z-Transform	ROC
		$x[n]$	$X(z)$	R
		$x_1[n]$	$X_1(z)$	R_1
		$x_2[n]$	$X_2(z)$	R_2
10.5.1	Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least the intersection of R_1 and R_2
10.5.2	Time shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	R , except for the possible addition or deletion of the origin
10.5.3	Scaling in the z-domain	$e^{j\omega_0 n}x[n]$	$X(e^{-j\omega_0}z)$	R
		$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$z_0 R$
		$a^n x[n]$	$X(a^{-1}z)$	Scaled version of R (i.e., $ a R = \{ a z \}$ for z in R)
10.5.4	Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R (i.e., $R^{-1} = \{ z^{-1} \}$ where z is in R)
10.5.5	Time expansion	$x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer r	$X(z^k)$	$R^{1/k}$ (i.e., the set of points $z^{1/k}$, where z is in R)
10.5.6	Conjugation	$x^*[n]$	$X^*(z^*)$	R
10.5.7	Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of R_1 and R_2
10.5.7	First difference	$x[n] - x[n - 1]$	$(1 - z^{-1})X(z)$	At least the intersection of R and $ z > 0$
10.5.7	Accumulation	$\sum_{k=-\infty}^n x[k]$	$\frac{1}{1 - z^{-1}}X(z)$	At least the intersection of R and $ z > 1$
10.5.8	Differentiation in the z-domain	$nx[n]$	$-z \frac{dX(z)}{dz}$	R
10.5.9		Initial Value Theorem If $x[n] = 0$ for $n < 0$, then $x[0] = \lim_{z \rightarrow \infty} X(z)$		

- Example – Use z-Transform to find system response

- Consider the difference equation:

$$y[k] - \frac{1}{2}y[k-1] = u[k] + \frac{1}{3}u[k-1]$$

$$\Rightarrow Y(z) - \frac{1}{2}z^{-1}Y(z) = U(z) + \frac{1}{3}z^{-1}U(z)$$

$$\begin{aligned}\Rightarrow G(z) &= \frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}} &= \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{\frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}} \\ &= \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{\frac{1}{3}}{1 - \frac{1}{2}z^{-1}}z^{-1} \\ &= \frac{z + \frac{1}{3}}{z - \frac{1}{2}} &= \frac{z}{z - \frac{1}{2}} + \frac{\frac{1}{3}}{z - \frac{1}{2}} \\ &= \frac{z}{z - \frac{1}{2}} + \frac{\frac{1}{3}z}{z - \frac{1}{2}}z^{-1}\end{aligned}$$

■ Example – Use z-Transform to find system response

- If $u[k]$ = unit impulse function:

$$\Rightarrow U(z) = 1$$

$$\Rightarrow Y(z) = G(z)U(z)$$

$$\begin{aligned} &= \frac{z}{z - \frac{1}{2}} + \frac{\frac{1}{3}z}{z - \frac{1}{2}} z^{-1} \\ &= \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{\frac{1}{3}}{1 - \frac{1}{2}z^{-1}} z^{-1} \end{aligned}$$

$$\Rightarrow y[k] = \left(\frac{1}{2}\right)^k s[k] + \frac{1}{3} \left(\frac{1}{2}\right)^{k-1} s[k-1]$$

1. $\delta[n]$	$\frac{1}{1 - z^{-1}}$	All z
2. $u[n]$	$\frac{1}{1 - z^{-1}}$	$ z > 1$
3. $-u[-n - 1]$	$\frac{1}{1 - z^{-1}}$	$ z < 1$
4. $\delta[n - m]$	z^{-m}	All z , except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $\alpha^n u[n]$	$\frac{1}{1 - \alpha z^{-1}}$	$ z > \alpha $
6. $-\alpha^n u[-n - 1]$	$\frac{1}{1 - \alpha z^{-1}}$	$ z < \alpha $

$f(kT)$	$F(z)$
$1, k = 0; 0, k \neq 0$	1
$1, k = k_o; 0, k \neq k_o$	z^{-k_o}
$1(kT)$	$\frac{z}{z - 1}$
e^{-akT}	$\frac{z}{z - e^{-aT}}$

■ Example – Use z-Transform to find system response

- If $u[k]$ = unit step function:

$$\Rightarrow U(z) = \frac{z}{z-1}$$

$$\Rightarrow Y(z) = G(z)U(z)$$

$$= \frac{z + \frac{1}{3}}{z - \frac{1}{2}} \cdot \frac{z}{z-1}$$

$$= \frac{-\frac{5}{3}z}{z - \frac{1}{2}} + \frac{\frac{8}{3}z}{z-1}$$

$$= \left(-\frac{5}{3}\right) \frac{z}{z - \frac{1}{2}} + \left(\frac{8}{3}\right) \frac{z}{z-1}$$

$$\Rightarrow y[k] = \left(-\frac{5}{3}\right) \left(\frac{1}{2}\right)^k s[k] + \left(\frac{8}{3}\right) s[k]$$

1. $\delta[n]$	$\frac{1}{1-z^{-1}}$	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$ z > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$ z < 1$
4. $\delta[n-m]$	z^{-m}	All z , except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $\alpha^n u[n]$	$\frac{1}{1-\alpha z^{-1}}$	$ z > \alpha $
6. $-\alpha^n u[-n-1]$	$\frac{1}{1-\alpha z^{-1}}$	$ z < \alpha $

$f(kT)$	$F(z)$
$1, k = 0; 0, k \neq 0$	1
$1, k = k_o; 0, k \neq k_o$	z^{-k_o}
$1(kT)$	$\frac{z}{z-1}$
e^{-akT}	$\frac{z}{z - e^{-aT}}$

In Summary

■ Example – Use z-Transform to find system response

- Step Response:

$$\Rightarrow y_s[k] = \left(-\frac{5}{3}\right) \left(\frac{1}{2}\right)^k s[k] + \left(\frac{8}{3}\right) s[k]$$

- Impulse Response:

$$\Rightarrow y_i[k] = \left(\frac{1}{2}\right)^k s[k] + \frac{1}{3} \left(\frac{1}{2}\right)^{k-1} s[k-1]$$

k	y_s	y_i
0	1	1
1	11/6	5/6
2	27/12	5/12

■ Relationship between s and z:

$$f(t) = e^{-at}, \quad t > 0 \quad \Rightarrow \quad F(s) = \frac{1}{s + a}$$
$$\Rightarrow \text{Pole: } s = -a$$

$$f[kT] = e^{-a k T}, \quad k \in N \quad \Rightarrow \quad F(z) = \mathcal{Z} \left\{ e^{-a k T} \right\}$$
$$= \frac{z}{z - e^{-a T}}$$
$$\Rightarrow \text{Pole: } z = e^{-a T}$$

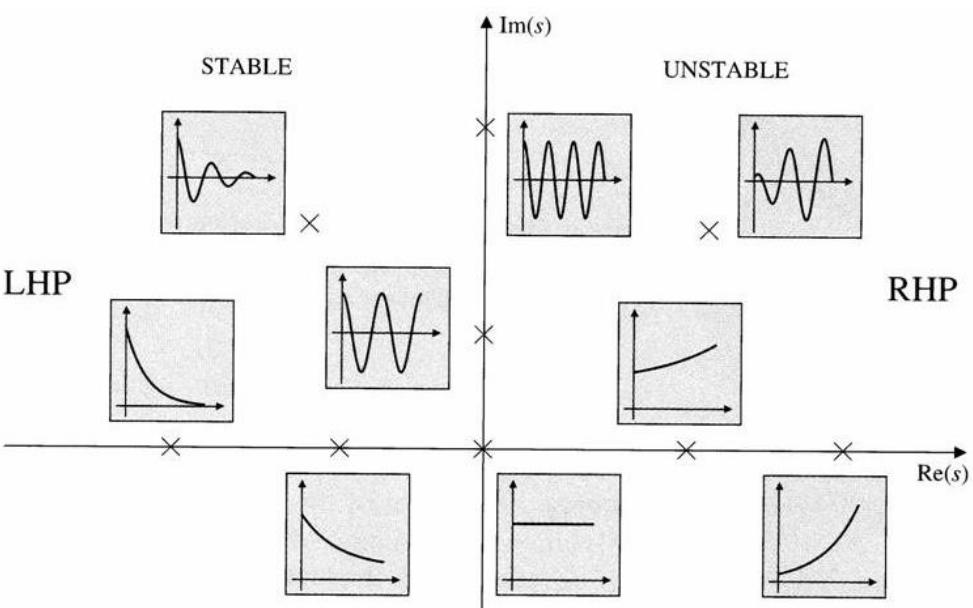
$$\Rightarrow z = e^{sT}$$

■ T: sampling period

Franklin et al. 2002

- Pole location and response between s and z:

s plane



z plane

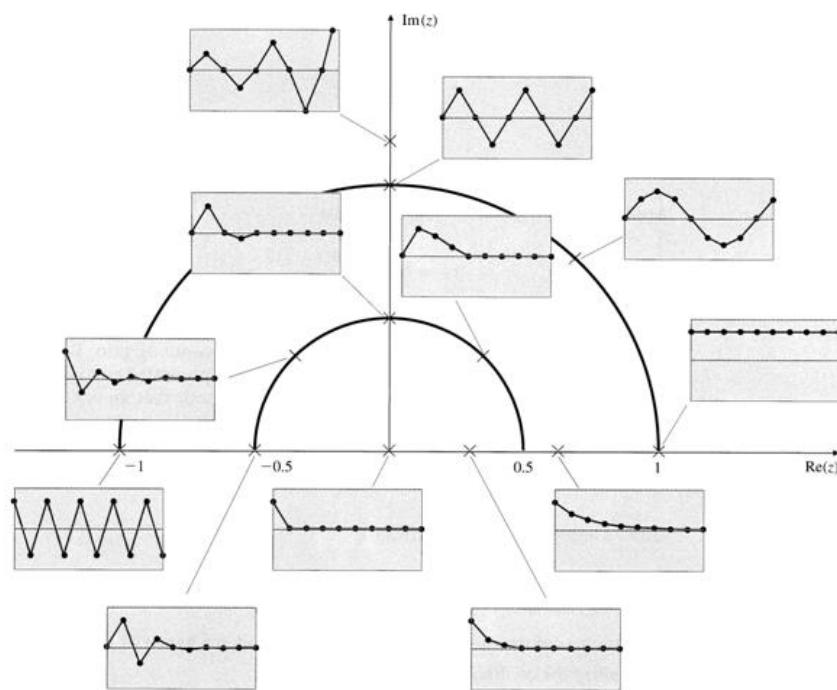
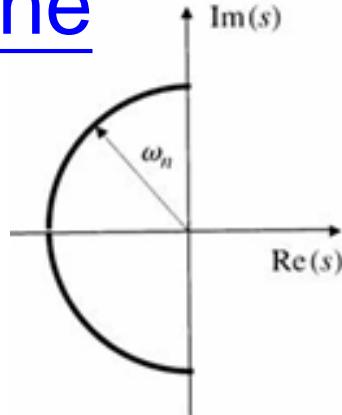
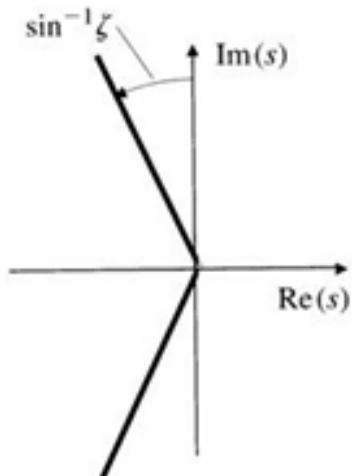


Figure 8.5 Time sequences associated with points in the z -plane

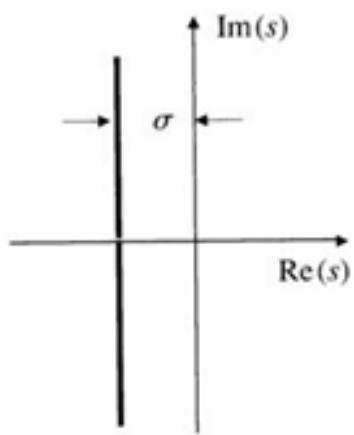
■ Dynamic Properties between s and z: s plane



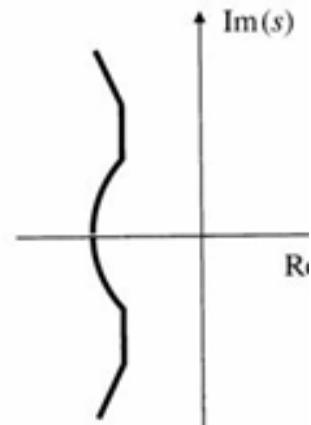
(a)



(b)



(c)



(d)

z plane

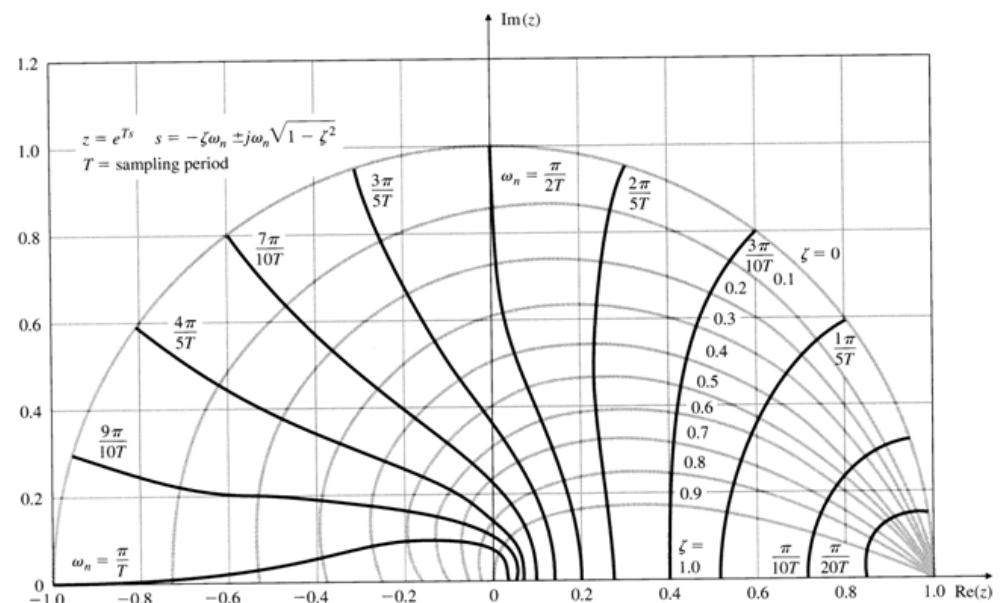
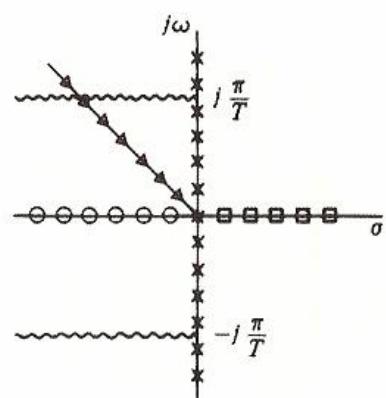


Figure 8.4 Natural frequency (solid color) and damping loci (light color) in the z-plane; the portion below the $\text{Re}(z)$ -axis (not shown) is the mirror image of the upper half shown.

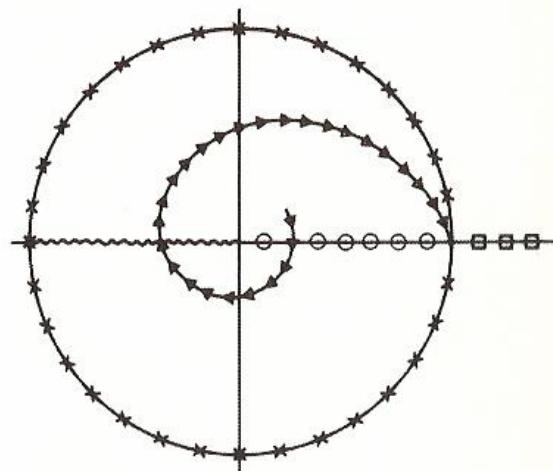
Franklin et al. 2002

Description of corresponding lines in s -plane and z -plane

s -plane	Symbol	z -plane
$s = j\omega$	$\times \times \times$	$ z = 1$
Real frequency axis		Unit circle
$s = \sigma \geq 0$	$\square \square \square$	$z = r \geq 1$
$s = \sigma \leq 0$	$\circ \circ \circ$	$z = r, 0 \leq r \leq 1$
$s = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$ $= -a + jb$ Constant damping ratio if ζ is fixed and ω_n varies	$\triangle \triangle \triangle$	$z = re^{j\theta}$ where $r = \exp(-\zeta\omega_n T)$ $\theta = \omega_n T \sqrt{1-\zeta^2}$ Logarithmic spiral
$s = \pm j(\pi/T) + \sigma,$	$\sigma \leq 0$	$z = -r$

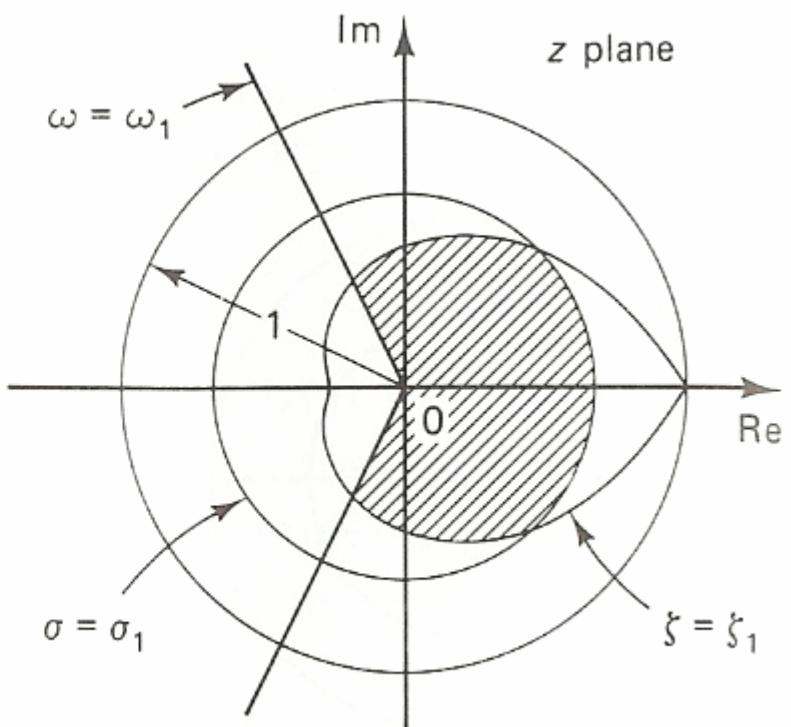
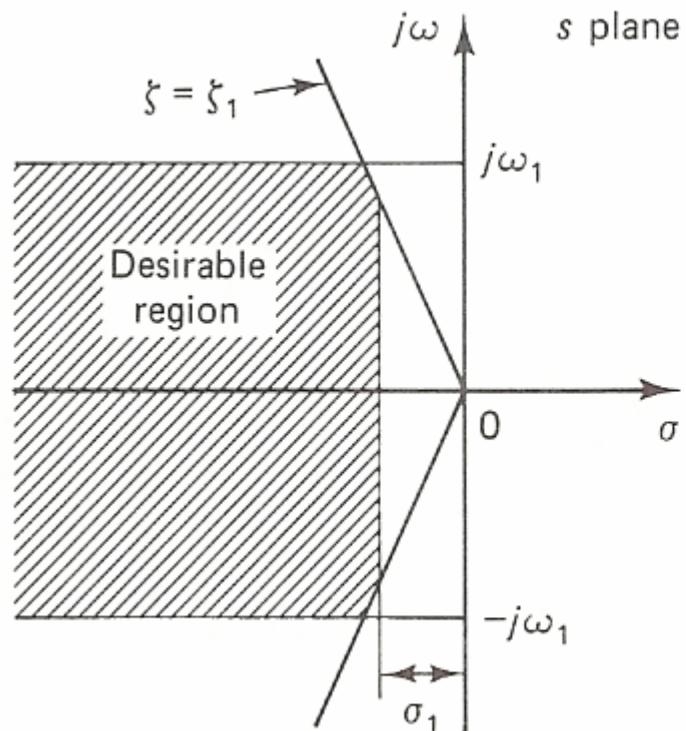


(a)
 s -plane



(b)
 z -plane

- Relationship between s and z :



- Pole Location and Impulse Responses:

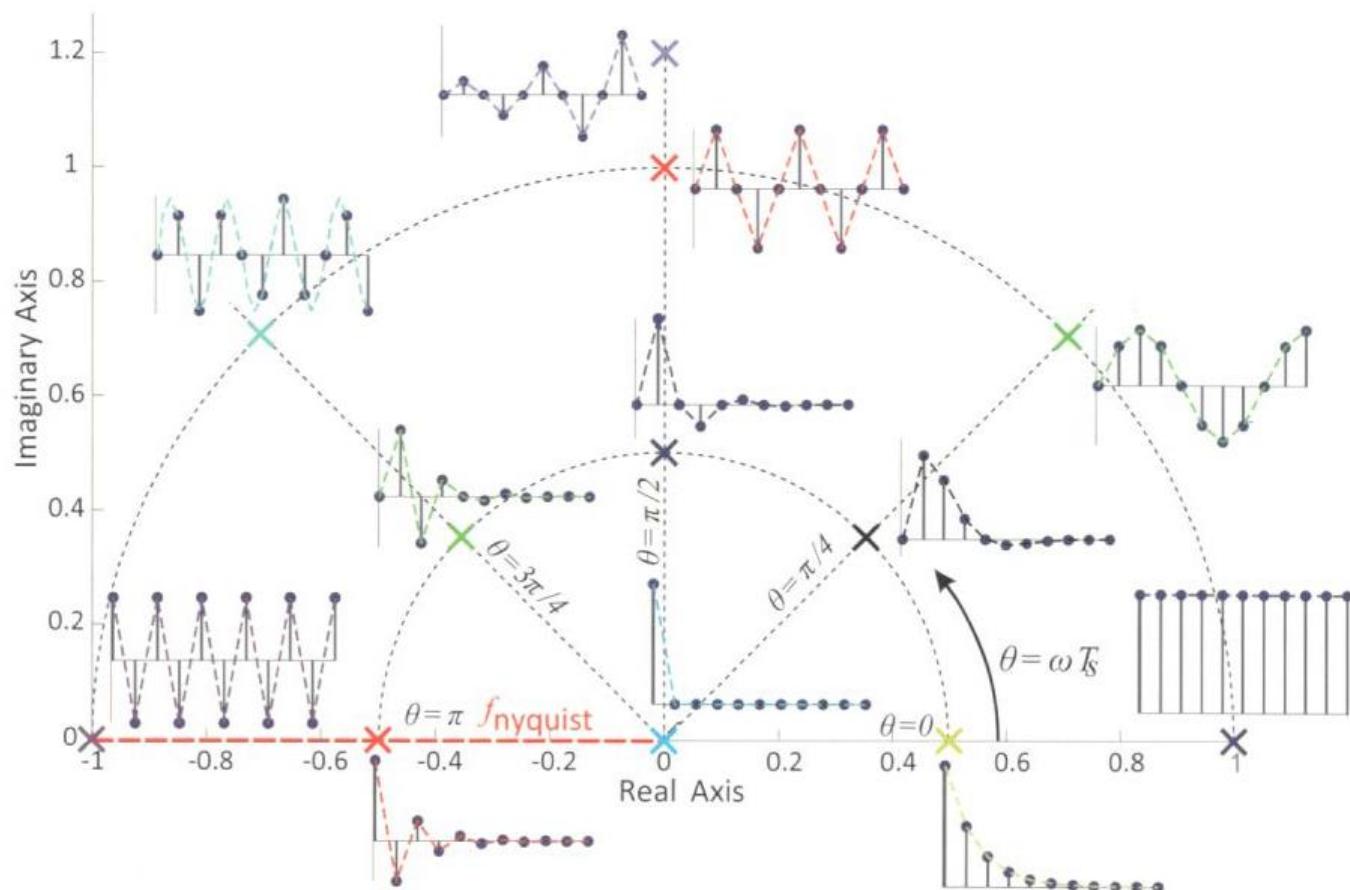


Figure 4.43: Pole map with related time-domain impulse responses, clearly indicating the discretised values of the sampled output signal.

- Final Value Theorem:

$$\lim_{t \rightarrow \infty} x(t) = x_{ss} = \lim_{s \rightarrow 0} s X(s)$$

If all the poles of $s X(s)$ are in LHP

$$\lim_{k \rightarrow \infty} x[k] = x_{ss} = \lim_{z \rightarrow 1} (1 - z^{-1}) X(z)$$

If all the poles of $(1 - z^{-1}) X(z)$
are inside the unit circle