

Spring 2021

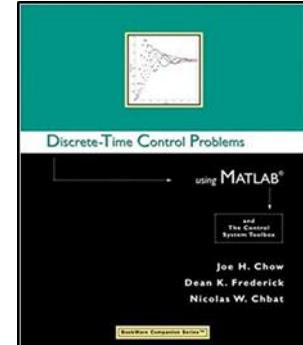
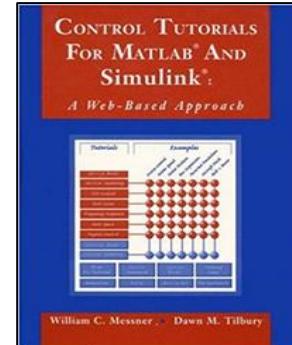
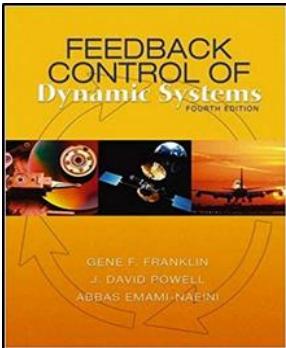
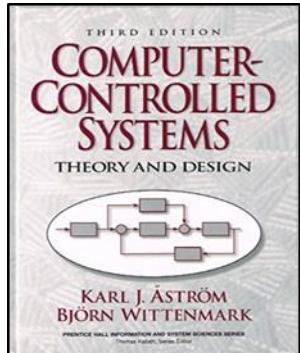
數位控制系統
Digital Control Systems

DCS-12
Discrete-Time Systems –
Input-Output (Transfer Function) Model

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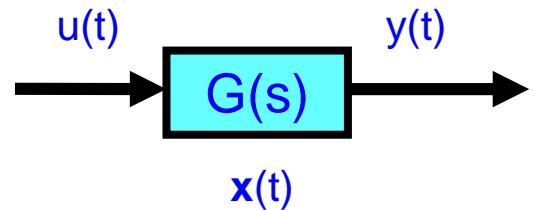
Feb – Jun, 2021



- Dynamic systems:

- Internal models

- State-space models



$$\begin{aligned}\frac{d}{dt}x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

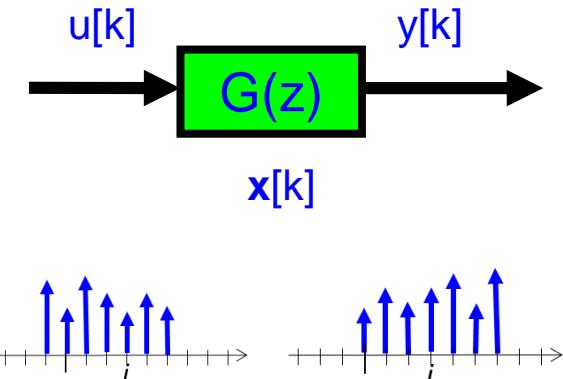
- External models:

- Input-output models
 - Pulse-response functions

$$Y(s) = G(s)U(s)$$

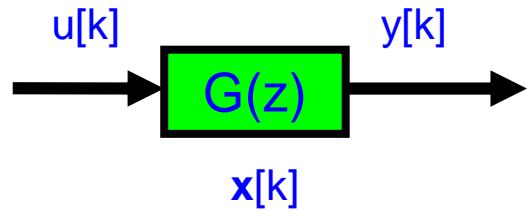
- The pulse response:

- Consider a discrete-time system with one input & one output
- Over a finite interval N , the input and output signals are:

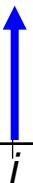


$$U = \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(N-1) \end{bmatrix} \quad Y = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N-1) \end{bmatrix}$$

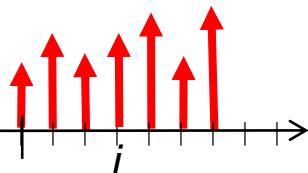
Input-Output Models: The Pulse Response



$u[i]$



$y[i]$



$\dots, y[i-2], y[i-1],$

$y[i+1], y[i+2], \dots$

$u[j]$



$y[j]$

$\dots, y[j-2], y[j-1],$

$y[j+1], y[j+2], \dots$

$\bar{g}[k, k-1] u[k-1]$

$\bar{g}[k, k+1] u[k+1]$

$y[k] = \bar{g}[k, k-2] u[k-2]$

$\bar{g}[k, k] u[k]$

$\bar{g}[k, k+2] u[k+2]$

\vdots

\vdots

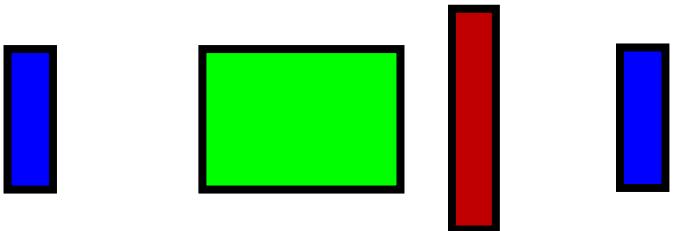
\vdots

\vdots

Input-Output Models: The Pulse Response

$$\begin{bmatrix} \vdots \\ \vdots \\ y[k] \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} \cdots & \bar{g}[k, k-1] & \bar{g}[k, k] & \bar{g}[k, k+1] & \cdots \end{bmatrix} \begin{bmatrix} \vdots \\ u[k-1] \\ u[k] \\ u[k+1] \\ \vdots \end{bmatrix}$$

$$\Rightarrow Y = \bar{G} U + Y_p$$



- $\bar{G} \in \mathcal{R}^{N \times N}$,
- $\bar{G} = [\bar{g}(k, m)]$
- Y_p for initial conditions

- $\bar{g}(k, m)$:
 - pulse-response or weighting function
 - gives the output at time k for a unit pulse at time m

Input-Output Models: The Pulse Response

- If $U \rightarrow Y$ is causal Current input affects current output or future output
BUT does not affect past output

$$\bar{g}[k, k-1] u[k-1]$$

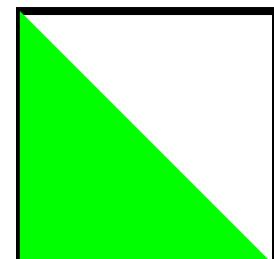
$$y[k] = \bar{g}[k, k-2] u[k-2] \quad \bar{g}[k, k] u[k]$$

⋮

⋮

$$\begin{bmatrix} \vdots \\ y[k] \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} \cdots & \bar{g}[k, k-1] & \bar{g}[k, k] & \bar{g}[k, k+1] & \cdots \end{bmatrix} \begin{bmatrix} u[k-1] \\ u[k] \\ u[k+1] \\ \vdots \end{bmatrix}$$

- \bar{G} : - lower triangular
- $g[k, m] = '0'$ if $m > k$.



$$\Rightarrow y[k] = \sum_{m=0}^k \bar{g}[k, m] u[m] + y_p[k]$$

Input-Output Models: The Pulse Response

- For time-invariant systems:
 $\Rightarrow \bar{g}[k, m] = g[k - m]$ Outputs of time-different inputs
only the function of relative time-difference,
but NOT at absolute time instants

$$\begin{bmatrix} & & & & & \vdots \\ \cdots & \bar{g}[k-1, k-2] & \bar{g}[k-1, k-1] & 0 & 0 & \cdots \\ \cdots & \bar{g}[k, k-2] & \bar{g}[k, k-1] & \bar{g}[k, k] & 0 & \cdots \\ & & & & \vdots & \\ & & & & \vdots & \end{bmatrix}$$

$$\begin{bmatrix} & & & & & \vdots \\ \cdots & \bar{g}[1] & \bar{g}[0] & 0 & 0 & \cdots \\ \cdots & \bar{g}[2] & \bar{g}[1] & \bar{g}[0] & 0 & \cdots \\ & & & & \vdots & \\ & & & & \vdots & \end{bmatrix}$$

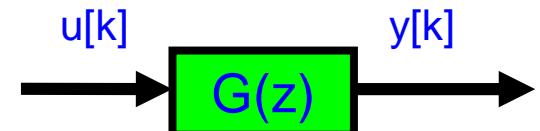
$$\Rightarrow y[k] = \sum_{m=0}^k \bar{g}[k-m]u[m] + y_p[k]$$

Input-Output Models: The Pulse Response

- Then, the state-space model:

$$\mathbf{x}[k+1] = \mathbf{F}\mathbf{x}[k] + \mathbf{H}u[k]$$

$$y[k] = \mathbf{C}\mathbf{x}[k] + \mathbf{D}u[k]$$



$$U = \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(N-1) \end{bmatrix} \quad Y = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N-1) \end{bmatrix}$$

$$\Rightarrow y[k] = \mathbf{C}\mathbf{F}^{k-k_0} \mathbf{x}[k_0] + \sum_{j=k_0}^{k-1} \mathbf{C}\mathbf{F}^{k-j-1} \mathbf{H}u[j] + \mathbf{D}u[k]$$

$$\Rightarrow y[k] = \sum_{m=0}^k \bar{g}[k-m]u[m] + y_p[k]$$

- The **pulse response** for the D.T. system:

$$\Rightarrow g[i] = \begin{cases} 0 & i < 0 \\ \mathbf{D} & i = 0 \\ \mathbf{C}\mathbf{F}^{i-1} \mathbf{H} & i \geq 1 \end{cases}$$

- All signals are considered as **doubly infinite sequences**:

$$\{f(k) : k = \dots, -1, 0, 1, \dots\}$$

Discrete-time signals
 $f[k] = f(k)$, $k = \text{integer}$

$$\{\dots, f(-3), f(-2), f(-1), f(0), f(1), f(2), f(3), \dots\}$$

- Forward-shift operator: q

$$qf(k) = f(k + 1)$$

- Backward-shift operator: q^{-1}
or delay operator:

$$q^{-1}f(k) = f(k - 1)$$

- Norm of a signal:

$$\|f\| = \sup_k |f(k)|$$

$$\text{or } \|f\|^2 = \sum_{k=-\infty}^{\infty} f^2(k)$$

⇒ Shift operator has unit norm

e.x. $\|\mathbf{q}f(k)\| = \|f(k+1)\|$

$$\|\mathbf{q}\| \|f(k)\| = \|f(k+1)\|$$

$$\|\mathbf{q}\| = 1$$

- A higher-order difference eqn:

$$\begin{aligned}y(k + n_a) + a_1 y(k + n_a - 1) + \cdots + a_{n_a} y(k) \\= b_0 u(k + n_b) + b_1 u(k + n_b - 1) + \cdots + b_{n_b} u(k)\end{aligned}$$

where $n_a \geq n_b$

- Use shift operator: $\textcolor{red}{q}$ $\textcolor{red}{q}f(k) = f(k + 1)$

$$\begin{aligned}\textcolor{red}{q}^{n_a} y(k) + a_1 \textcolor{red}{q}^{n_a-1} y(k) + \cdots + a_{n_a} y(k) \\= b_0 \textcolor{red}{q}^{n_b} u(k) + b_1 \textcolor{red}{q}^{n_b-1} u(k) + \cdots + b_{n_b} u(k)\end{aligned}$$

$$\begin{aligned}(\textcolor{red}{q}^{n_a} + a_1 \textcolor{red}{q}^{n_a-1} + \cdots + a_{n_a}) y(k) \\= (b_0 \textcolor{red}{q}^{n_b} + b_1 \textcolor{red}{q}^{n_b-1} + \cdots + b_{n_b}) u(k)\end{aligned}$$

- Let:

$$A(z) = z^{n_a} + a_1 z^{n_a-1} + \cdots + a_{n_a}$$

$$B(z) = b_0 z^{n_b} + b_1 z^{n_b-1} + \cdots + b_{n_b}$$

$$\Rightarrow A(q)y(k) = B(q)u(k)$$

- On the other hand, it is equivalent to write:

$$y(k) + a_1 y(k-1) + \cdots + a_{n_a} y(k-n_a)$$

$$= b_0 u(k-d) + \cdots + b_{n_b} u(k-d-n_b)$$

where $d = n_a - n_b$: pole excess of the system

- The reciprocal polynomial:

$$A(z) = z^{n_a} + a_1 z^{n_a-1} + \cdots + a_{n_a}$$

$$A^*(z) = 1 + a_1 z + \cdots + a_{n_a} z^{n_a} = z^{n_a} A(z^{-1})$$

$$\Rightarrow A^*(q^{-1})y(k) = B^*(q^{-1})u(k-d)$$

- Note that $A^{**} = (A^*)^* =$ or $\neq A$

e.x. $A(z) = z \quad A^*(z) = z z^{-1} = 1$

but, $(A^*(z))^* = 1 \neq A(z)$

- self-reciprocal: $A^*(z) = A(z)$

- Example :

$$y(k+1) - ay(k) = u(k) \quad |a| < 1$$

$$\Rightarrow q(y(k)) - ay(k) = u(k)$$

$$\Rightarrow (q - a)y(k) = u(k)$$

$$\Rightarrow y(k) = \frac{1}{(q - a)}u(k)$$

$$= \frac{q^{-1}}{(1 - q^{-1}a)}u(k) = q^{-1} \frac{1}{(1 - aq^{-1})}u(k)$$

$$= q^{-1} [1 + aq^{-1} + a^2q^{-2} + \dots] u(k)$$

$$= [q^{-1} + aq^{-2} + a^2q^{-3} + \dots] u(k)$$

$$= q^{-1}u(k) + aq^{-2}u(k) + a^2q^{-3}u(k) + \dots$$

$$= \sum_{i=1}^{\infty} a^{i-1}u(k-i) = u(k-1) + au(k-2) + a^2u(k-3) + \dots$$

$$y(k+1) = ay(k) + u(k)$$

IF $y(k_0) = y_0$

$$y(k_0 + 1) = ay(k_0) + u(k_0)$$

$$y(k_0 + 2) = ay(k_0 + 1) + u(k_0 + 1)$$

$$= a[ay(k_0) + u(k_0)] + u(k_0 + 1)$$

$$= a^2y(k_0) + au(k_0) + u(k_0 + 1)$$

$$y(k_0 + 3) = ay(k_0 + 2) + u(k_0 + 2)$$

$$= a^3y(k_0) + a^2u(k_0) + au(k_0 + 1) + u(k_0 + 2)$$

$$\Rightarrow y(k) = a^{k-k_0}y_0 + \sum_{j=k_0}^{k-1} a^{k-j-1}u(j)$$

$$= a^{k-k_0}y_0 + \sum_{i=1}^{k-k_0} a^{i-1}u(k-i)$$

- $\textcolor{red}{q}x(k) = x(k + 1)$

$$x(k + 1) = \mathbf{F}x(k) + \mathbf{H}u(k)$$

- Hence, $\textcolor{red}{q}x(k) = \mathbf{F}x(k) + \mathbf{H}u(k)$

that is, $(\textcolor{red}{q}\mathbf{I} - \mathbf{F})x(k) = \mathbf{H}u(k)$

or, $x(k) = (\textcolor{red}{q}\mathbf{I} - \mathbf{F})^{-1}\mathbf{H}u(k)$

- This gives

$$y(k) = \mathbf{C}x(k) + \mathbf{D}u(k) = (\mathbf{C}(\textcolor{red}{q}\mathbf{I} - \mathbf{F})^{-1}\mathbf{H} + \mathbf{D})u(k)$$

- The pulse-transfer operator:

$$G(q) = (\mathbf{C}(\textcolor{red}{q}\mathbf{I} - \mathbf{F})^{-1}\mathbf{H} + \mathbf{D}) = \frac{B(q)}{A(q)}$$

- If the system is of dimension n ,
- If $A(q), B(q)$ do not have common factors,
 $\Rightarrow A(q)$ is of degree n .

$$A(q) = q^n + a_1q^{n-1} + a_2q^{n-2} + \cdots + a_n$$

- Since $A(q)$ is also the characteristic polynomial of \mathbf{F} :
 $\Rightarrow y(k) + a_1y(k-1) + \cdots + a_ny(k-n)$
 $= b_0u(k) + b_1u(k-1) + \cdots + b_nu(k-n)$

- Example: Double integrator $G(s) = \frac{1}{s^2}$

$$\begin{aligned}\dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}\end{aligned}$$

$$\begin{aligned}\mathbf{F} &= \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \\ \mathbf{H} &= \begin{bmatrix} h^2/2 \\ h \end{bmatrix} \quad h = 1\end{aligned}$$

$$G(q) = \mathbf{C}(q\mathbf{I} - \mathbf{F})^{-1} \mathbf{H} + \mathbf{D} = \frac{B(q)}{A(q)}$$

$$\begin{aligned}G(q) &= [1 \ 0] \begin{bmatrix} q-1 & -1 \\ 0 & q-1 \end{bmatrix}^{-1} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \quad \frac{y[k]}{u[k]} = G(q) \\ &= \frac{0.5(q+1)}{(q-1)^2} \quad = \frac{0.5q+0.5}{q^2-2q+1} \quad = \frac{0.5(q^{-1}+q^{-2})}{1-2q^{-1}+q^{-2}}\end{aligned}$$

$$y[k+2] - 2y[k+1] + y[k] = 0.5u[k+1] + 0.5u[k]$$

$$y[k] - 2y[k-1] + y[k-2] = 0.5u[k-1] + 0.5u[k-2]$$

- Example: Double integrator with time delay

$$\mathbf{F} = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \quad \mathbf{H}_1 = \begin{bmatrix} d(h - d/2) \\ d \end{bmatrix} \quad \mathbf{H}_0 = \begin{bmatrix} (h - d^2)/2 \\ h - d \end{bmatrix}$$

$$h = 1, d = 0.5$$

▪ Homework 2-4

$$\begin{aligned}
 G(q) &= \mathbf{C}(q\mathbf{I} - \mathbf{F})^{-1}(\mathbf{H}_0 + \mathbf{H}_1 q^{-1}) & \frac{y[k]}{u[k]} &= G(q) \\
 &= [1 \ 0] \frac{\begin{bmatrix} q-1 & -1 \\ 0 & q-1 \end{bmatrix}}{(q-1)^2} \begin{bmatrix} 0.125 + 0.375q^{-1} \\ 0.5 + 0.5q^{-1} \end{bmatrix} \\
 &= \frac{0.125(q^2 + 6q + 1)}{q(q^2 - 2q + 1)} & &= \frac{0.125(q^{-1} + 6q^{-2} + q^{-3})}{1 - 2q^{-1} + q^{-2}}
 \end{aligned}$$

$$y[k+3] - 2y[k+2] + y[k+1] = 0.125(u[k+2] + 6u[k+1] + u[k])$$

$$y[k] - 2y[k-1] + y[k-2] = 0.125(u[k-1] + 6u[k-2] + u[k-3])$$

- The pulse-transfer operator:

$$G(q) = (\mathbf{C}(q\mathbf{I} - \mathbf{F})^{-1}\mathbf{H} + \mathbf{D}) = \frac{B(q)}{A(q)}$$

- Poles of a system: \Rightarrow roots of $A(q) = 0$
- Zeros of a system: \Rightarrow roots of $B(q) = 0$
- Time delay: \Rightarrow poles at the origin
- Order of a system:
 - \Rightarrow the dim of a state-space representation
 - \Rightarrow the number of poles of the system

The Table of Pulse-Transfer Operator

Table 2.1 Zero-order hold sampling of a continuous-time system, $G(s)$. The table gives the zero-order-hold equivalent of the continuous-time system, $G(s)$, preceded by a zero-order hold. The sampled system is described by its pulse-transfer operator. The pulse-transfer operator is given in terms of the coefficients of

$$H(q) = \frac{b_1 q^{n-1} + b_2 q^{n-2} + \cdots + b_n}{q^n + a_1 q^{n-1} + \cdots + a_n}$$

$G(s)$	$H(q)$ or the coefficients in $H(q)$
$\frac{1}{s}$	$\frac{h}{q - 1}$
$\frac{1}{s^2}$	$\frac{h^2(q + 1)}{2(q - 1)^2}$
$\frac{1}{s^m}$	$\frac{q - 1}{q} \lim_{a \rightarrow 0} \frac{(-1)^m}{m!} \frac{\partial^m}{\partial a^m} \left(\frac{q}{q - e^{-ah}} \right)$
e^{-sh}	q^{-1}
$\frac{a}{s + a}$	$\frac{1 - \exp(-ah)}{q - \exp(-ah)}$
$\frac{a}{s(s + a)}$	$b_1 = \frac{1}{a}(ah - 1 + e^{-ah}) \quad b_2 = \frac{1}{a}(1 - e^{-ah} - ah e^{-ah})$ $a_1 = -(1 + e^{-ah}) \quad a_2 = e^{-ah}$
$\frac{a^2}{(s + a)^2}$	$b_1 = 1 - e^{-ah}(1 + ah) \quad b_2 = e^{-ah}(e^{-ah} + ah - 1)$ $a_1 = -2e^{-ah} \quad a_2 = e^{-2ah}$
$\frac{s}{(s + a)^2}$	$\frac{(q - 1)he^{-ah}}{(q - e^{-ah})^2}$
$\frac{ab}{(s + a)(s + b)}$ $a \neq b$	$b_1 = \frac{b(1 - e^{-ah}) - a(1 - e^{-bh})}{b - a}$ $b_2 = \frac{a(1 - e^{-bh})e^{-ah} - b(1 - e^{-ah})e^{-bh}}{b - a}$ $a_1 = -(e^{-ah} + e^{-bh})$ $a_2 = e^{-(a+b)h}$

Table 2.1 *continued*

$G(s)$	$H(q)$ or the coefficients in $H(q)$
$\frac{(s + c)}{(s + a)(s + b)}$ $a \neq b$	$b_1 = \frac{e^{-bh} - e^{-ah} + (1 - e^{-bh})c/b - (1 - e^{-ah})c/a}{a - b}$ $b_2 = \frac{c}{ab} e^{-(a+b)h} + \frac{b - c}{b(a - b)} e^{-ah} + \frac{c - a}{a(a - b)} e^{-bh}$ $a_1 = -e^{-ah} - e^{-bh} \quad a_2 = e^{-(a+b)h}$
$\frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$	$b_1 = 1 - \alpha \left(\beta + \frac{\zeta\omega_0}{\omega} \gamma \right) \quad \omega = \omega_0 \sqrt{1 - \zeta^2} \quad \zeta < 1$ $b_2 = \alpha^2 + \alpha \left(\frac{\zeta\omega_0}{\omega} \gamma - \beta \right) \quad \alpha = e^{-\zeta\omega_0 h}$ $a_1 = -2\alpha\beta \quad \beta = \cos(\omega h)$ $a_2 = \alpha^2 \quad \gamma = \sin(\omega h)$
$\frac{s}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$	$b_1 = \frac{1}{\omega} e^{-\zeta\omega_0 h} \sin(\omega h) \quad b_2 = -b_1$ $a_1 = -2e^{-\zeta\omega_0 h} \cos(\omega h) \quad a_2 = e^{-2\zeta\omega_0 h}$ $\omega = \omega_0 \sqrt{1 - \zeta^2}$
$\frac{a^2}{s^2 + a^2}$	$b_1 = 1 - \cos ah \quad b_2 = 1 - \cos ah$ $a_1 = -2 \cos ah \quad a_2 = 1$
$\frac{s}{s^2 + a^2}$	$b_1 = \frac{1}{a} \sin ah \quad b_2 = -\frac{1}{a} \sin ah$ $a_1 = -2 \cos ah \quad a_2 = 1$
$\frac{a}{s^2(s + a)}$	$b_1 = \frac{1 - \alpha}{a^2} + h \left(\frac{h}{2} - \frac{1}{a} \right) \quad \alpha = e^{-ah}$ $b_2 = (1 - \alpha) \left(\frac{h^2}{2} - \frac{2}{a^2} \right) + \frac{h}{a} (1 + \alpha)$ $b_3 = - \left[\frac{1}{a^2} (\alpha - 1) + \alpha h \left(\frac{h}{2} + \frac{1}{a} \right) \right]$ $a_1 = -(\alpha + 2) \quad a_2 = 2\alpha + 1 \quad a_3 = -\alpha$