

Spring 2021

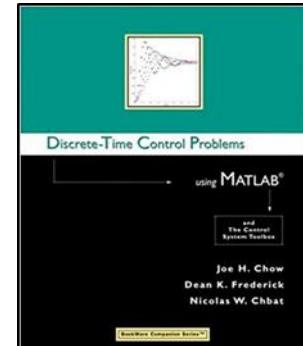
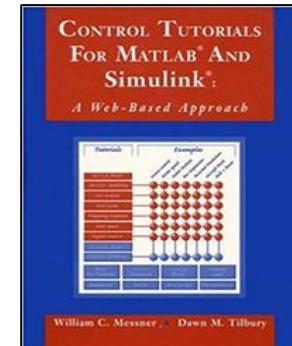
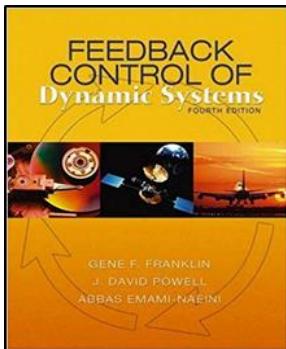
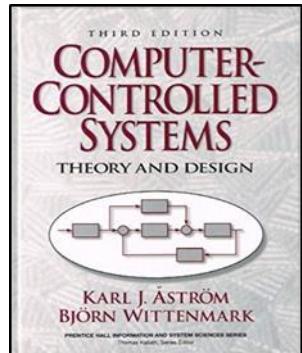
# 數位控制系統 Digital Control Systems

## DCS-11 Discrete-Time Systems – State Space Model

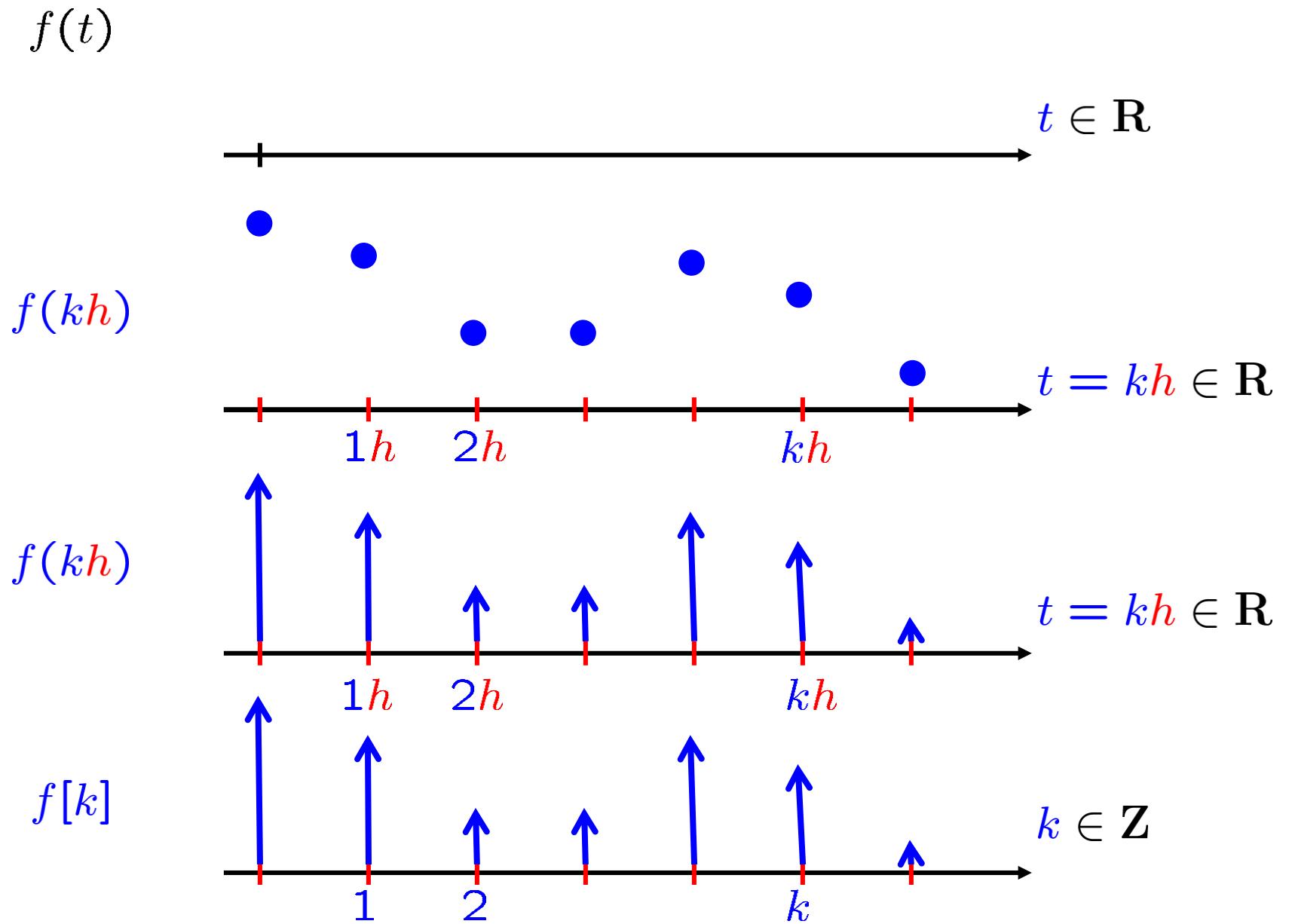
Feng-Li Lian

NTU-EE

Feb – Jun, 2021



# Sampling a CT Signal



# Sampling a CT Signal

$$\mathbf{Z} = \{\dots, -1, 0, 1, 2, \dots\}$$

- Sampling Period:  $h$  (sec)

- Sampling Rate:  $f_s = \frac{1}{h}$  (Hz)

- Sampling Frequency:  $w_s = \frac{2\pi}{h}$  (rad/s)

- Sampling Instants:  $t_k$

$$\mathbf{T} = \{t_k = kh, k \in \mathbf{Z}\}$$

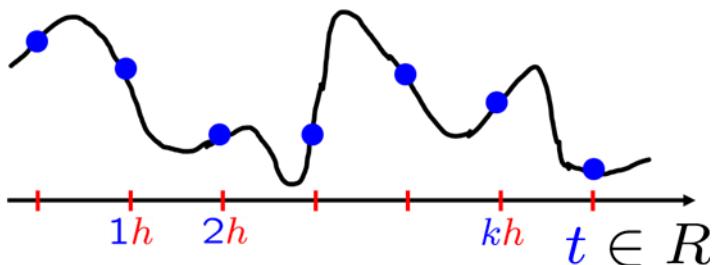
- CT Signals:  $f(t) : f(t) \in \mathbf{R}, t \in \mathbf{R}$

- DT Signals:  $f(t_k) : f(t_k) \in \mathbf{R}, t_k \in \mathbf{T}$

$$f(t_k) \in \mathbf{D}$$

- DT Signals:  $f[k] : f[k] \in \mathbf{R}, k \in \mathbf{Z}$

$$f[k] \in \mathbf{D}$$

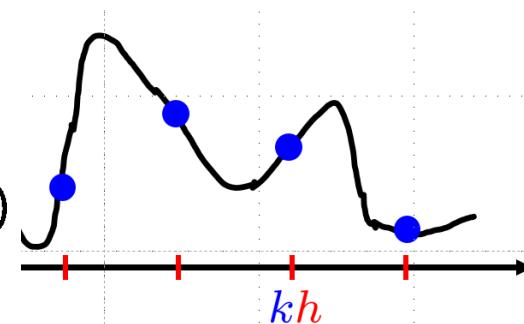


- Consider the following LTI system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

$$y(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t)$$

$$\dot{\mathbf{x}}(t) = \frac{d}{dt}\mathbf{x}(t)$$



$$\mathbf{x}(t) \in \mathbb{R}^n \quad \mathbf{A} \in \mathbb{R}^{n \times n}$$

$$\mathbf{u}(t) \in \mathbb{R}^r \quad \mathbf{B} \in \mathbb{R}^{n \times r}$$

$$\mathbf{y}(t) \in \mathbb{R}^p \quad \mathbf{C} \in \mathbb{R}^{p \times n}$$

$$\mathbf{D} \in \mathbb{R}^{p \times r}$$

For SISO system,  
 $r = 1, p = 1$

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

See: Solution of 1st-order DE

$$\Rightarrow \mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^t e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)d\tau$$

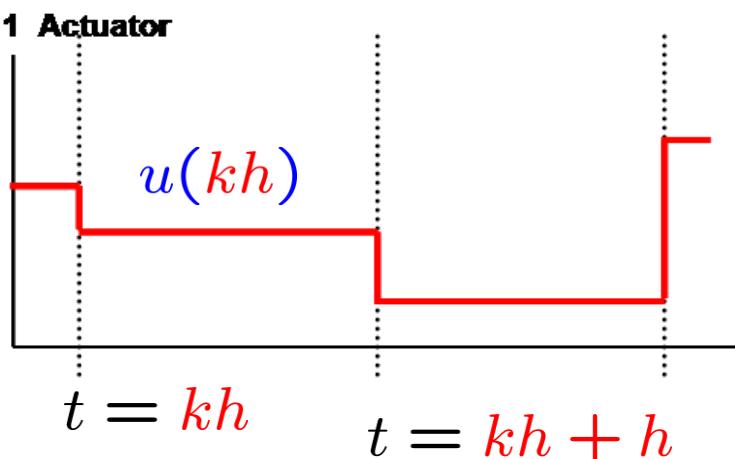
$$\text{Let } t = t_{k+1} = kh + h \quad \& \quad t_0 = t_k = kh$$

$$\Rightarrow \mathbf{x}(kh + h) = e^{\mathbf{A}(h)}\mathbf{x}(kh) + \int_{kh}^{kh+h} e^{\mathbf{A}(kh+h-\tau)}\mathbf{B}u(\tau)d\tau$$

Let  $u(\tau)$  be piecewise constant through  $h$

$$u(\tau) = u(kh), \quad kh \leq \tau < kh + h$$

$$\text{Let } \eta = kh + h - \tau$$



$$\Rightarrow \mathbf{x}(kh + h) = e^{\mathbf{A}h} \mathbf{x}(kh) + \left( \int_0^h e^{\mathbf{A}\eta} d\eta \right) \mathbf{B} u(kh)$$

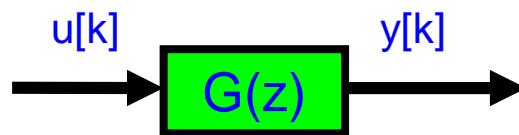
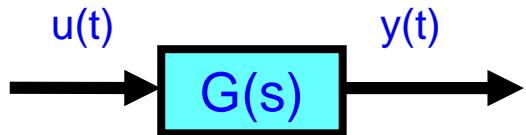
$$\text{Let } \mathbf{F} = e^{\mathbf{A}h} \quad \& \quad \mathbf{H} = \left( \int_0^h e^{\mathbf{A}\eta} d\eta \right) \mathbf{B}$$

$$\Rightarrow \mathbf{x}((k+1)h) = \mathbf{F} \mathbf{x}(kh) + \mathbf{H} u(kh)$$

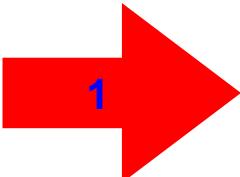
Then,

$$\begin{aligned} \mathbf{x}[k+1] &= \mathbf{F} \mathbf{x}[k] + \mathbf{H} u[k] \\ y[k] &= \mathbf{C} \mathbf{x}[k] + \mathbf{D} u[k] \end{aligned}$$

# Sampling a CT State-Space System: CT ( $A, B$ ) to DT ( $F, H$ )



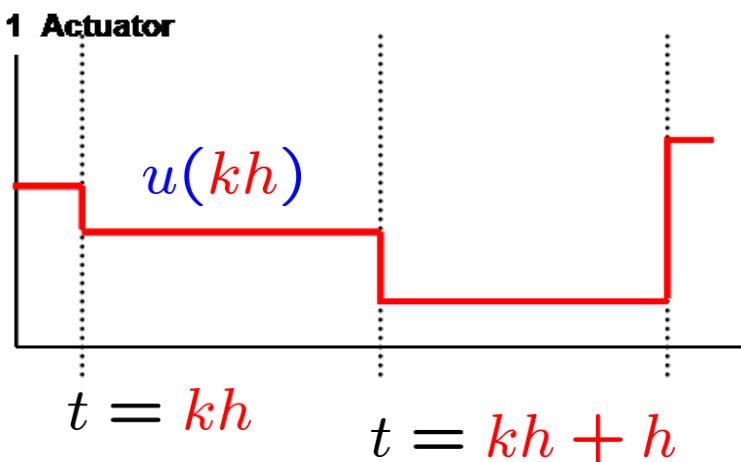
$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\ y(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t)\end{aligned}$$



$$\begin{aligned}\mathbf{x}[k+1] &= \mathbf{F}\mathbf{x}[k] + \mathbf{H}u[k] \\ y[k] &= \mathbf{C}\mathbf{x}[k] + \mathbf{D}u[k]\end{aligned}$$

- No approximation
- Control signal = Piecewise constant

$$\mathbf{F} = e^{\mathbf{A}h} \quad \& \quad \mathbf{H} = \left( \int_0^h e^{\mathbf{A}\eta} d\eta \right) \mathbf{B}$$



$$\Rightarrow \mathbf{F}(h) = e^{\mathbf{A}h} \quad \& \quad \mathbf{H}(h) = \left( \int_0^h e^{\mathbf{A}\eta} d\eta \right) \mathbf{B}$$

# Sampling a CT State-Space System: CT (A, B) to DT (F, H)

$$\Rightarrow F(h) = e^{Ah} \quad \& \quad H(h) = \left( \int_0^h e^{A\eta} d\eta \right) B$$

$$\begin{aligned} \Rightarrow \frac{d}{dh} F(h) &= \frac{d}{dh} (e^{Ah}) = A(e^{Ah}) = AF(h) \\ &\qquad\qquad\qquad = (e^{Ah})A = F(h)A \end{aligned}$$

$$\begin{aligned} \Rightarrow \frac{d}{dh} H(h) &= \frac{d}{dh} \left( \int_0^h e^{A\eta} d\eta \right) B \qquad\qquad\qquad MN \neq NM \\ &\qquad\qquad\qquad = (e^{Ah})B = F(h)B \qquad\qquad\qquad \neq BF(h) \end{aligned}$$

$$\Rightarrow \frac{d}{dh} \begin{bmatrix} F(h) & H(h) \\ 0 & I \end{bmatrix} = \begin{bmatrix} F(h) & H(h) \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$$

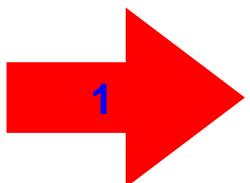
$$\Rightarrow \begin{bmatrix} F(h) & H(h) \\ 0 & I \end{bmatrix} = \exp \left( \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} h \right)$$

$$\Rightarrow \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} = \frac{1}{h} \ln \begin{bmatrix} F(h) & H(h) \\ 0 & I \end{bmatrix}$$

# Sampling a CT State-Space System: CT (A, B) to DT (F, H)

$$\dot{\mathbf{x}}(t) = \mathbf{Ax}(t) + \mathbf{Bu}(t)$$

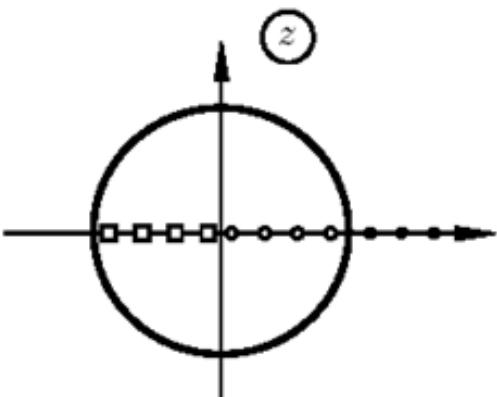
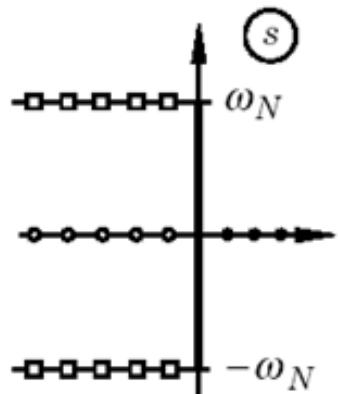
$$y(t) = \mathbf{Cx}(t) + \mathbf{Du}(t)$$



$$\mathbf{x}[k+1] = \mathbf{Fx}[k] + \mathbf{Hu}[k]$$

$$y[k] = \mathbf{Cx}[k] + \mathbf{Du}[k]$$

$$\Rightarrow \mathbf{F}(h) = e^{\mathbf{A}h} \quad \& \quad \mathbf{H}(h) = \left( \int_0^h e^{\mathbf{A}\eta} d\eta \right) \mathbf{B}$$



- Consider the following SS Model (Double Integrator):

$$\begin{aligned}\dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u & = \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x} & = \mathbf{C}\mathbf{x}\end{aligned}$$

$$\begin{aligned}\mathbf{F} &= e^{\mathbf{A}h} = \mathbf{I} + \mathbf{A}h + \frac{\mathbf{A}^2h^2}{2!} + \dots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} h + \frac{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 h^2}{2!} + \dots \\ &= \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}\end{aligned}$$

- Consider the following SS Model (Double Integrator):

$$\begin{aligned}\dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u & = \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x} & = \mathbf{C}\mathbf{x}\end{aligned}$$

$$\mathbf{H} = \left( \int_0^h e^{\mathbf{A}\eta} d\eta \right) \mathbf{B} \quad e^{\mathbf{A}\tau} = \mathbf{I} + \mathbf{A}\tau + \frac{\mathbf{A}^2\tau^2}{2!} + \dots$$

$$\sum_{k=0}^{\infty} \frac{\mathbf{A}^k h^{k+1}}{(k+1)!}$$

$$\mathbf{H} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k h^{k+1}}{(k+1)!} \mathbf{B} = \mathbf{I}h\mathbf{B} + \mathbf{A}\frac{h^2}{2!}\mathbf{B} + \mathbf{A}^2\frac{h^3}{3!}\mathbf{B} + \dots$$

$$= \left( \begin{bmatrix} h & 0 \\ 0 & h \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \frac{h^2}{2} \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} h^2/2 \\ h \end{bmatrix}$$

# Sampling a CT State-Space System with Time Delay

- Consider the LTI system with delayed input:

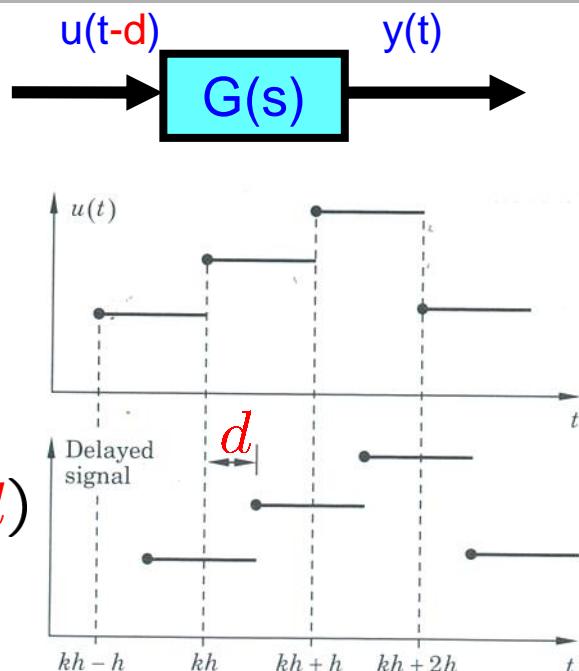
$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t-d)$$

$$y(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t-d)$$

$$\Rightarrow \mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0)$$

$$+ \int_{t_0}^t e^{\mathbf{A}(t-\tau)} \mathbf{B}u(\tau-d) d\tau$$

$$0 \leq d < h$$



$$\text{Let } t = t_{k+1} = kh + h \quad \& \quad t_0 = t_k = kh$$

$$\Rightarrow \mathbf{x}(kh+h) = e^{\mathbf{A}(h)}\mathbf{x}(kh) + \int_{kh}^{kh+h} e^{\mathbf{A}(kh+h-\tau)} \mathbf{B}u(\tau-d) d\tau$$

$$\int_{kh}^{kh+h} \Rightarrow \int_{kh}^{kh+d} + \int_{kh+d}^{kh+h}$$

$$+ \int_{kh}^{kh+d} e^{\mathbf{A}(kh+h-\tau)} \mathbf{B}u(\tau-d) d\tau + \int_{kh+d}^{kh+h} e^{\mathbf{A}(kh+h-\tau)} \mathbf{B}u(\tau-d) d\tau$$

# Sampling a CT State-Space System with Time Delay

$$\Rightarrow \mathbf{x}(kh + h)$$

$$= e^{\mathbf{A}(h)} \mathbf{x}(kh)$$

$$+ \int_{kh}^{kh+d} e^{\mathbf{A}(kh+h-\tau)} \mathbf{B} u(\tau-d) d\tau$$

$$u((k-1)h) = u[k-1]$$

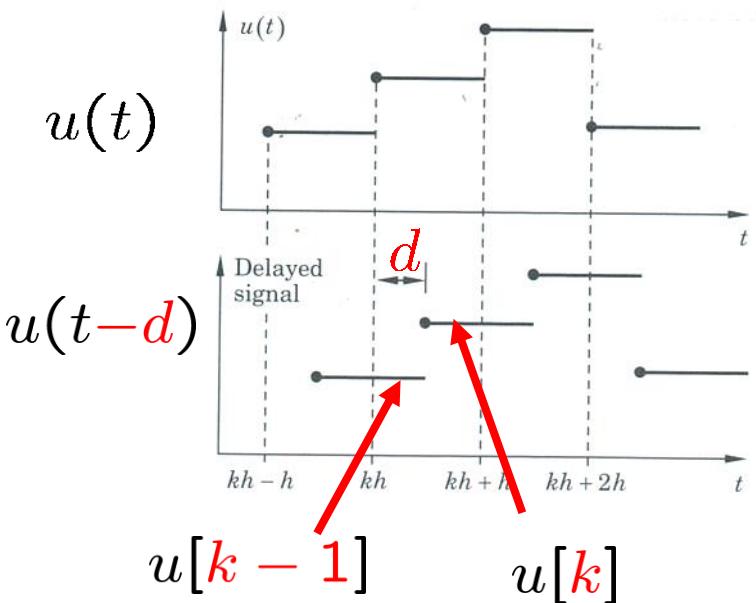
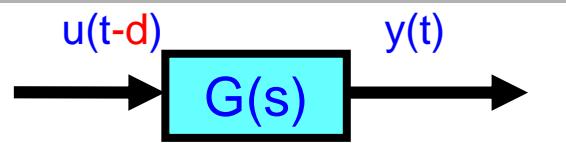
$$+ \int_{kh+d}^{kh+h} e^{\mathbf{A}(kh+h-\tau)} \mathbf{B} u(\tau-d) d\tau$$

$$u(kh) = u[k]$$

$$= \mathbf{F} \mathbf{x}(kh) + \mathbf{H}_1 u((k-1)h) + \mathbf{H}_0 u(kh)$$

$$\mathbf{H}_1 = \int_{kh}^{kh+d} e^{\mathbf{A}(kh+h-\tau)} \mathbf{B} d\tau = e^{\mathbf{A}(h-d)} \int_0^d e^{(\mathbf{A}\eta)} \mathbf{B} d\eta$$

$$\mathbf{H}_0 = \int_{kh+d}^{kh+h} e^{\mathbf{A}(kh+h-\tau)} \mathbf{B} d\tau = \int_0^{h-d} e^{(\mathbf{A}\eta)} \mathbf{B} d\eta$$



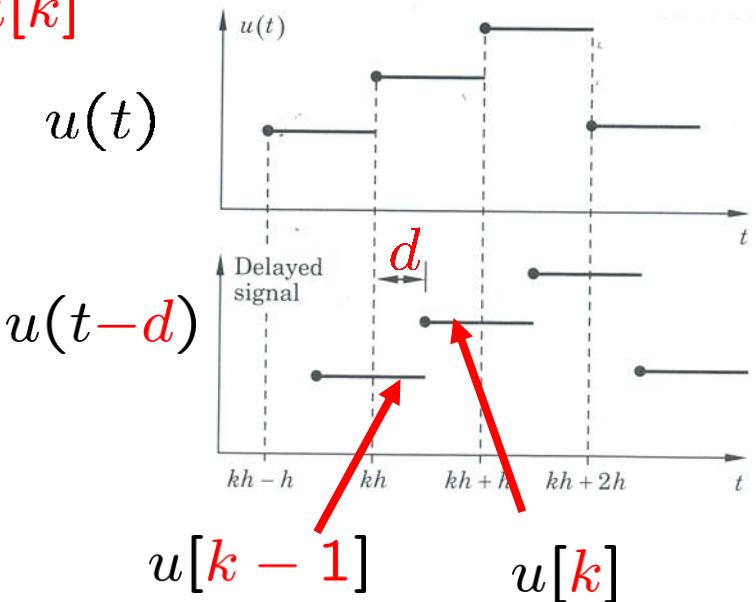
# Sampling a CT State-Space System with Time Delay

$$\Rightarrow \mathbf{x}[k+1]$$

$$= \mathbf{F}\mathbf{x}[k] + \mathbf{H}_1 u[k-1] + \mathbf{H}_0 u[k]$$

$$\mathbf{H}_1 = e^{\mathbf{A}(h-d)} \int_0^d e^{(\mathbf{A}\eta)} \mathbf{B} d\eta$$

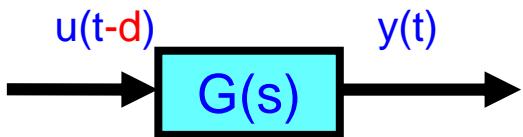
$$\mathbf{H}_0 = \int_0^{h-d} e^{(\mathbf{A}\eta)} \mathbf{B} d\eta$$



$$\begin{bmatrix} \mathbf{x}[k+1] \\ \mathbf{u}[k] \end{bmatrix} = \begin{bmatrix} \mathbf{F} & \mathbf{H}_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}[k] \\ \mathbf{u}[k-1] \end{bmatrix} + \begin{bmatrix} \mathbf{H}_0 \\ \mathbf{I} \end{bmatrix} \mathbf{u}[k]$$

# Sampling a CT State-Space System with Longer Time Delay

- Consider the LTI system with longer delayed input:



If  $d \geq h$

$$\text{Then, } d = (m - 1)h + \eta$$

$$0 \leq \eta < h$$

$$m \in N$$

$$\Rightarrow \mathbf{x}(kh + h) = \mathbf{F}\mathbf{x}(kh)$$

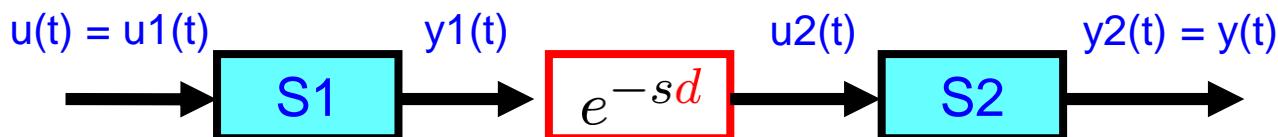
$$+ \mathbf{H}_1 u((kh - (m - 1)h)) + \mathbf{H}_0 u(kh - mh)$$

$$\mathbf{H}_1 = \int_{kh + (m - 1)h}^{kh + (m - 1)h + d} e^{\mathbf{A}(kh + h - \tau)} \mathbf{B} d\tau$$

$$\mathbf{H}_0 = \int_{kh + (m - 1)h + d}^{kh + (m - 1)h + h} e^{\mathbf{A}(kh + h - \tau)} \mathbf{B} d\tau$$

$$\begin{bmatrix} \mathbf{x}[k + 1] \\ \mathbf{u}[k - (m - 1)] \\ \mathbf{u}[k - m] \\ \vdots \\ \mathbf{u}[k] \end{bmatrix} = \begin{bmatrix} \mathbf{F} & \mathbf{H}_1 & \mathbf{H}_0 & 0 & \cdots & 0 \\ 0 & 0 & \mathbf{I} & 0 & \cdots & 0 \\ 0 & & & \ddots & & \vdots \\ 0 & & & & \ddots & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}[k] \\ \mathbf{u}[k - m] \\ \mathbf{u}[k - m - 1] \\ \vdots \\ \mathbf{u}[k - 1] \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \mathbf{I} \end{bmatrix} \mathbf{u}[k]$$

- Consider the LTI system with inner time delay:



$$\mathbf{S1}: \dot{\mathbf{x}}_1(t) = \mathbf{A}_1 \mathbf{x}_1(t) + \mathbf{B}_1 u_1(t)$$

$$y_1(t) = \mathbf{C}_1 \mathbf{x}_1(t) + \mathbf{D}_1 u_1(t)$$

$$\mathbf{S2}: \dot{\mathbf{x}}_2(t) = \mathbf{A}_2 \mathbf{x}_2(t) + \mathbf{B}_2 u_2(t)$$

$$y_2(t) = \mathbf{C}_2 \mathbf{x}_2(t) + \mathbf{D}_2 u_2(t) \quad u_2(t) = y_1(t - \bar{d})$$

- By using sampling interval,  $h$ , and  $0 < \bar{d} \leq h$
- Then, the sampled-data representation is as follows:

$$\mathbf{x}_1(kh + h) = \mathbf{F}_1(h) \mathbf{x}_1(kh) + \mathbf{H}_1(h) u(kh)$$

$$\mathbf{x}_2(kh + h) = \mathbf{F}_{21} \mathbf{x}_1(kh - h) + \mathbf{F}_2(h) \mathbf{x}_2(kh)$$

$$+ \mathbf{H}_{21} u(kh - h) + \mathbf{H}_2(h - \bar{d}) u(kh)$$

## ■ Where

$$F_i(\eta) = e^{\mathbf{A}_i \eta}, \quad i = 1, 2$$

$$F_{21}^a(\eta) = \int_0^\eta e^{\mathbf{A}_2 s} \mathbf{B}_2 \mathbf{C}_1 e^{\mathbf{A}_1(\eta-s)} ds$$

$$\mathbf{H}_1(\eta) = \int_0^\eta e^{\mathbf{A}_1 s} \mathbf{B}_1 ds$$

$$\mathbf{H}_2(\eta) = \int_0^\eta e^{\mathbf{A}_2 s} \mathbf{B}_2 \mathbf{C}_1 \mathbf{H}_1(\tau - s) ds$$

$$F_{21} = F_{21}^a(h) F_1(h - d)$$

$$H_{21} = F_{21}^a(h) H_1(h - d) + F_{21}^a(h - d) H_1(d)$$

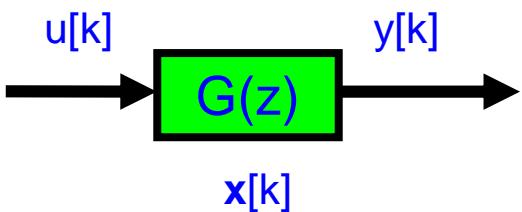
$$+ F_2(h - d) H_2(d)$$

## ■ Reference:

- Bjorn Wittenmark, "Sampling of a system with a time delay,"  
IEEE Transactions on Automatic Control, Vol. 30, No. 5, pp. 507-510, May 1985.
- <https://ieeexplore.ieee.org/document/1103985>

■ Homework 2-1

# Solution of DT State-Space System



$$\begin{aligned} \mathbf{x}[k+1] &= \mathbf{F}\mathbf{x}[k] + \mathbf{H}u[k] \\ y[k] &= \mathbf{C}\mathbf{x}[k] + \mathbf{D}u[k] \end{aligned}$$

$$\mathbf{x}[k_0 + 1] = \mathbf{F}\mathbf{x}[k_0] + \mathbf{H}u[k_0]$$

$$\begin{aligned} \mathbf{x}[k_0 + 2] &= \mathbf{F}\mathbf{x}[k_0 + 1] + \mathbf{H}u[k_0 + 1] \\ &= \mathbf{F}\{\mathbf{F}\mathbf{x}[k_0] + \mathbf{H}u[k_0]\} + \mathbf{H}u[k_0 + 1] \\ &= \mathbf{F}^2\mathbf{x}[k_0] + \mathbf{F}\mathbf{H}u[k_0] + \mathbf{H}u[k_0 + 1] \end{aligned}$$

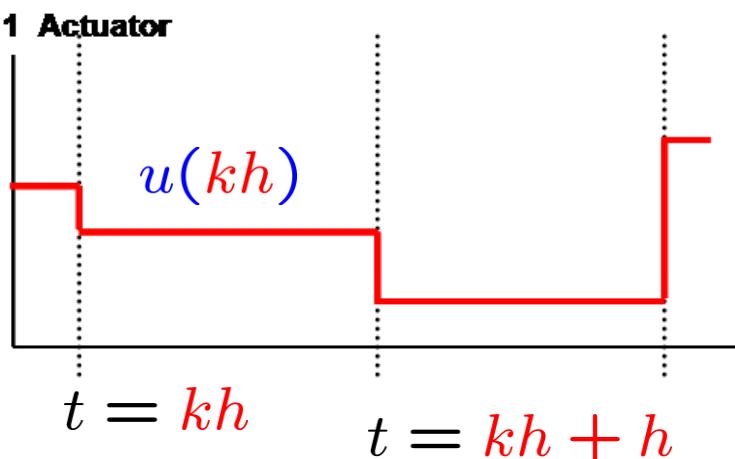
$$\mathbf{x}[k_0 + (k - k_0)]$$

$$\begin{aligned} \mathbf{x}[k] &= \mathbf{F}^{k-k_0}\mathbf{x}[k_0] + \mathbf{F}^{k-k_0-1}\mathbf{H}u[k_0] + \cdots + \mathbf{H}u[k-1] \\ &= \mathbf{F}^{k-k_0}\mathbf{x}[k_0] + \sum_{j=k_0}^{k-1} \mathbf{F}^{k-j-1}\mathbf{H}u[j] \\ y[k] &= \mathbf{C}\mathbf{F}^{k-k_0}\mathbf{x}[k_0] + \sum_{j=k_0}^{k-1} \mathbf{C}\mathbf{F}^{k-j-1}\mathbf{H}u[j] + \mathbf{D}u[k] \end{aligned}$$

Let  $u(\tau)$  be piecewise constant through  $h$

$$u(\tau) = u(kh), \quad kh \leq \tau < kh + h$$

$$\text{Let } \eta = kh + h - \tau$$



$$\Rightarrow \mathbf{x}(kh + h) = e^{\mathbf{A}h} \mathbf{x}(kh) + \left( \int_0^h e^{\mathbf{A}\eta} d\eta \right) \mathbf{B} u(kh)$$

$$\text{Let } \mathbf{F} = e^{\mathbf{A}h} \quad \& \quad \mathbf{H} = \left( \int_0^h e^{\mathbf{A}\eta} d\eta \right) \mathbf{B}$$

$$\Rightarrow \mathbf{x}((k+1)h) = \mathbf{F} \mathbf{x}(kh) + \mathbf{H} u(kh)$$

Then,

$$\begin{aligned} \mathbf{x}[k+1] &= \mathbf{F} \mathbf{x}[k] + \mathbf{H} u[k] \\ y[k] &= \mathbf{C} \mathbf{x}[k] + \mathbf{D} u[k] \end{aligned}$$