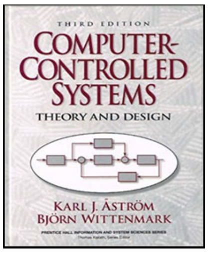
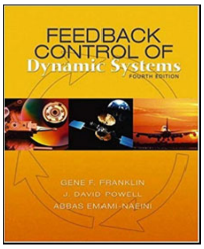
Spring 2019

數位控制系統
Digital Control Systems

DCS-22
Stability





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Feb19 – Jun19

Introduction: The Design Philosophy of Control Science

- The Research Procedure in Control Science

Physical Process → Math Model → System Analysis → Control Design

Physical Process:
- Plant
- Sensor
- Actuator
- Computer
- Communication
- Noise
- Disturbance

Math Model:
- Differential eqn
- Laplace transform
- Transfer function
- State space form

↓

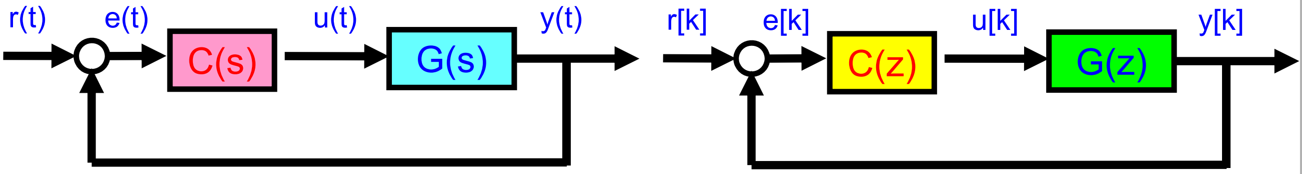
- Difference eqn
- z transform
- Transfer function
- State space form

System Analysis:
- Root locus
- Bode diagram
- Nyquist plot

- Stability
- Robustness
- Sensitivity
- Controllability
- Observability

Control Design:
- Estimator
- Identification
- Regulation
- Tracking
- PID
- Pole placement
- Optimal Control LQR/LQG
- Adaptive control
- Robust control
- Decentralized (or Multi-person) Control

Introduction: From CT Plant to DT Plant



- Plant (CT):
  - Input-Output Model:

$$u(t)$$
$$y(t)$$

$$\xrightarrow{2}$$

$$G(s) = \frac{Y(s)}{U(s)}$$

$$\xrightarrow{3}$$

  - State-Space Model:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\ y(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t) \end{aligned}$$

$$\xrightarrow{1}$$

- Plant (DT):
  - Input-Output Model:

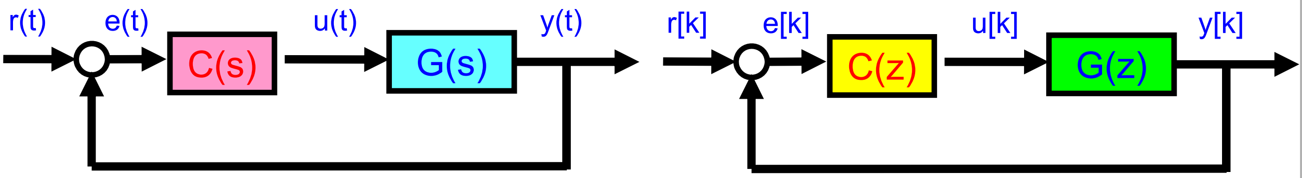
$$u[k]$$
$$y[k]$$

$$G(z) = \frac{Y(z)}{U(z)}$$

  - State-Space Model:

$$\begin{aligned} \mathbf{x}[k+1] &= \mathbf{F}\mathbf{x}[k] + \mathbf{H}u[k] \\ y[k] &= \mathbf{C}\mathbf{x}[k] + \mathbf{D}u[k] \end{aligned}$$

Introduction: Model and Analysis



- Plant (CT):
  - Input-Output Model:

$$\frac{Y_c(s)}{U_c(s)} = G_c(s) = \frac{B_c(s)}{A_c(s)}$$

  - State-Space Model:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\ y(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t) \end{aligned}$$

- Plant (DT):
  - Input-Output Model:

$$\frac{Y_d(z)}{U_d(z)} = G_d(z) = \boxed{\frac{B_d(z)}{A_d(z)}}$$

  - State-Space Model:

$$\begin{aligned} \mathbf{x}[k+1] &= \boxed{\mathbf{F}}\mathbf{x}[k] + \boxed{\mathbf{H}}u[k] \\ y[k] &= \boxed{\mathbf{C}}\mathbf{x}[k] + \boxed{\mathbf{D}}u[k] \end{aligned}$$

- System Properties:
  - Stability
  - Controllability and Reachability
  - Observability and Detectability

Outline

- Solution of a System
- Stability and Asymptotic Stability
- Input-Output Stability
- Stability Tests:
  - Jury's Stability Criterion
  - Nyquist and Bode Diagrams
  - Nyquist Criterion
  - Relative Stability
- Lyapunov's Second Stability

Solution of a System

- CT: $$\frac{d}{dt}x(t) = f(x(t), t)$$
- DT: $$x(k+1) = f(x(k), k)$$

• linear or nonlinear
• time-invariant or time-varying

- Initial Condition:

$$x_{10} = x_1(k_0)$$

$$x_{20} = x_2(k_0)$$

- Solution:

$$\begin{aligned} x_1(k) &= p_1(k, k_0, x_{10}) \\ x_2(k) &= p_2(k, k_0, x_{20}) \end{aligned}$$

▪ Definition 3.1: Stability

$x_1(k)$

. $x_2(k)$

- $x_1(k)$ is stable

  if for a given $\epsilon > 0$

  there exists a $\delta(\epsilon, k_0)$

  $x_1(k_0)$

  . $x_2(k_0)$

  such that
  all solutions with $||x_2(k_0) - x_1(k_0)|| < \delta$

$$\Rightarrow \quad ||x_2(k) - x_1(k)|| < \epsilon, \qquad \forall k \geq k_0$$

▪ Definition 3.2: Asymptotic Stability

$x_1(k)$

. $x_2(k)$

- $x_1(k)$ is asymptotic stable

  if it is stable , and

  if δ can be chosen

  $x_1(k_0)$

  . $x_2(k_0)$

  such that $||x_2(k_0) - x_1(k_0)|| < \delta$

$$\Rightarrow \quad ||x_2(k) - x_1(k)|| \rightarrow 0, \qquad \text{when } k \rightarrow \infty$$

Stability of Linear Discrete-Time Systems

$$x_1(k+1) = \mathbf{F}\, x_1(k), \quad x_1(0) = a_1$$

$$x_2(k+1) = \mathbf{F}\, x_2(k), \quad x_2(0) = a_2$$

$$\Rightarrow \tilde{x} = x_1 - x_2$$

$$\Rightarrow x_1(k+1) - x_2(k+1) = \mathbf{F}\, x_1(k) - \mathbf{F}\, x_2(k)$$

$$\Rightarrow \tilde{x}(k+1) = \mathbf{F}\, \tilde{x}(k), \qquad \tilde{x}(0) = a_1 - a_2$$

$\Rightarrow$ If x_1 is stable

$\Rightarrow$ every other solution is also stable

$\Rightarrow$ Hence, for LTI systems,
stability is a property of the system and
not of a special solution

Solution of LTI DT Systems

$$\tilde{x}(k+1) = \mathbf{F}\, \tilde{x}(k), \qquad \tilde{x}(0) = a_1 - a_2$$

$$\Rightarrow \tilde{x}(k) = \mathbf{F}^k\, \tilde{x}(0)$$

Let $\lambda_i = \text{eig}(\ \mathbf{F}\ )$

$$\mathbf{F} = \mathbf{U} \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \mathbf{U}^{-1}$$

$$\mathbf{F}^k = \mathbf{U} \begin{bmatrix} \lambda_1^k & & * \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix} \mathbf{U}^{-1}$$

Asymptotic stable $\Rightarrow$ $|\lambda_i| < 1, \ i = 1, \cdots, n$

▪ Theorem 3.1: Asymptotic Stability of Linear Systems

- A DT LTI system is asymptotic stable

  $\Leftrightarrow$ all eig($\mathbf{F}$) are strictly inside the unit disc

▪ Stability of Linear Continuous-Time Systems

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$$

$$\Rightarrow \mathbf{x}(t) = e^{(\mathbf{A}(t-t_0))}\,\mathbf{x}(t_0)$$

Let $\lambda_i = \text{eig}(\ \mathbf{A}\ )$ $\qquad \Rightarrow \mathbf{A} = \mathbf{U}\begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}\mathbf{U}^{-1}$

$$\Rightarrow \mathbf{x}(t) = \mathbf{U}\begin{bmatrix} e^{\lambda_1(t-t_0)} & & * \\ & \ddots & \\ 0 & & e^{\lambda_n(t-t_0)} \end{bmatrix}\mathbf{U}^{-1}\,\mathbf{x}(t_0)$$

Asymptotic stable $\Rightarrow$ $\quad \text{Real}(\lambda_i) < 0,\ i = 1, \cdots, n$

- Definition 3.3: Bounded-Input-Bounded-Output Stability

  - A LTI system is defined as BIBO stable

    if a bounded input gives a bounded output
    for every initial value

- Theorem 3.2: Relation between Stability Concept

  Asymptotic stable $\Rightarrow$ stable and BIBO stable

- Example 3.1: Harmonic Oscillator

$$
\begin{aligned}
x(k+1) &= \begin{bmatrix} \cos wh & \sin wh \\ -\sin wh & \cos wh \end{bmatrix} x(k) + \begin{bmatrix} 1-\cos wh \\ \sin wh \end{bmatrix} u(k) \\
y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(k)
\end{aligned}
$$

  - mag(eig($\mathbf{F}$)) $= 1$
  - if $u(k) = 0$ $\Rightarrow$ $||x(k+1)|| = ||x(0)||$

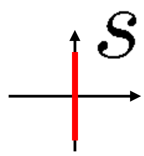
    $\Rightarrow$ the system is stable
  - But, if input is a cos or sin signal with $w\ rad/s$

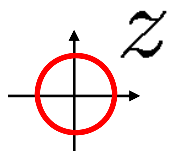
    $\Rightarrow$ the output contains a sinusoidal function
    with growing amplitude

    $\Rightarrow$ the system is not BIBO stable

Stability Test

- Eigenvalues of $\mathbf{F}$ $\qquad \lambda_i = \text{eig}(\mathbf{F})$
- Characteristic Polynomials $\qquad A(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$ $\qquad a_i \Leftrightarrow \lambda_i$
- Root locus method





- Nyquist criterion
- Lyapunov's method

Stability Test: Jury's Stability Criterion

(1918) (1922) (1961)

Schur-Cohn-Jury

$$A(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n = 0$$

$$\begin{array}{ccccc}
a_0 & a_1 & \cdots & a_{n-1} & a_n \\
a_n & a_{n-1} & \cdots & a_1 & a_0 \\
\hline
a_0^{n-1} & a_1^{n-1} & \cdots & a_{n-1}^{n-1} & \\
 & & & & \\
a_{n-1}^{n-1} & a_{n-2}^{n-1} & \cdots & a_0^{n-1} & \\
\hline
\vdots & & & & \\
a_0^0 & & & &
\end{array}
\qquad
\begin{array}{l}
\alpha_n = \dfrac{a_n}{a_0} \\
 \\
\alpha_{n-1} = \dfrac{a_{n-1}^{n-1}}{a_0^{n-1}}
\end{array}$$

$$a_i^{k-1} = a_i^k - \alpha_k a_{k-i}^k$$

$$\alpha_k = a_k^k / a_0^k$$

Stability Test: Routh's Stability Criterion (for CT)

- Routh in 1874
- Hurwitz in 1895

3.6.2 Routh's Stability Criterion

$$a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n. \quad (3.65)$$

A necessary (but not sufficient) condition for stability is that *all* the coefficients of the characteristic polynomial be positive.

A system is stable if and only if *all* the elements in the first column of the Routh array are positive.

We then add subsequent rows to complete the **Routh array**:

Row n	s^n:	1	a_2	a_4	$\cdots$
Row $n-1$	s^{n-1}:	a_1	a_3	a_5	$\cdots$
Row $n-2$	s^{n-2}:	b_1	b_2	b_3	$\cdots$
Row $n-3$	s^{n-3}:	c_1	c_2	c_3	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
Row 2	s^2:	$*$	$*$		
Row 1	s:	$*$			
Row 0	s^0:	$*$			

$$b_1 = -\frac{\det\begin{bmatrix} 1 & a_2 \\ a_1 & a_3 \end{bmatrix}}{a_1} = \frac{a_1 a_2 - a_3}{a_1},$$

$$b_2 = -\frac{\det\begin{bmatrix} 1 & a_4 \\ a_1 & a_5 \end{bmatrix}}{a_1} = \frac{a_1 a_4 - a_5}{a_1},$$

$$b_3 = -\frac{\det\begin{bmatrix} 1 & a_6 \\ a_1 & a_7 \end{bmatrix}}{a_1} = \frac{a_1 a_6 - a_7}{a_1},$$

$$c_1 = -\frac{\det\begin{bmatrix} a_1 & a_3 \\ b_1 & b_2 \end{bmatrix}}{b_1} = \frac{b_1 a_3 - a_1 b_2}{b_1},$$

$$c_2 = -\frac{\det\begin{bmatrix} a_1 & a_5 \\ b_1 & b_3 \end{bmatrix}}{b_1} = \frac{b_1 a_5 - a_1 b_3}{b_1},$$

$$c_3 = -\frac{\det\begin{bmatrix} a_1 & a_7 \\ b_1 & b_4 \end{bmatrix}}{b_1} = \frac{b_1 a_7 - a_1 b_4}{b_1}.$$

Franklin, Powell, Emami-Naeini 2002

Stability Test: Jury's Stability Criterion

- **Theorem 3.3:** Jury's Stability Test
  - If $a_0 > 0$,
    then, $A(z) = 0$ has all roots inside unit disc
    $\Longleftrightarrow$ all $a_0^k > 0$, $k = 0, 1, \cdots, n-1$
  - If no a_0 is zero,
    then, the number of negative a_0^k
    $=$ the number of roots outside the unit disc
  - Remark:
  - If all $a_0^k > 0$,
    then,

$$a_0^0 > 0 \Longleftrightarrow \begin{cases} A(1) > 0 \\ (-1)^n A(-1) > 0 \end{cases}$$

Stability Test: Jury's Stability Criterion

- Example: Jury's Stability Test

$$\alpha_n = \frac{a_n}{a_0} \qquad a_i^{k-1} = a_i^k - \alpha_k a_{k-i}^k \qquad \alpha_k = a_k^k / a_0^k$$

$$A(z) = z^2 + a_1 z + a_2$$

$$\begin{array}{ccc}
\boxed{1} & a_1 & a_2 \\
a_2 & a_1 & 1 \\
\hline
\boxed{1 - a_2^2} & a_1(1-a_2) & \\
a_1(1-a_2) & 1 - a_2^2 & \\
\hline
\boxed{1 - a_2^2 - \frac{a_1^2(1-a_2)}{1+a_2}} & & \\
\hline
\end{array}$$

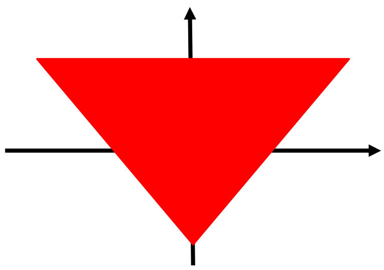
$$\alpha_2 = a_2$$

$$\alpha_1 = \frac{a_1}{1 + a_2}$$

All the roots are inside the unit circle if

$$\begin{aligned}
1 - a_2^2 &> 0 \\
1 - a_2^2 - \frac{a_1^2(1-a_2)}{1+a_2} &> 0
\end{aligned}$$

$$\Rightarrow \quad \begin{aligned}
a_2 &< 1 \\
a_2 &> -1 + a_1 \\
a_2 &> -1 - a_1
\end{aligned}$$



Stability Test: Nyquist and Bode Diagrams

- DT pulse-transfer function: $G(z)$
- Nyquist or Frequency curve

  $G(e^{jwh})$, for $wh \in [0, \pi]$

  upto to the Nyquist frequency, $w_N = \pi / h$

- Note that it is sufficient to consider the map in $wh \in [-\pi, \pi]$

- Because $G(e^{jwh})$ is periodic with period $2\pi / h$

- Example 3.3: Frequency Responses

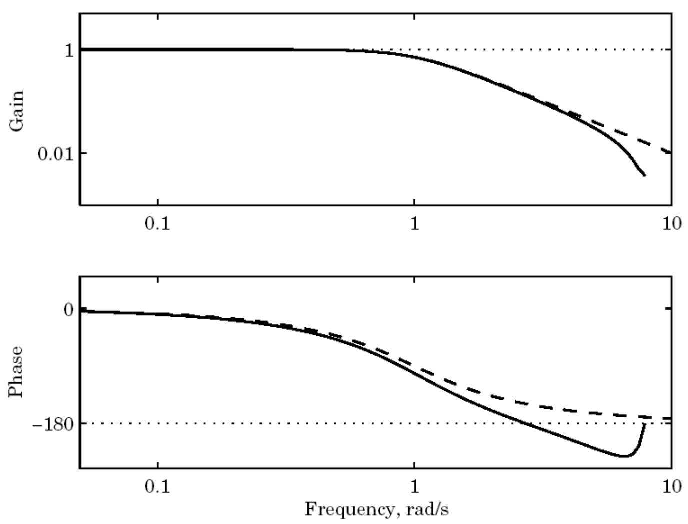
$$G(s) = \frac{1}{s^2 + 1.4s + 1}$$

Zero-order hold sampling $h = 0.4$

$$G(z) = \frac{0.066z + 0.055}{z^2 - 1.450z + 0.571}$$

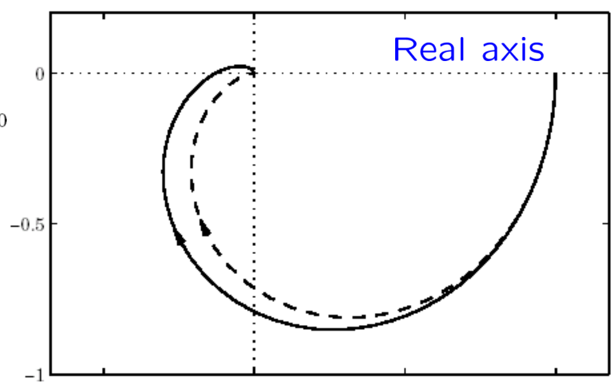
- Example 3.3: Frequency Responses

Bode Diagram



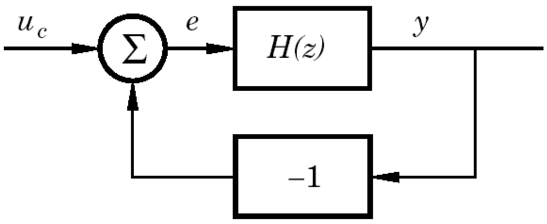
- CT: - - -
- DT: —

Imaginary axis



Nyquist Diagram

- Nyquist Criterion



Closed-loop system

$$Y(z) = H_{cl}(z)U_c(z) = \frac{H(z)}{1+H(z)}\,U_c(z)$$

Closed-loop system characteristic equation

$$1 + H(z) = 0$$

Franklin, Powell, Emami-Naeini 2002

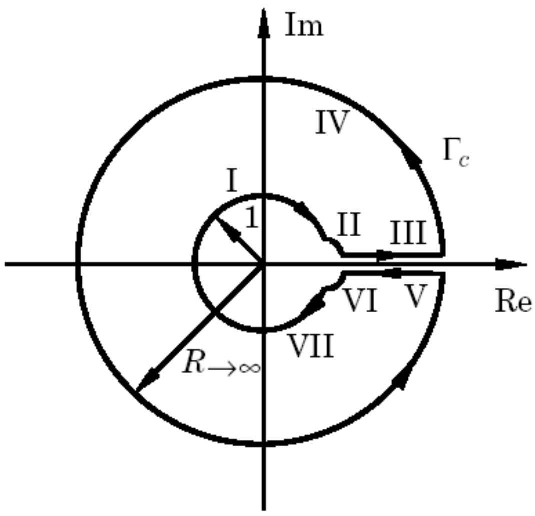
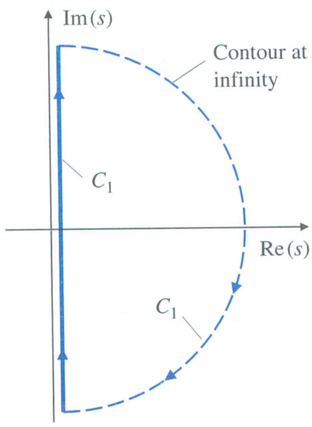


Figure 6.17
An s-plane plot of a contour C_1 that encircles the entire RHP



Principle of arguments states

$$N = Z - P$$

Z and P are the number of zeros and poles of $1 + H(z)$ outside the unit disc.

- Example 3.4: A Second-order system

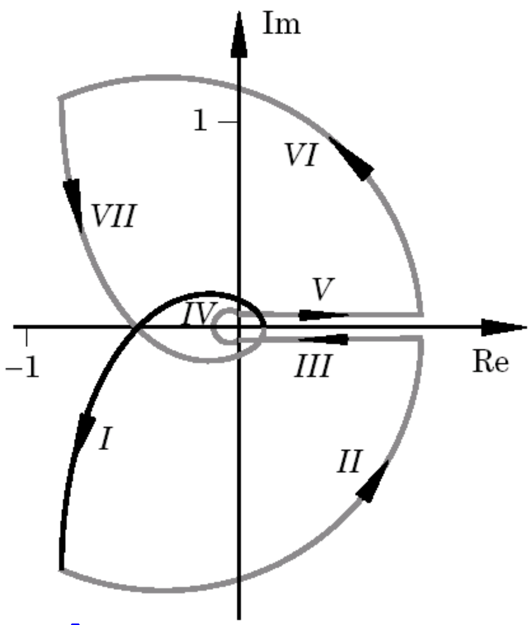
$$h = 1$$

$$H(z) = \frac{0.25K}{(z-1)(z-0.5)}$$

then

$$H(e^{i\omega}) = \frac{0.25K\left(1.5(1-\cos\omega) - 2\sin^2\omega - i\sin\omega(2\cos\omega - 1.5)\right)}{(2-2\cos\omega)(1.25+\cos\omega)}$$

- Example 3.4: A Second-order system



$H(e^{jw})$, for $w \in [0, \pi]$

- At some w, phase shift $> 180^0$
- Stable if $K > 2$

Definitions 3.4 & 3.5: Gain & Phase Margins

- The amplitude or gain margin:

$$\arg G(e^{iw_0 h}) = -\pi \qquad A_{\text{marg}} = \frac{1}{|G(e^{iw_0 h})|}$$

- The phase margin:

$$|G(e^{iw_c h})| = 1 \qquad \phi_{\text{marg}} = \pi + \arg G(e^{iw_c h})$$

Definition 3.6: Lyapunov Function

- $V(x)$ is a Lyapunov function for

$$x(k+1) = f(x(k)) \quad f(0) = 0$$

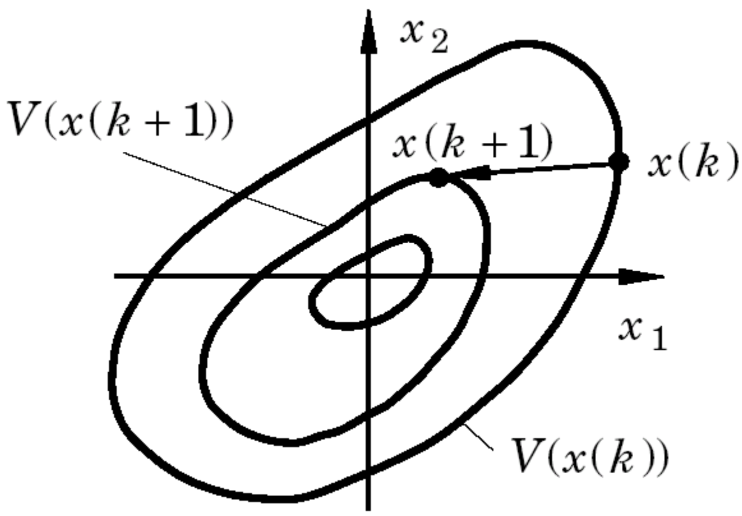
- If:
  1. $V(x)$ is continuous in x and $V(0) = 0$
  2. $V(x)$ is positive definite
  3. $\triangle V(x) = V(f(x)) - V(x)$ is negative definite
  4. $V(x) \rightarrow \infty$ as $|x| \rightarrow \infty$

- Existence of Lyapunov function implies asymptotic stability for the solution $x = 0$

Geometric Illustration

1. $V(x)$ is continuous in x and $V(0)=0$
2. $V(x)$ is positive definite
3. $\triangle V(x) = V(f(x)) - V(x)$ is negative definite
4. $V(x) \to \infty$ as $|x| \to \infty$

$$x(k+1) = f(x(k)), \qquad f(0) = 0$$



Example 3.6: Lyapunov function

1. $V(x)$ is continuous in x and $V(0)=0$
2. $V(x)$ is positive definite
3. $\triangle V(x) = V(f(x)) - V(x)$ is negative definite
4. $V(x) \to \infty$ as $|x| \to \infty$

$$\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k)$$

$$V(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x} \qquad \mathbf{P} > 0$$

$$\begin{aligned}\Delta V(\mathbf{x}) &= V(\mathbf{x}(k+1)) - V(\mathbf{x}(k)) \\ &= V(\mathbf{F}\mathbf{x}(k)) - V(\mathbf{x}(k)) \\ &= (\mathbf{F}\mathbf{x}(k))^T\mathbf{P}(\mathbf{F}\mathbf{x}(k)) \quad -\mathbf{x}^T\mathbf{P}\mathbf{x} \\ &= \mathbf{x}^T\mathbf{F}^T\mathbf{P}\mathbf{F}\mathbf{x} \quad -\mathbf{x}^T\mathbf{P}\mathbf{x} \\ &= \mathbf{x}^T(\mathbf{F}^T\mathbf{P}\mathbf{F} - \mathbf{P})\mathbf{x} \quad = \mathbf{x}^T(-\mathbf{Q})\mathbf{x} \quad = -\mathbf{x}^T(\mathbf{Q})\mathbf{x}\end{aligned}$$

V is a Lyapunov function

iff there exists a $P > 0$ $\qquad \mathbf{F}^T\mathbf{P}\mathbf{F} - \mathbf{P} = -\mathbf{Q} \qquad \mathbf{Q} > 0$

that satisfies the *Lyapunov equation*

- Example 3.6: Lyapunov function for CT case

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$$

$$V(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x} \qquad \mathbf{P} > 0$$

$$\begin{aligned}\dot{V}(\mathbf{x}) &= \mathbf{x}^T\mathbf{P}\dot{\mathbf{x}} + \dot{\mathbf{x}}^T\mathbf{P}\mathbf{x}\\ &= \mathbf{x}^T\mathbf{P}\mathbf{A}\mathbf{x} + (\mathbf{A}\mathbf{x})^T\mathbf{P}\mathbf{x}\\ &= \mathbf{x}^T\mathbf{P}\mathbf{A}\mathbf{x} + \mathbf{x}^T\mathbf{A}^T\mathbf{P}\mathbf{x}\\ &= \mathbf{x}^T(\mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P})\mathbf{x}\\ &= \mathbf{x}^T(-\mathbf{Q})\mathbf{x}\\ &= -\mathbf{x}^T(\mathbf{Q})\mathbf{x}\end{aligned}$$

$$\mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P} = -\mathbf{Q}$$

Khalil 2002

- Example 3.6: Lyapunov function

$$\Phi^T P \Phi - P = -Q \qquad Q > 0$$

$$\Phi = \begin{bmatrix} 0.4 & 0 \\ -0.4 & 0.6 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \Rightarrow P = \begin{bmatrix} 1.19 & -0.25 \\ -0.25 & 2.05 \end{bmatrix}$$