Spring 2019

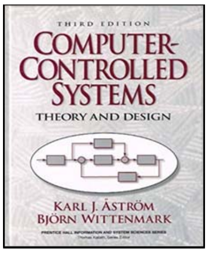
# 數位控制系統
# Digital Control Systems

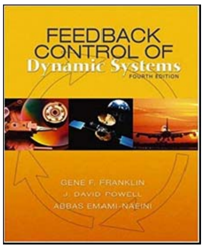
# DCS-13
# z Transform

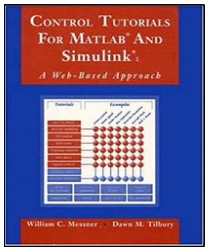

Feng-Li Lian

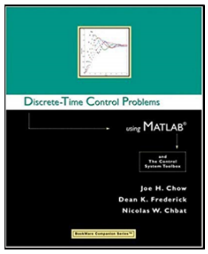
NTU-EE


Feb19 – Jun19




## The z-Transform



- Continuous-time systems: $\Rightarrow$ Laplace transform
- Discrete-time systems: $\Rightarrow$ z transform
- z transform maps a semi-infinite time sequence into a function of a complex variable
- Range of z-transform and operator calculus
  - operator calculus: $\Rightarrow \{f(k) : k = \cdots, -1, 0, 1, \cdots\}$
  - z transform: $\Rightarrow \{f(k) : k = 0, 1, 2, \cdots\}$

$\Rightarrow$ also take the initial values into consideration

- Definition: z-transform
- Consider the discrete-time signal

$$\{f(k) : k = 0, 1, 2, \cdots\}$$

- The z transform of $f(k)$ is defined as:

$$\mathcal{Z}\{f(k)\} = F(z) \quad = \sum_{k=0}^{\infty} f(k) z^{-k}$$

where $z$ is a complex variable.

- The inverse z-transform is given by

$$f(k) = \frac{1}{2\pi i} \oint F(z) z^{k-1} dz$$

where the contour of integration encloses the singularities of $F(z)$.



- Example: Transform of a ramp

$$\mathcal{Z}\{f(k)\} = F(z) = \sum_{k=0}^{\infty} f(k) z^{-k}$$

- Consider a ramp signal:

$$y(k) = kh \text{ for } k \geq 0$$

$$\Rightarrow Y(z) = y(0)z^0 + y(1)z^{-1} + y(2)z^{-2} + \cdots$$

$$= 0 + hz^{-1} + 2hz^{-2} + \cdots$$

$$= h(z^{-1} + 2z^{-2} + \cdots)$$

$$= \frac{hz}{(z-1)^2}$$

## The z-Transform: Table



**Table 2.3** Some time functions and corresponding Laplace and $z$-transforms. Warning: Use the table only as prescribed!

| $f$ | $\mathcal{L}f$ | $\mathcal{Z}f$ |
|---|---|---|
| $\delta(k)$ (pulse) | – | $1$ |
| $1 \quad k \geq 0$ (step) | $\frac{1}{s}$ | $\frac{z}{z-1}$ |
| $kh$ | $\frac{1}{s^2}$ | $\frac{hz}{(z-1)^2}$ |
| $\frac{1}{2}(kh)^2$ | $\frac{1}{s^3}$ | $\frac{h^2 z(z+1)}{2(z-1)^3}$ |
| $e^{-kh/T}$ | $\frac{T}{1+sT}$ | $\frac{z}{z-e^{-h/T}}$ |
| $1-e^{-kh/T}$ | $\frac{1}{s(1+sT)}$ | $\frac{z(1-e^{-h/T})}{(z-1)(z-e^{-h/T})}$ |
| $\sin \omega kh$ | $\frac{\omega}{s^2+\omega^2}$ | $\frac{z \sin \omega h}{z^2 - 2z\cos \omega h + 1}$ |

Astrom & Wittenmark 1997



## The z-Transform: Properties

1. Definition:

$$F(z) = \sum_{k=0}^{\infty} f(k) z^{-k}$$

2. Inversion:

$$f(k) = \frac{1}{2\pi i} \oint F(z) z^{k-1} dz$$

3. Linearity:

$$\mathcal{Z}\{af + bg\} = \mathcal{Z}af + \mathcal{Z}bg$$

4. Time shift:

$$\mathcal{Z}\{q^{-n} f\} = z^{-n} F$$

$$\mathcal{Z}\{q^{n} f\} = z^{n}(F - F_1)$$

where $F_1(z) = \sum_{j=0}^{n-1} f(j) z^{-j}$

5 Initial-value theorem:

$$f(0) = \lim_{z \to \infty} F(z)$$

6 Final-value theorem:

If $(1 - z^{-1})F(z)$ does not have any poles on or outside the unit circle:

$$\lim_{k \to \infty} f(k) = \lim_{z \to 1} (1 - z^{-1}) F(z)$$

7 Convolution:

$$\mathcal{Z}\{f * g\} = (\mathcal{Z}f)(\mathcal{Z}g)$$

$$f * g = \sum_{n=0}^{k} f(n) g(k-n)$$

Astrom & Wittenmark 1997

## The z-Transform: From State Space to Pulse Transfer Function

$$\begin{cases} x(k+1) = \mathbf{F}x(k) + \mathbf{H}u(k) \\ y(k) = \mathbf{C}x(k) + \mathbf{D}u(k) \end{cases}$$

$$\begin{cases} z\left(X(z) - x(0)\right) = \mathbf{F}X(z) + \mathbf{H}U(z) \\ Y(z) = \mathbf{C}X(z) + \mathbf{D}U(z) \end{cases}$$

$$\Rightarrow \quad zX(z) - \mathbf{F}X(z) = zx(0) + \mathbf{H}U(z)$$

$$\Rightarrow \quad (z\mathbf{I} - \mathbf{F})X(z) = zx(0) + \mathbf{H}U(z)$$

$$\Rightarrow \quad X(z) = (z\mathbf{I} - \mathbf{F})^{-1}[zx(0) + \mathbf{H}U(z)]$$

$$\Rightarrow \quad Y(z) = \mathbf{C}(z\mathbf{I} - \mathbf{F})^{-1}[zx(0) + \mathbf{H}U(z)] + \mathbf{D}U(z)$$

$$\Rightarrow \quad Y(z) = \mathbf{C}(z\mathbf{I} - \mathbf{F})^{-1}zx(0) + \left(\mathbf{C}(z\mathbf{I} - \mathbf{F})^{-1}\mathbf{H} + \mathbf{D}\right)U(z)$$

$$\Rightarrow \quad G(z) = \mathbf{C}(z\mathbf{I} - \mathbf{F})^{-1}\mathbf{H} + \mathbf{D}$$

## The z-Transform: From State Space to Pulse Transfer Function



$$y(k) + a_1 y(k-1) + \cdots + a_n y(k-n) = b_0 u(k) + b_1 u(k-1) + \cdots + b_n u(k-n)$$

- Take z-transform of both sides:

$$\underbrace{[z^n + a_1 z^{n-1} + \cdots + a_n]}_{A(z)} Y(z) = \underbrace{[b_0 z^n b_1 z^{n-1} + b_2 z^{n-2} + \cdots + b_n]}_{B(z)} U(z)$$

$\Rightarrow \dfrac{B(z)}{A(z)}$ is the pulse transfer function: $u \rightarrow y$

- The corresponding state-space form:

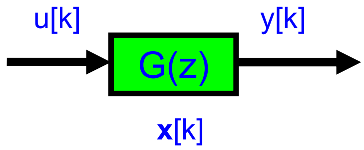
$$x(k+1) = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & \cdots & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u(k)$$

$$y(k) = [b_1 \; b_2 \; \cdots \; b_n]\, x(k)$$

## The z-Transform: From State Space to Pulse Transfer Function



- Theorem 2.5:




$$U = \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(N-1) \end{bmatrix} \quad Y = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N-1) \end{bmatrix}$$

- The pulse response $g(k)$ and the pulse-transfer function $G(z)$ are a $z$-transform pair, that is, $\mathcal{Z}\{g(k)\} = G(z)$

- The Pulse Response for the D.T. system:

$$g(k) = \begin{cases} 0 & k < 0 \\ \mathbf{D} & k = 0 \\ \mathbf{C}\mathbf{F}^{k-1}\mathbf{H} & k \geq 1 \end{cases}$$

- Pulse Transfer Function

$$G(z) = \mathbf{C}(z\mathbf{I} - \mathbf{F})^{-1}\mathbf{H} + \mathbf{D}$$

## The z-Transform: Computation of G(z) from G(s)



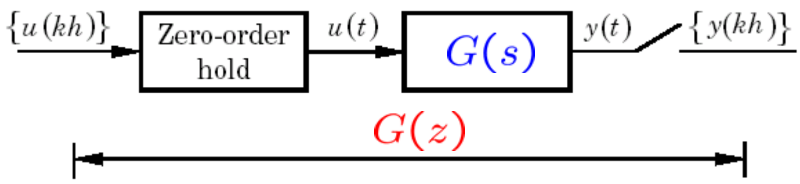
- Method 1:
  - Make state-space realization
  - Calculate $\mathbf{F}$ and $\mathbf{H}$
  - Find $G(z)$

$$\begin{cases} x(k+1) = \mathbf{F}x(k) + \mathbf{H}u(k) \\ y(k) = \mathbf{C}x(k) + \mathbf{D}u(k) \end{cases}$$

$$\Rightarrow \quad G(z) = \mathbf{C}(z\mathbf{I} - \mathbf{F})^{-1}\mathbf{H} + \mathbf{D}$$

## The z-Transform: Computation of G(z) from G(s)

- Method 2:




  - Step response of $G(s)$
  - $z$-transform of $\frac{G(s)}{s}$
  - Divide by the $z$-transform of step function

$$\Rightarrow \; Y(s) = \frac{G(s)}{s}$$

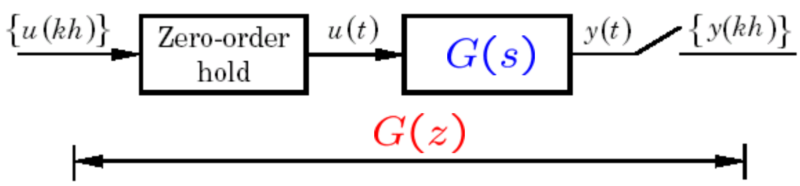
$$\Rightarrow \; \hat{Y}(z) = \mathcal{Z}\{\mathcal{L}^{-1} Y(s)\}$$

$$\Rightarrow \; G(z) = (1 - z^{-1}) \hat{Y}(z)$$



## The z-Transform: Computation of H(z) from G(s)

- Method 2:




$$\Rightarrow \; G(z) = \frac{z-1}{z} \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{sh}}{z - e^{sh}} \frac{G(s)}{s} \, ds$$

- If $G(s)$ goes to zero at least as fast as $|s|^{-1}$ for large $s$
- and $G(s)$ has distinct poles, none are at the origin

$$\Rightarrow \; G(z) = \sum_{s=s_i} \frac{1}{z - e^{sh}} \; \text{Res} \left\{ \frac{e^{sh} - 1}{s} \right\} G(s)$$

where $s_i$ are the poles of $G(s)$
and Res denotes the residue.

- Example:
  - Consider the difference equation:

$$y[k+1] + ay[k] = u[k+1] + au[k]$$

  - By using z-transform, Its pulse-transfer function:

$$G(z) = \frac{z+a}{z+a} \qquad = 1 \qquad \Rightarrow \; y[k] = u[k]$$

  - But, the solution of the difference equation is:

$$y[k] = (-a)^k y[0] + u[k], \quad k \geq 1$$

  - That is, by using the shift-operator calculus:

$$(q+a)y[k] = (q+a)u[k]$$

  - Different conclusions for: $\begin{cases} |a| < 0 & \text{stable mode} \\ |a| \geq 0 & \text{unstable mode} \end{cases}$



- Use modified $z$-transform
  to study the intersampling behavor

- Defintion: The Modified z-Transform

$$\tilde{F}(z,m) = \sum_{k=0}^{\infty} z^{-k} f(kh - h + mh) \qquad 0 \leq m \leq 1$$

- The inverse transform:

$$f(kh - h + mh) = \frac{1}{2\pi i} \int_{\Gamma} \tilde{F}(z,m)\, z^{n-1}\, dz$$

  where the contour $\Gamma$ encloses
  the singularities of integrand.

- Poles and zeros

$$Y(z) = \underbrace{\left[\mathbf{C}(z\mathbf{I}-\mathbf{F})^{-1}\mathbf{H}+\mathbf{D}\right]}_{G(z)} U(z)$$

  - Poles:
    - The points $p \in C$ where $G(p) = \infty$ are the poles of $G(z)$
    - They are eigenvalues of $\mathbf{F}$ and determine stability
  - Zeros:
    - The points $p \in C$ where $G(p) = 0$ are the zeros of $G(z)$



- Interpretation of poles and zeros
  - Poles:
    - A pole $z = a$ is associated with the time function $z(k) = a^k$
    - A pole $z = a$ is an eigenvalue of $\mathbf{F}$
  - Zeros:
    - A zero $z = b$ implies that the transmission of $u(k) = b^k$ is blocked by the system
    - A zero is related to how inputs and outputs are coupled to the states

## Poles and Zeros



- Poles determine stability
  - All poles of $G(z) = \mathbf{C}(z\mathbf{I} - \mathbf{F})^{-1}\mathbf{H} + \mathbf{D}$ are eigenvalues of $\mathbf{F}$
  - The matrix $\mathbf{F}$ can always be written as:

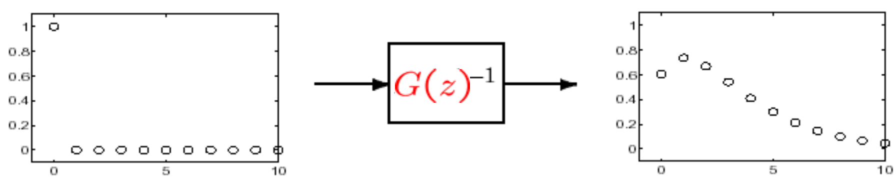
$$\mathbf{F} = U \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} U^{-1} \qquad \mathbf{F}^k = U \begin{bmatrix} \lambda_1^k & & * \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix} U^{-1}$$

  - The diagonal elements $\lambda_k$ are the eig$(\mathbf{F})$
  - $\mathbf{F}^k$ decays exponentially iff $|\lambda_k| < 1, \; \forall k$

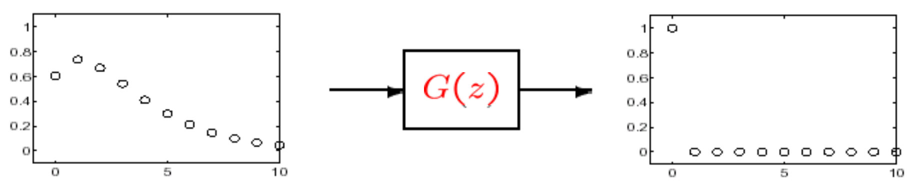
## Poles and Zeros



- Zeros: blocking of signals
  - Assume that $G(z) = \frac{B(z)}{A(z)}$ and $G(z)^{-1}$ has pole at $z = b, \; b < 1$




  - The pulse response of $G(z)^{-1}$ decays as $b^k$




  - $u(k) = b^k$ to $G(z) \Rightarrow$ zero output

- Non-minimum phase
  - Unstable zeros are sometimes called non-minimum phase zeros
  - Because:

$$\frac{s-3}{(s+1)(s+2)} \text{ and } \frac{s+3}{(s+1)(s+2)}$$

    have Bode diagrams with equal amplitude:

$$|iw - 3| = \sqrt{w^2 + 3^3} = |iw + 3|$$

  - But the first has more phase lag
  - This is similar to a time delay and make control harder



- Non-minimum phase zeros make control harder
  - To get a pulse as output, one should use an input which is the impulse response of $G^{-1}$
  - If there are non-minimum phase zeros, the input needs to grow exponentially
  - For a stable system with one unstable zero, the step response will initially have different sign than the stationary value.

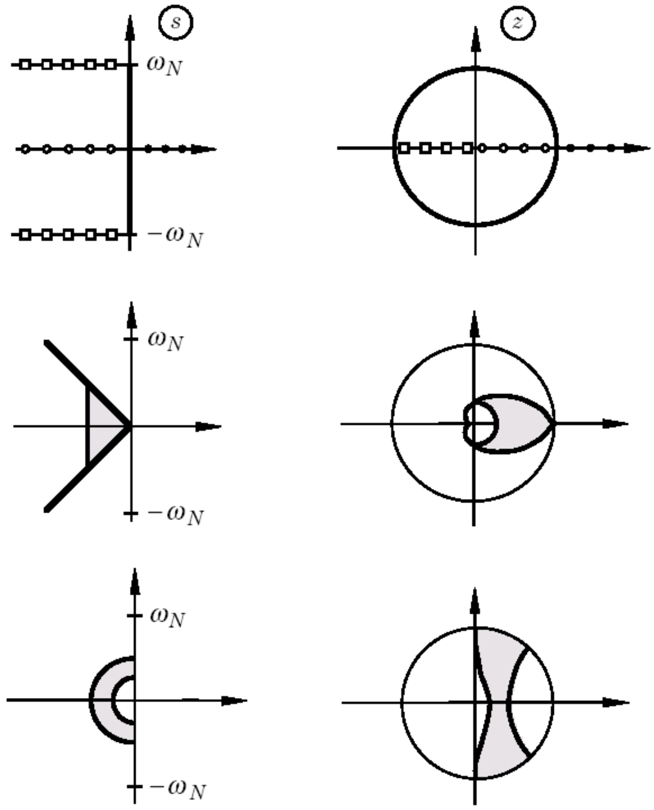
    $\Rightarrow$ This make control harder

## Poles and Zeros

- Mapping between $s$ & $z$
  - Because $\mathbf{F} = \exp(\mathbf{A}h)$

    $\Rightarrow \quad \lambda_i(\mathbf{F}) = e^{\lambda_i(\mathbf{A})h}$

    $\Rightarrow \quad z = e^{sh}$



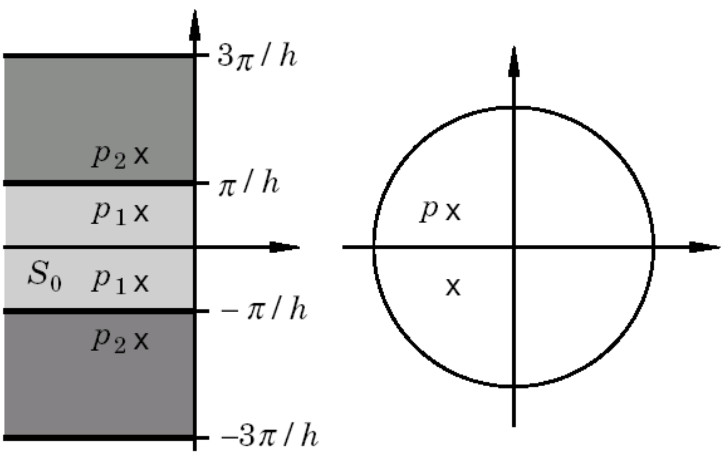



## Poles and Zeros

- Alias Problem

  $z = e^{sh} \qquad \Rightarrow$ Several points in the $s$-plane map into the same point in the $z$-plane

  $\Rightarrow$ The map is not bijective



- Sampling of a second-order system

$$\frac{w_0{}^2}{s^2+2\zeta w_0 s+w_0{}^2}$$

- Poles of the D.T. system:

$$z^2 + a_1 z + a_2 = 0$$

where

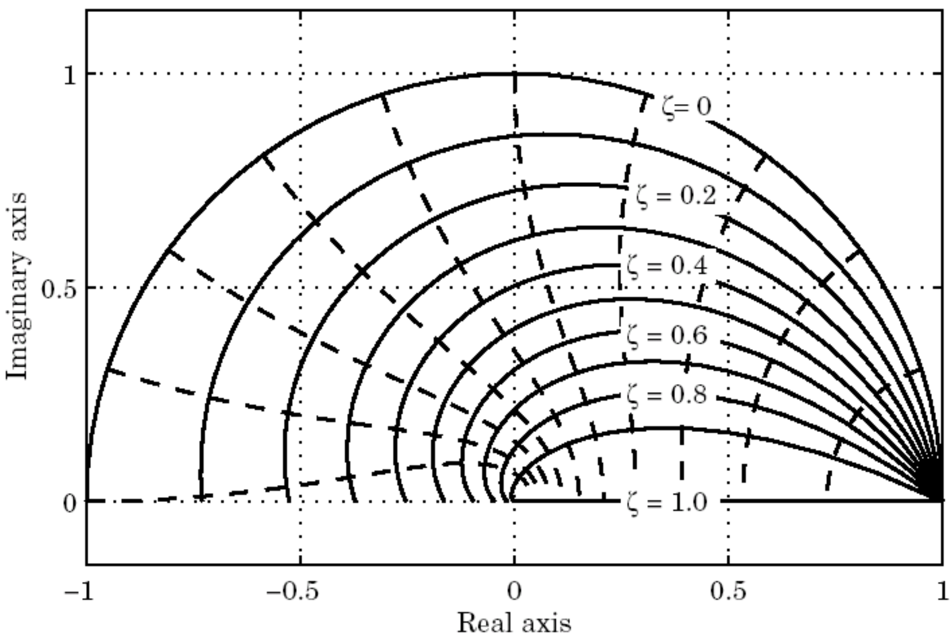
$$\begin{aligned} a_1 &= -2e^{-\zeta w_0 h}\cos\left(\sqrt{1-\zeta^2}\,w_0 h\right) \\ a_2 &= e^{-2\zeta w_0 h} \end{aligned}$$



$$z^2 + a_1 z + a_2 = 0$$

- Sampling of a second-order system

$$\begin{aligned} a_1 &= -2e^{-\zeta w_0 h}\cos\left(\sqrt{1-\zeta^2}\,w_0 h\right) \\ a_2 &= e^{-2\zeta w_0 h} \end{aligned}$$




$w_0 h$: - - -

## Poles and Zeros

| $d$ | $Z_d$ |
|---|---|
| 1 | $1$ |
| 2 | $z+1$ |
| 3 | $z^2+4z+1$ |
| 4 | $z^3+11z^2+11z+1$ |
| 5 | $z^4+26z^3+66z^2+26z+1$ |

d = #(p) - #(z)

- Transformation of zeros
  - More difficult than poles
  - In general,
    more sampled zeros than continuous zeros
  - For short sampling periods $z_i \approx e^{s_i h}$
  - For large $s$ then $G(s) \approx e^{-d}$
    where $d =$ deg $A(s)$ - deg $B(s)$
  - The $r = d - 1$ zeros
    introduced by the sampling
    go to the zeros of the polynomials $z_d$



## Poles and Zeros

$$H(q) = \frac{b_1 q^{n-1} + b_2 q^{n-2} + \cdots + b_n}{q^n + a_1 q^{n-1} + \cdots + a_n}$$

| | |
|---|---|
| $\dfrac{ab}{(s+a)(s+b)}$ <br> $a \neq b$ | $b_1 = \dfrac{b(1-e^{-ah}) - a(1-e^{-bh})}{b-a}$ <br> $b_2 = \dfrac{a(1-e^{-bh})e^{-ah} - b(1-e^{-ah})e^{-bh}}{b-a}$ <br> $a_1 = -(e^{-ah} + e^{-bh})$ <br> $a_2 = e^{-(a+b)h}$ |

- Example:
  - Consider the CT transfer function:

$$\frac{2}{(s+1)(s+2)}$$

  - The zero of the pulse-transfer function:

$$z = \frac{(1-e^{-2h})e^{-h} - 2(1-e^{-h})e^{-2h}}{2(1-e^{-h}) - (1-e^{-2h})}$$

  - When h is small:

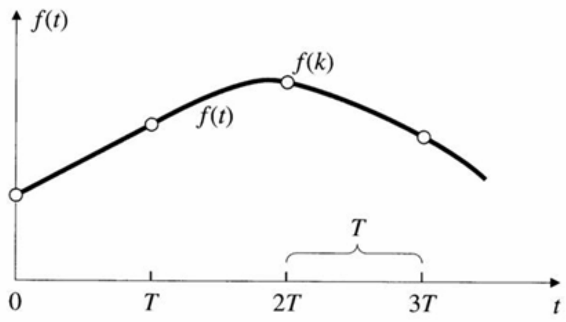
$$z \approx -1 + 3h$$

  - When h approaches 0:

$$z \to -1$$

  - When h increases:

$$z \to 0$$




- Laplace transform:

$$\mathcal{L}\left\{f(t)\right\} = F(s) \triangleq \int_0^\infty f(t)\, e^{-st}\, dt$$

$$\mathcal{L}\left\{\dot{f}(t)\right\} = s\, F(s)$$

- z-Transform:

$$\mathcal{Z}\left\{f[k]\right\} = F(z) \triangleq \sum_{k=0}^{\infty} f[k]\, z^{-k}$$

$$\mathcal{Z}\left\{f[k-1]\right\} = z^{-1}\, F(z)$$

Franklin et al. 2002



- Analysis of discrete systems by the z-transform:

$$y[k] + a_1\, y[k-1] + a_2\, y[k-2]$$
$$= b_0\, u[k] + b_1\, u[k-1] + b_2\, u[k-2]$$

$$\Rightarrow\; Y(z) + a_1\, z^{-1}\, Y(z) + a_2\, z^{-2}\, Y(z)$$
$$= b_0\, U(z) + b_1\, z^{-1}\, U(z) + b_2\, z^{-2}\, U(z)$$

$$\Rightarrow\; \frac{Y(z)}{U(z)} = \frac{b_0 + b_1\, z^{-1} + b_2\, z^{-2}}{1 + a_1\, z^{-1} + a_2\, z^{-2}} = \frac{b_0\, z^2 + b_1\, z + b_2}{z^2 + a_1\, z + a_2}$$

Franklin et al. 2002

- The z-transform inversion:
  - Long division

$$\frac{Y(z)}{U(z)} = \frac{1}{1 - a z^{-1}}$$

- For a unit pulse input: $u[0] = 1, \;\&\; u[k] = 0, k \neq 0$

$$\Rightarrow U(z) = 1$$

$$\Rightarrow Y(z) = \frac{1}{1 - a z^{-1}}$$

$$= 1 + az^{-1} + a^2 z^{-2} + a^3 z^{-3} + \cdots$$

$$= y[0] + y[1]z^{-1} + y[2]z^{-2} + y[3]z^{-3} + \cdots$$

Franklin et al. 2002



- The z-transform inversion:
  - The z-transform table

TABLE 10.2 SOME COMMON z-TRANSFORM PAIRS

| Signal | Transform | ROC |
|---|---|---|
| 1. $\delta[n]$ | $1$ | All $z$ |
| 2. $u[n]$ | $\frac{1}{1 - z^{-1}}$ | $\lvert z\rvert > 1$ |
| 3. $-u[-n-1]$ | $\frac{1}{1 - z^{-1}}$ | $\lvert z\rvert < 1$ |
| 4. $\delta[n-m]$ | $z^{-m}$ | All $z$, except 0 (if $m > 0$) or $\infty$ (if $m < 0$) |
| 5. $\alpha^n u[n]$ | $\frac{1}{1 - \alpha z^{-1}}$ | $\lvert z\rvert > \lvert\alpha\rvert$ |
| 6. $-\alpha^n u[-n-1]$ | $\frac{1}{1 - \alpha z^{-1}}$ | $\lvert z\rvert < \lvert\alpha\rvert$ |
| 7. $n\alpha^n u[n]$ | $\frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$ | $\lvert z\rvert > \lvert\alpha\rvert$ |
| 8. $-n\alpha^n u[-n-1]$ | $\frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$ | $\lvert z\rvert < \lvert\alpha\rvert$ |
| 9. $[\cos \omega_0 n]u[n]$ | $\frac{1 - [\cos \omega_0]z^{-1}}{1 - [2\cos \omega_0]z^{-1} + z^{-2}}$ | $\lvert z\rvert > 1$ |
| 10. $[\sin \omega_0 n]u[n]$ | $\frac{[\sin \omega_0]z^{-1}}{1 - [2\cos \omega_0]z^{-1} + z^{-2}}$ | $\lvert z\rvert > 1$ |
| 11. $[r^n \cos \omega_0 n]u[n]$ | $\frac{1 - [r\cos \omega_0]z^{-1}}{1 - [2r\cos \omega_0]z^{-1} + r^2 z^{-2}}$ | $\lvert z\rvert > r$ |
| 12. $[r^n \sin \omega_0 n]u[n]$ | $\frac{[r\sin \omega_0]z^{-1}}{1 - [2r\cos \omega_0]z^{-1} + r^2 z^{-2}}$ | $\lvert z\rvert > r$ |

Oppenheim et al. 1997

# In Summary



**Laplace Transforms and $z$-Transforms of Simple Discrete Time Functions**

F$(s)$ is the Laplace transform of $f(t)$, and $F(z)$ is the $z$-transform of $f(kT)$. *Note:* $f(t) = 0$ for $t = 0$.

| No. | F$(s)$ | $f(kT)$ | $F(z)$ |
|---|---|---|---|
| 1 | | $1, k=0; 0, k \neq 0$ | $1$ |
| 2 | | $1, k=k_o; 0, k \neq k_o$ | $z^{-k_o}$ |
| 3 | $\frac{1}{s}$ | $1(kT)$ | $\frac{z}{z-1}$ |
| 4 | $\frac{1}{s^2}$ | $kT$ | $\frac{Tz}{(z-1)^2}$ |
| 5 | $\frac{1}{s^3}$ | $\frac{1}{2!}(kT)^2$ | $\frac{T^2}{2}\left[\frac{z(z+1)}{(z-1)^3}\right]$ |
| 6 | $\frac{1}{s^4}$ | $\frac{1}{3!}(kT)^3$ | $\frac{T^3}{6}\left[\frac{z(z^2+4z+1)}{(z-1)^4}\right]$ |
| 7 | $\frac{1}{s^m}$ | $\lim_{a\to 0}\frac{(-1)^{m-1}}{(m-1)!}\left(\frac{\partial^{m-1}}{\partial a^{m-1}}e^{-akT}\right)$ | $\lim_{a\to 0}\frac{(-1)^{m-1}}{(m-1)!}\left(\frac{\partial^{m-1}}{\partial a^{m-1}}\frac{z}{z-e^{-aT}}\right)$ |
| 8 | $\frac{1}{s+a}$ | $e^{-akT}$ | $\frac{z}{z-e^{-aT}}$ |
| 9 | $\frac{1}{(s+a)^2}$ | $kTe^{-akT}$ | $\frac{Tze^{-aT}}{(z-e^{-aT})^2}$ |
| 10 | $\frac{1}{(s+a)^3}$ | $\frac{1}{2}(kT)^2e^{-akT}$ | $\frac{T^2}{2}e^{-aT}z\frac{(z+e^{-aT})}{(z-e^{-aT})^3}$ |
| 11 | $\frac{1}{(s+a)^m}$ | $\frac{(-1)^{m-1}}{(m-1)!}\left(\frac{\partial^{m-1}}{\partial a^{m-1}}e^{-akT}\right)$ | $\frac{(-1)^{m-1}}{(m-1)!}\left(\frac{\partial^{m-1}}{\partial a^{m-1}}\frac{z}{z-e^{-aT}}\right)$ |

- T: sampling period

Franklin et al. 2002

# In Summary



| No. | F$(s)$ | $f(kT)$ | $F(z)$ |
|---|---|---|---|
| 12 | $\frac{a}{s(s+a)}$ | $1-e^{-akT}$ | $\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$ |
| 13 | $\frac{a}{s^2(s+a)}$ | $\frac{1}{a}(akT-1+e^{-akT})$ | $\frac{z[(aT-1+e^{-aT})z+(1-e^{-aT}-aTe^{-aT})]}{a(z-1)^2(z-e^{-aT})}$ |
| 14 | $\frac{b-a}{(s+a)(s+b)}$ | $e^{-akT}-e^{-bkT}$ | $\frac{(e^{-aT}-e^{-bT})z}{(z-e^{-aT})(z-e^{-bT})}$ |
| 15 | $\frac{s}{(s+a)^2}$ | $(1-akT)e^{-akT}$ | $\frac{z[z-e^{-aT}(1+aT)]}{(z-e^{-aT})^2}$ |
| 16 | $\frac{a^2}{s(s+a)^2}$ | $1-e^{-akT}(1+akT)$ | $\frac{z[z(1-e^{-aT}-aTe^{-aT})+e^{-2aT}-e^{-aT}+aTe^{-aT}]}{(z-1)(z-e^{-aT})^2}$ |
| 17 | $\frac{(b-a)s}{(s+a)(s+b)}$ | $be^{-bkT}-ae^{-akT}$ | $\frac{z[z(b-a)-(be^{-aT}-ae^{-bT})]}{(z-e^{-aT})(z-e^{-bT})}$ |
| 18 | $\frac{a}{s^2+a^2}$ | $\sin akT$ | $\frac{z\sin aT}{z^2-(2\cos aT)z+1}$ |
| 19 | $\frac{s}{s^2+a^2}$ | $\cos akT$ | $\frac{z(z-\cos aT)}{z^2-(2\cos aT)z+1}$ |
| 20 | $\frac{s+a}{(s+a)^2+b^2}$ | $e^{-akT}\cos bkT$ | $\frac{z(z-e^{-aT}\cos bT)}{z^2-2e^{-aT}(\cos bT)z+e^{-2aT}}$ |
| 21 | $\frac{b}{(s+a)^2+b^2}$ | $e^{-akT}\sin bkT$ | $\frac{ze^{-aT}\sin bT}{z^2-2e^{-aT}(\cos bT)z+e^{-2aT}}$ |
| 22 | $\frac{a^2+b^2}{s[(s+a)^2+b^2]}$ | $1-e^{-akT}\left(\cos bkT+\frac{a}{b}\sin bkT\right)$ | $\frac{z(Az+B)}{(z-1)[z^2-2e^{-aT}(\cos bT)z+e^{-2aT}]}$ <br> $A = 1-e^{-aT}\cos bT-\frac{a}{b}e^{-aT}\sin bT$ <br> $B = e^{-2aT}+\frac{a}{b}e^{-aT}\sin bT-e^{-aT}\cos bT$ |

- T: sampling period

Franklin et al. 2002

## Properties of the z-transform:

1. Definition:

$$F(z) = \sum_{k=0}^{\infty} f[k] z^{-k}$$

2. Inversion:

$$f[k] = \frac{1}{2\pi i} \oint F(z) z^{k-1} dz$$

3. Linearity:

$$\mathcal{Z}\{af + bg\} = a\mathcal{Z}\{f\} + b\mathcal{Z}\{g\}$$

4. Time shift:

$$\mathcal{Z}\{q^{-n} f\} = z^{-n} F$$

$$\mathcal{Z}\{q^{n} f\} = z^{n} (F - F_1)$$

where $F_1(z) = \sum_{j=0}^{n-1} f[j] z^{-j}$

5 Initial-value theorem:

$$f[0] = \lim_{z\to\infty} F(z)$$

6 Final-value theorem:

If $(1 - z^{-1})F(z)$ does not have any poles on or outside the unit circle:

$$\lim_{k\to\infty} f[k] = \lim_{z\to 1} (1 - z^{-1}) F(z)$$

7 Convolution:

$$\mathcal{Z}\{f * g\} = (\mathcal{Z}f)(\mathcal{Z}g)$$

$$f * g = \sum_{n=0}^{k} f[n] g[k-n]$$

Astrom & Wittenmark 1997



## Properties of the z-transform:

| Section | Property | Signal | z-Transform | ROC |
|---|---|---|---|---|
| | | $x[n]$ | $X(z)$ | $R$ |
| | | $x_1[n]$ | $X_1(z)$ | $R_1$ |
| | | $x_2[n]$ | $X_2(z)$ | $R_2$ |
| 10.5.1 | Linearity | $ax_1[n] + bx_2[n]$ | $aX_1(z) + bX_2(z)$ | At least the intersection of $R_1$ and $R_2$ |
| 10.5.2 | Time shifting | $x[n - n_0]$ | $z^{-n_0} X(z)$ | $R$, except for the possible addition or deletion of the origin |
| 10.5.3 | Scaling in the $z$-domain | $e^{j\omega_0 n} x[n]$ | $X(e^{-j\omega_0} z)$ | $R$ |
| | | $z_0^n x[n]$ | $X\left(\frac{z}{z_0}\right)$ | $z_0 R$ |
| | | $a^n x[n]$ | $X(a^{-1} z)$ | Scaled version of $R$ (i.e., $\|a\|R$ = the set of points $\{\|a\|z\}$ for $z$ in $R$) |
| 10.5.4 | Time reversal | $x[-n]$ | $X(z^{-1})$ | Inverted $R$ (i.e., $R^{-1}$ = the set of points $z^{-1}$, where $z$ is in $R$) |
| 10.5.5 | Time expansion | $x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer $r$ | $X(z^k)$ | $R^{1/k}$ (i.e., the set of points $z^{1/k}$, where $z$ is in $R$) |
| 10.5.6 | Conjugation | $x^*[n]$ | $X^*(z^*)$ | $R$ |
| 10.5.7 | Convolution | $x_1[n] * x_2[n]$ | $X_1(z) X_2(z)$ | At least the intersection of $R_1$ and $R_2$ |
| 10.5.7 | First difference | $x[n] - x[n-1]$ | $(1 - z^{-1}) X(z)$ | At least the intersection of $R$ and $\|z\| > 0$ |
| 10.5.7 | Accumulation | $\sum_{k=-\infty}^{n} x[k]$ | $\frac{1}{1 - z^{-1}} X(z)$ | At least the intersection of $R$ and $\|z\| > 1$ |
| 10.5.8 | Differentiation in the $z$-domain | $nx[n]$ | $-z \frac{dX(z)}{dz}$ | $R$ |
| 10.5.9 | | Initial Value Theorem<br>If $x[n] = 0$ for $n < 0$, then<br>$x[0] = \lim_{z\to\infty} X(z)$ | | |

Oppenheim et al. 1997

## Example – Use z-Transform to find system response

- Consider the difference equation:

$$y[k] - \frac{1}{2}y[k-1] = u[k] + \frac{1}{3}u[k-1]$$

$$\Rightarrow \; Y(z) - \frac{1}{2}z^{-1}Y(z) = U(z) + \frac{1}{3}z^{-1}U(z)$$

$$\Rightarrow \; G(z) = \frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}} = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}$$

$$= \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{\frac{1}{3}}{1-\frac{1}{2}z^{-1}}z^{-1}$$

$$= \frac{z+\frac{1}{3}}{z-\frac{1}{2}} = \frac{z}{z-\frac{1}{2}} + \frac{\frac{1}{3}}{z-\frac{1}{2}}$$

$$= \frac{z}{z-\frac{1}{2}} + \frac{\frac{1}{3}z}{z-\frac{1}{2}}z^{-1}$$



## Example – Use z-Transform to find system response

- If u[k] = unit impulse function:

$$\Rightarrow \; U(z) = 1$$

$$\Rightarrow \; Y(z) = G(z)U(z)$$

$$= \frac{z}{z-\frac{1}{2}} + \frac{\frac{1}{3}z}{z-\frac{1}{2}}z^{-1}$$

$$= \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{\frac{1}{3}}{1-\frac{1}{2}z^{-1}}z^{-1}$$

$$\Rightarrow \; y[k] = \left(\frac{1}{2}\right)^{k} s[k] + \frac{1}{3}\left(\frac{1}{2}\right)^{k-1} s[k-1]$$

| | | |
|---|---|---|
| 1. $\delta[n]$ | $1$ | All $z$ |
| 2. $u[n]$ | $\frac{1}{1-z^{-1}}$ | $\lvert z\rvert > 1$ |
| 3. $-u[-n-1]$ | $\frac{1}{1-z^{-1}}$ | $\lvert z\rvert < 1$ |
| 4. $\delta[n-m]$ | $z^{-m}$ | All $z$, except 0 (if $m > 0$) or $\infty$ (if $m < 0$) |
| 5. $\alpha^n u[n]$ | $\frac{1}{1-\alpha z^{-1}}$ | $\lvert z\rvert > \lvert\alpha\rvert$ |
| 6. $-\alpha^n u[-n-1]$ | $\frac{1}{1-\alpha z^{-1}}$ | $\lvert z\rvert < \lvert\alpha\rvert$ |

| $f(kT)$ | $F(z)$ |
|---|---|
| $1, k=0; 0, k \neq 0$ | $1$ |
| $1, k=k_o; 0, k \neq k_o$ | $z^{-k_o}$ |
| $1(kT)$ | $\frac{z}{z-1}$ |
| $e^{-akT}$ | $\frac{z}{z-e^{-aT}}$ |

- Example – Use z-Transform to find system response
  - If u[k] = unit step function:

$$\Rightarrow U(z) = \frac{z}{z-1}$$

$$\Rightarrow Y(z) = G(z)U(z)$$

$$= \frac{z+\frac{1}{3}}{z-\frac{1}{2}} \frac{z}{z-1}$$

$$= \frac{-\frac{5}{3}z}{z-\frac{1}{2}} + \frac{\frac{8}{3}z}{z-1}$$

$$= \left(-\frac{5}{3}\right)\frac{z}{z-\frac{1}{2}} + \left(\frac{8}{3}\right)\frac{z}{z-1}$$

$$\Rightarrow y[k] = \left(-\frac{5}{3}\right)\left(\frac{1}{2}\right)^k s[k] + \left(\frac{8}{3}\right) s[k]$$

| | | |
|---|---|---|
| 1. $\delta[n]$ | $1$ | All $z$ |
| 2. $u[n]$ | $\frac{1}{1-z^{-1}}$ | $\|z\| > 1$ |
| 3. $-u[-n-1]$ | $\frac{1}{1-z^{-1}}$ | $\|z\| < 1$ |
| 4. $\delta[n-m]$ | $z^{-m}$ | All $z$, except 0 (if $m > 0$) or $\infty$ (if $m < 0$) |
| 5. $\alpha^n u[n]$ | $\frac{1}{1-\alpha z^{-1}}$ | $\|z\| > \|\alpha\|$ |
| 6. $-\alpha^n u[-n-1]$ | $\frac{1}{1-\alpha z^{-1}}$ | $\|z\| < \|\alpha\|$ |

| $f(kT)$ | $F(z)$ |
|---|---|
| $1, k=0; 0, k \neq 0$ | $1$ |
| $1, k=k_o; 0, k \neq k_o$ | $z^{-k_o}$ |
| $1(kT)$ | $\frac{z}{z-1}$ |
| $e^{-akT}$ | $\frac{z}{z-e^{-aT}}$ |



- Example – Use z-Transform to find system response
  - Step Response:

$$\Rightarrow y_s[k] = \left(-\frac{5}{3}\right)\left(\frac{1}{2}\right)^k s[k] + \left(\frac{8}{3}\right) s[k]$$

  - Impulse Response:

$$\Rightarrow y_i[k] = \left(\frac{1}{2}\right)^k s[k] + \frac{1}{3}\left(\frac{1}{2}\right)^{k-1} s[k-1]$$

| *k* | *ys* | *yi* |
|---|---|---|
| 0 | 1 | 1 |
| 1 | 11/6 | 5/6 |
| 2 | 27/12 | 5/12 |

- Relationship between s and z:

$$f(t) = e^{-at}, \quad t > 0 \quad \Rightarrow \quad F(s) = \frac{1}{s+a}$$

$$\Rightarrow \quad \text{Pole: } s = -a$$

$$f[kT] = e^{-akT}, \quad k \in N \quad \Rightarrow \quad F(z) = \mathcal{Z}\left\{e^{-akT}\right\}$$

$$= \frac{z}{z - e^{-aT}}$$

$$\Rightarrow \quad \text{Pole: } z = e^{-aT}$$

$$\Rightarrow \quad z = e^{sT}$$

- T: sampling period

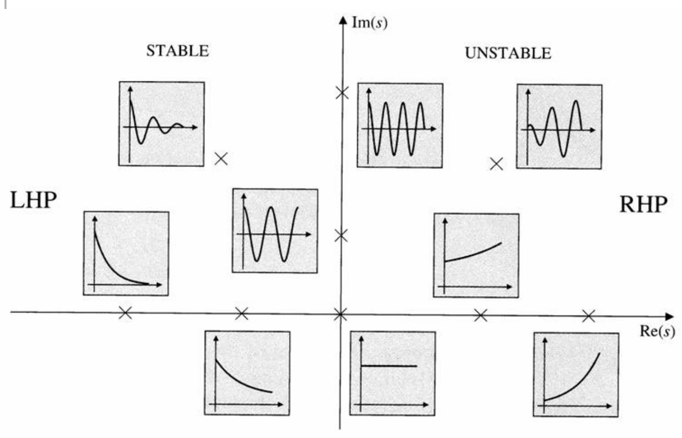
Franklin et al. 2002



- Pole location and response between s and z:

s plane

z plane



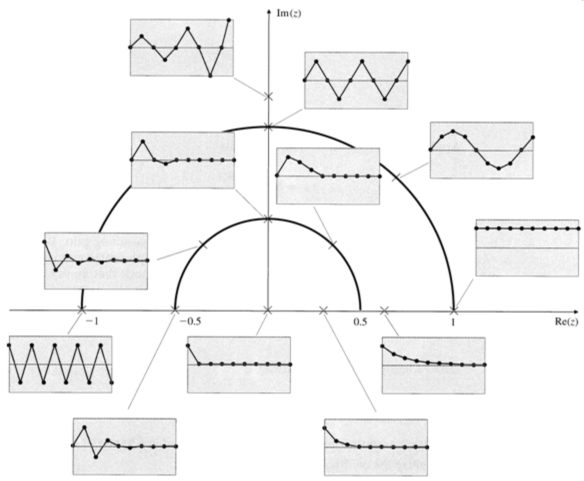




**Figure 8.5** Time sequences associated with points in the z-plane

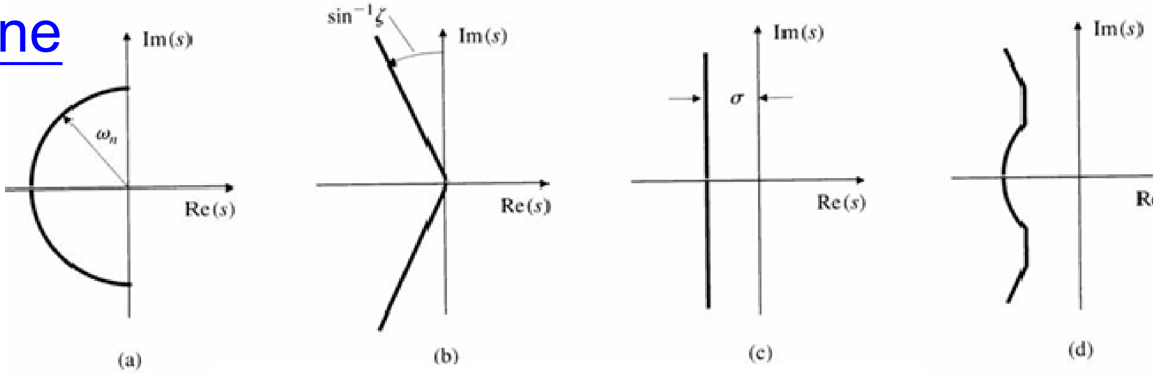
Franklin et al. 2002

## In Summary



- Dynamic Properties between s and z:

s plane




z plane

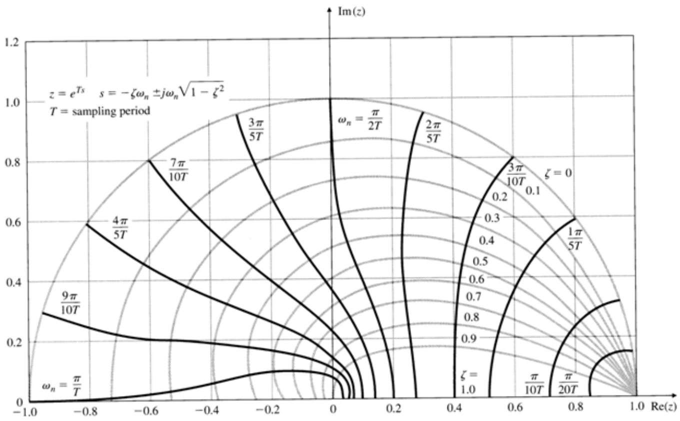



**Figure 8.4** Natural frequency (solid color) and damping loci (light color) in the $z$-plane; the portion below the Re($z$)-axis (not shown) is the mirror image of the upper half shown.

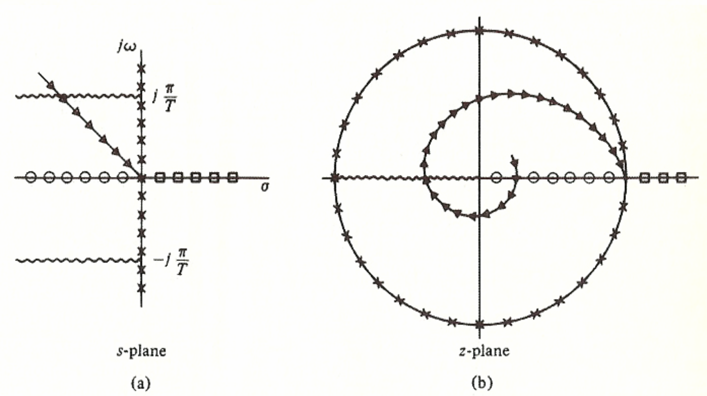
Franklin et al. 2002

## In Summary



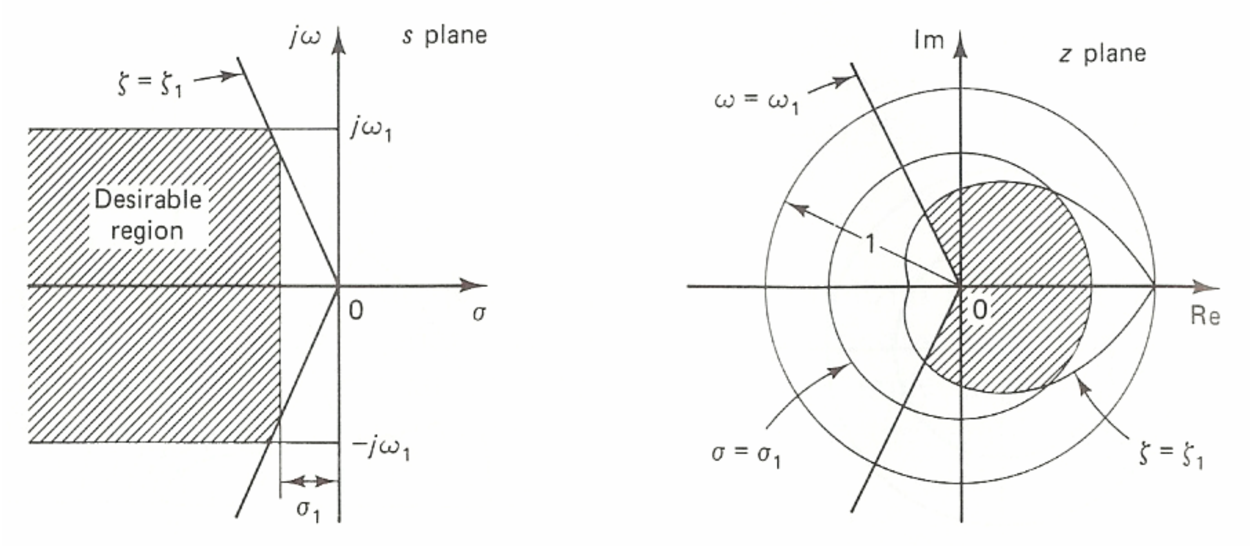
Description of corresponding lines in $s$-plane and $z$-plane

| *s-plane* | *Symbol* | *z-plane* |
|---|---|---|
| $s = j\omega$ Real frequency axis | × × × | $\lvert z \rvert = 1$ Unit circle |
| $s = \sigma \geq 0$ | □□□ | $z = r \geq 1$ |
| $s = \sigma \leq 0$ | ○○○ | $z = r, 0 \leq r \leq 1$ |
| $s = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$. $= -a + jb$ Constant damping ratio if $\zeta$ is fixed and $\omega_n$ varies | △△△ | $z = re^{j\theta}$ where $r = \exp(-\zeta\omega_n T) = e^{-aT}$, $\theta = \omega_n T\sqrt{1-\zeta^2} = bT$ Logarithmic spiral |
| $s = \pm j(\pi/T) + \sigma$, | $\sigma \leq 0$ | $z = -r$ |




Franklin et al. 2002

- Relationship between s and z:




Ogata 1995



- Final Value Theorem:

$$\lim_{t\to\infty} x(t) = x_{ss} = \lim_{s\to 0} s\,X(s)$$

If all the poles of $s\,X(s)$ are in LHP

$$\lim_{k\to\infty} x[k] = x_{ss} = \lim_{z\to 1} (1 - z^{-1})\,X(z)$$

If all the poles of $(1 - z^{-1})\,X(z)$
are inside the unit circle

Franklin et al. 2002