Spring 2019

數位控制系統

Digital Control Systems

DCS-12
Discrete-Time Systems –
Input-Output (Transfer Function) Model

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NTU-EE

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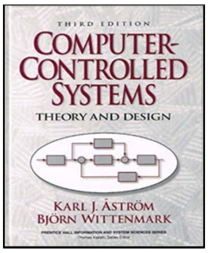
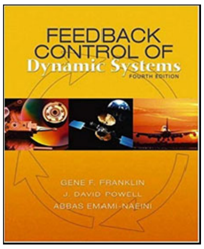
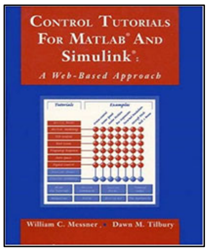
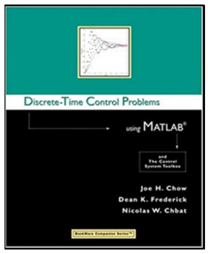
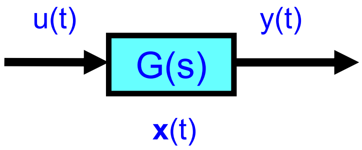








Introduction: External and Internal Models



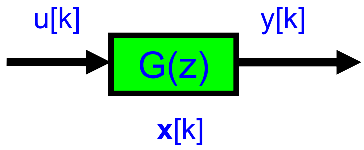
- Dynamic systems:
  - Internal models
    - State-space models

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

  - External models:
    - Input-output models
    - Pulse-response functions

$$Y(s) = G(s)U(s)$$

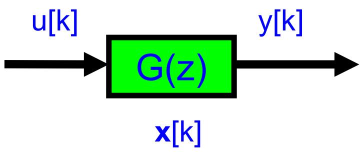
Input-Output Models: The Pulse Response



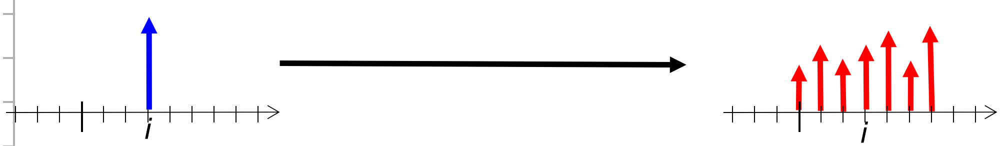
- The pulse response:
  - Consider a discrete-time system with one input & one output
  - Over a finite interval N, the input and output signals are:

$$U = \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(N-1) \end{bmatrix} \qquad Y = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N-1) \end{bmatrix}$$

Input-Output Models: The Pulse Response



$u[i]$ $\qquad\qquad$ $y[i]$



$$\cdots\ y[i-2],\ y[i-1],\qquad y[i+1],\ y[i+2],\ \cdots$$

$$u[j] \longrightarrow y[j]$$

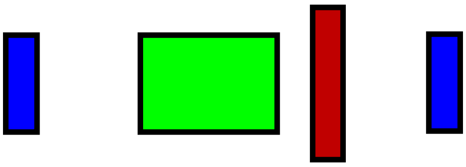
$$\cdots\ y[j-2],\ y[j-1],\qquad y[j+1],\ y[j+2],\ \cdots$$

$$\begin{array}{lccc} & \bar{g}[k,k-1]\,u[k-1] & & \bar{g}[k,k+1]\,u[k+1] \\ y[k] = & \bar{g}[k,k-2]\,u[k-2] & \bar{g}[k,k]\,u[k] & \bar{g}[k,k+2]\,u[k+2] \\ & \vdots \quad \vdots & & \vdots \quad \vdots \end{array}$$

Input-Output Models: The Pulse Response

$$\begin{bmatrix} \vdots \\ \vdots \\ y[k] \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} & & \vdots & & \\ & & \vdots & & \\ \cdots & \bar{g}[k,k-1] & \bar{g}[k,k] & \bar{g}[k,k+1] & \cdots \\ & & \vdots & & \\ & & \vdots & & \end{bmatrix} \begin{bmatrix} \vdots \\ u[k-1] \\ u[k] \\ u[k+1] \\ \vdots \end{bmatrix}$$

$$\Rightarrow Y = \bar{G}\, U + Y_p$$



- $\bar{G} \in \mathcal{R}^{N \times N}$,
- $\bar{G} = \left[\bar{g}(k,m)\right]$
- Y_p for initial conditions

- $\bar{g}(k,m)$:
  - pulse-response or weighting function
  - gives the output at time k for a unit pulse at time m

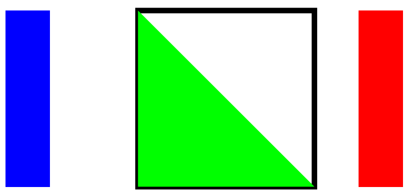
Input-Output Models: The Pulse Response

- If $U \to Y$ is causal

$$y[k] = \begin{matrix} \bar{g}[k,k-1]\,u[k-1] & & \bar{g}[k,k+1]\,u[k+1] \\ \bar{g}[k,k-2]\,u[k-2] & \bar{g}[k,k]\,u[k] & \bar{g}[k,k+2]\,u[k+2] \\ \vdots \quad \vdots & & \vdots \quad \vdots \end{matrix}$$

$$\begin{bmatrix} \vdots \\ \vdots \\ y[k] \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} & & \vdots & & \\ & & \vdots & & \\ \cdots & \bar{g}[k,k-1] & \bar{g}[k,k] & \bar{g}[k,k+1] & \cdots \\ & & \vdots & & \\ & & \vdots & & \end{bmatrix} \begin{bmatrix} \vdots \\ u[k-1] \\ u[k] \\ u[k+1] \\ \vdots \end{bmatrix}$$

- $\bar{G}$: - lower trianglar
  - $g[k,m]$ = '0' if $m > k$.



$$\Rightarrow y[k] = \sum_{m=0}^{k} \bar{g}[k,m]u[m] + y_p[k]$$

Input-Output Models: The Pulse Response

- For time-invariant systems:

$$\Rightarrow \bar{g}[k,m] = g[k-m]$$

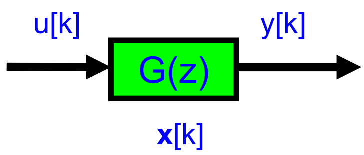
$$\begin{bmatrix} & & & \vdots & & \\ & & & \vdots & & \\ \cdots & \bar{g}[k-1,k-2] & \bar{g}[k-1,k-1] & 0 & 0 & \cdots \\ \cdots & \bar{g}[k,k-2] & \bar{g}[k,k-1] & \bar{g}[k,k] & 0 & \cdots \\ & & & \vdots & & \\ & & & \vdots & & \end{bmatrix}$$

$$\begin{bmatrix} & & & \vdots & & \\ & & & \vdots & & \\ \cdots & \bar{g}[1] & \bar{g}[0] & 0 & 0 & \cdots \\ \cdots & \bar{g}[2] & \bar{g}[1] & \bar{g}[0] & 0 & \cdots \\ & & & \vdots & & \\ & & & \vdots & & \end{bmatrix}$$

$$\Rightarrow y[k] = \sum_{m=0}^{k} \bar{g}[k-m]u[m] + y_p[k]$$

Input-Output Models: The Pulse Response

- Then, the state-space model:



$$\begin{aligned} \mathbf{x}[k+1] &= \mathbf{F}\mathbf{x}[k] + \mathbf{H}u[k] \\ y[k] &= \mathbf{C}\mathbf{x}[k] + \mathbf{D}u[k] \end{aligned}$$

$$U = \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(N-1) \end{bmatrix} \quad Y = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N-1) \end{bmatrix}$$

$$\Rightarrow y[k] = \mathbf{C}\mathbf{F}^{k-k_0}\, x[k_0] + \sum_{j=k_0}^{k-1} \mathbf{C}\mathbf{F}^{k-j-1}\mathbf{H}u[j] + \mathbf{D}\, u[k]$$

$$\Rightarrow y[k] = \sum_{m=0}^{k} \bar{g}[k-m]u[m] + y_p[k]$$

- The pulse response for the D.T. system:

$$\Rightarrow g[i] = \begin{cases} 0 & i < 0 \\ \mathbf{D} & i = 0 \\ \mathbf{C}\mathbf{F}^{i-1}\mathbf{H} & i \geq 1 \end{cases}$$

Input-Output Models: Shift-Operator Calculus

- All signals are considered as doubly infinite sequences:

$$\{f(k) : k = \cdots, -1, 0, 1, \cdots\}$$

$$\{\cdots, f(-3), f(-2), f(-1), f(0), f(1), f(2), f(3), \cdots\}$$

- Forward-shift operator: q

$$qf(k) = f(k+1)$$

- Backward-shift operator: q^{-1}
  or delay operator:

$$q^{-1}f(k) = f(k-1)$$

Input-Output Models: Shift-Operator Calculus

- Norm of a signal:

$$||f|| = \sup_k |f(k)|$$

$$\text{or } ||f||^2 = \sum_{k=-\infty}^{\infty} f^2(k)$$

$\Rightarrow$ Shift operator has unit norm

e.x. $||qf(k)|| = ||f(k+1)||$

$$||q|| \; ||f(k)|| = ||f(k+1)||$$

$$||q|| = 1$$

Input-Output Models: Shift-Operator Calculus

- A higher-order difference eqn:

$$y(k+n_a) + a_1 y(k+n_a-1) + \cdots + a_{n_a} y(k)$$
$$= b_0 u(k+n_b) + b_1 u(k+n_b-1) + \cdots + b_{n_b} u(k)$$

where $n_a \geq n_b$

- Use shift operator: q $\quad qf(k) = f(k+1)$

$$q^{n_a} y(k) + a_1 q^{n_a-1} y(k) + \cdots + a_{n_a} y(k)$$
$$= b_0 q^{n_b} u(k) + b_1 q^{n_b-1} u(k) + \cdots + b_{n_b} u(k)$$

$$(q^{n_a} + a_1 q^{n_a-1} + \cdots + a_{n_a}) y(k)$$
$$= (b_0 q^{n_b} + b_1 q^{n_b-1} + \cdots + b_{n_b}) u(k)$$

Input-Output Models: Shift-Operator Calculus

- Let:

$$A(z) = z^{n_a} + a_1 z^{n_a-1} + \cdots + a_{n_a}$$
$$B(z) = b_0 z^{n_b} + b_1 z^{n_b-1} + \cdots + b_{n_b}$$

$$\Rightarrow \quad A(q) y(k) = B(q) u(k)$$

- On the other hand, it is equivalent to write:

$$y(k) + a_1 y(k-1) + \cdots + a_{n_a} y(k-n_a)$$
$$= b_0 u(k-d) + \cdots + b_{n_b} u(k-d-n_b)$$

where $d = n_a - n_b$: pole excess of the system

- The reciprocal polynomial:

$$A(z) = z^{n_a} + a_1 z^{n_a - 1} + \cdots + a_{n_a}$$

$$A^*(z) = 1 + a_1 z + \cdots + a_{n_a} z^{n_a} = z^{n_a} A(z^{-1})$$

$$\Rightarrow \quad A^*(q^{-1}) y(k) = B^*(q^{-1}) u(k-d)$$

- Note that $A^{**} = (A^*)^* =$ or $\neq A$

e.x. $A(z) = z$ $\qquad A^*(z) = z\, z^{-1} = 1$

but, $(A^*(z))^* = 1 \neq A(z)$

- self-reciprocal: $\quad A^*(z) = A(z)$

- Example :

$$y(k+1) - a y(k) = u(k) \qquad |a| < 1$$

$$\Rightarrow \; q(y(k)) - a y(k) = u(k)$$

$$\Rightarrow \; (q - a) y(k) = u(k)$$

$$\Rightarrow \; y(k) = \frac{1}{(q-a)} u(k)$$

$$= \frac{q^{-1}}{(1 - q^{-1} a)} u(k) \quad = q^{-1} \frac{1}{(1 - a q^{-1})} u(k)$$

$$= q^{-1} \left[1 + a q^{-1} + a^2 q^{-2} + \cdots \right] u(k)$$

$$= \left[q^{-1} + a q^{-2} + a^2 q^{-3} + \cdots \right] u(k)$$

$$= q^{-1} u(k) + a q^{-2} u(k) + a^2 q^{-3} u(k) + \cdots$$

$$= \sum_{i=1}^{\infty} a^{i-1} u(k-i) \qquad = u(k-1) + a u(k-2) + a^2 u(k-3) + \cdots$$

Input-Output Models: Shift-Operator Calculus: Example

$$y(k+1) = ay(k) + u(k)$$

IF $y(k_0) = y_0$

$$y(k_0+1) = ay(k_0) + u(k_0)$$

$$\begin{aligned} y(k_0+2) &= ay(k_0+1) + u(k_0+1) \\ &= a[ay(k_0) + u(k_0)] + u(k_0+1) \\ &= a^2 y(k_0) + au(k_0) + u(k_0+1) \end{aligned}$$

$$\begin{aligned} y(k_0+3) &= ay(k_0+2) + u(k_0+2) \\ &= a^3 y(k_0) + a^2 u(k_0) + au(k_0+1) + u(k_0+2) \end{aligned}$$

$$\begin{aligned} \Rightarrow \; y(k) &= a^{k-k_0} y_0 + \sum_{j=k_0}^{k-1} a^{k-j-1} u(j) \\ &= a^{k-k_0} y_0 + \sum_{i=1}^{k-k_0} a^{i-1} u(k-i) \end{aligned}$$

Input-Output Models: The Pulse-Transfer Operator

- $qx(k) = x(k+1)$

$$x(k+1) = \mathbf{F}x(k) + \mathbf{H}u(k)$$

- Hence, $qx(k) = \mathbf{F}x(k) + \mathbf{H}u(k)$

  that is, $(qI - \mathbf{F})x(k) = \mathbf{H}u(k)$

  or, $x(k) = (q\mathbf{I} - \mathbf{F})^{-1}\mathbf{H}u(k)$

- This gives

$$y(k) = \mathbf{C}x(k) + \mathbf{D}u(k) \quad = \left(\mathbf{C}(q\mathbf{I} - \mathbf{F})^{-1}\mathbf{H} + \mathbf{D}\right) u(k)$$

- The pulse-tranfer operator:

$$G(q) = \left(\mathbf{C}(q\mathbf{I} - \mathbf{F})^{-1}\mathbf{H} + \mathbf{D}\right) \quad = \frac{B(q)}{A(q)}$$

Input-Output Models: The Pulse-Transfer Operator

- If the system is of dimension n,
- If $A(q), B(q)$ do not have common factors,

$\Rightarrow$ $A(q)$ is of degree n.

$$A(q) = q^n + a_1 q^{n-1} + a_2 q^{n-2} + \cdots + a_n$$

- Since $A(q)$ is also the characteristic polynomial of $\mathbf{F}$:

$$\Rightarrow \; y(k) + a_1 y(k-1) + \cdots + a_n y(k-n)$$

$$= b_0 u(k) + b_1 u(k-1) + \cdots + b_n u(k-n)$$

Input-Output Models: The Pulse-Transfer Operator - Example

- Example: Double integrator

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad \mathbf{F} = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x} \qquad \mathbf{H} = \begin{bmatrix} h^2/2 \\ h \end{bmatrix}$$

$$h = 1$$

$$G(q) = \mathbf{C}(q\mathbf{I} - \mathbf{F})^{-1}\mathbf{H} + \mathbf{D} \qquad = \frac{B(q)}{A(q)}$$

$$G(q) = [1 \; 0] \begin{bmatrix} q-1 & -1 \\ 0 & q-1 \end{bmatrix}^{-1} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

$$= \frac{0.5(q+1)}{(q-1)^2}$$

$$= \frac{0.5(q^{-1} + q^{-2})}{1 - 2q^{-1} + q^{-2}}$$

- Example: Double integrator with time delay

$$\mathbf{F} = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \quad \mathbf{H}_1 = \begin{bmatrix} d(h-d/2) \\ d \end{bmatrix} \quad \mathbf{H}_0 = \begin{bmatrix} (h-d^2)/2 \\ h-d \end{bmatrix}$$

$h = 1, d = 0.5$

$$G(q) = \mathbf{C}(q\mathbf{I}-\mathbf{F})^{-1}(\mathbf{H}_0 + \mathbf{H}_1 q^{-1})$$

$$= [1\ 0]\,\frac{\begin{bmatrix} q-1 & -1 \\ 0 & q-1 \end{bmatrix}}{(q-1)^2}\begin{bmatrix} 0.125+0.375q^{-1} \\ 0.5+0.5q^{-1} \end{bmatrix}$$

$$= \frac{0.125(q^2+6q+1)}{q(q^2-2q+1)}$$

$$= \frac{0.125(q^{-1}+6q^{-2}+q^{-3})}{1-2q^{-1}+q^{-2}}$$

- Homework 2-4

Input-Output Models: Poles and Zeros

- The pulse-transfer operator:

$$G(q) = \left(\mathbf{C}(q\mathbf{I}-\mathbf{F})^{-1}\mathbf{H} + \mathbf{D}\right) \quad = \frac{B(q)}{A(q)}$$

- Poles of a system: $\Rightarrow$ roots of $A(q) = 0$
- Zeros of a system: $\Rightarrow$ roots of $B(q) = 0$
- Time delay: $\Rightarrow$ poles at the origin
- Order of a system:

  $\Rightarrow$ the dim of a state-space representation

  $\Rightarrow$ the number of poles of the system

The Table of Pulse-Transfer Operator

Table 2.1 Zero-order hold sampling of a continuous-time system, $G(s)$. The table gives the zero-order-hold equivalent of the continuous-time system, $G(s)$, preceded by a zero-order hold. The sampled system is described by its pulse-transfer operator. The pulse-transfer operator is given in terms of the coefficients of

$$H(q) = \frac{b_1 q^{n-1} + b_2 q^{n-2} + \cdots + b_n}{q^n + a_1 q^{n-1} + \cdots + a_n}$$

$G(s)$	$H(q)$ or the coefficients in $H(q)$
$\frac{1}{s}$	$\frac{h}{q-1}$
$\frac{1}{s^2}$	$\frac{h^2(q+1)}{2(q-1)^2}$
$\frac{1}{s^m}$	$\frac{q-1}{q} \lim_{a \to 0} \frac{(-1)^m}{m!} \frac{\partial^m}{\partial a^m} \left(\frac{q}{q - e^{-ah}} \right)$
e^{-sh}	q^{-1}
$\frac{a}{s+a}$	$\frac{1 - \exp(-ah)}{q - \exp(-ah)}$
$\frac{a}{s(s+a)}$	$b_1 = \frac{1}{a}(ah - 1 + e^{-ah})$ $\quad b_2 = \frac{1}{a}(1 - e^{-ah} - ahe^{-ah})$ $a_1 = -(1 + e^{-ah})$ $\quad a_2 = e^{-ah}$
$\frac{a^2}{(s+a)^2}$	$b_1 = 1 - e^{-ah}(1 + ah)$ $\quad b_2 = e^{-ah}(e^{-ah} + ah - 1)$ $a_1 = -2e^{-ah}$ $\quad a_2 = e^{-2ah}$
$\frac{s}{(s+a)^2}$	$\frac{(q-1)he^{-ah}}{(q - e^{-ah})^2}$
$\frac{ab}{(s+a)(s+b)}$ $a \neq b$	$b_1 = \frac{b(1 - e^{-ah}) - a(1 - e^{-bh})}{b - a}$ $b_2 = \frac{a(1 - e^{-bh})e^{-ah} - b(1 - e^{-ah})e^{-bh}}{b - a}$ $a_1 = -(e^{-ah} + e^{-bh})$ $a_2 = e^{-(a+b)h}$

Table 2.1 *continued*

$G(s)$	$H(q)$ or the coefficients in $H(q)$
$\frac{(s+c)}{(s+a)(s+b)}$ $a \neq b$	$b_1 = \frac{e^{-bh} - e^{-ah} + (1 - e^{-bh})c/b - (1 - e^{-ah})c/a}{a - b}$ $b_2 = \frac{c}{ab} e^{-(a+b)h} + \frac{b-c}{b(a-b)} e^{-ah} + \frac{c-a}{a(a-b)} e^{-bh}$ $a_1 = -e^{-ah} - e^{-bh}$ $\quad a_2 = e^{-(a+b)h}$
$\frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$	$b_1 = 1 - \alpha \left(\beta + \frac{\zeta\omega_0}{\omega} \gamma \right)$ $\quad \omega = \omega_0 \sqrt{1 - \zeta^2}$ $\quad \zeta < 1$ $b_2 = \alpha^2 + \alpha \left(\frac{\zeta\omega_0}{\omega} \gamma - \beta \right)$ $\quad \alpha = e^{-\zeta\omega_0 h}$ $a_1 = -2\alpha\beta$ $\quad \beta = \cos(\omega h)$ $a_2 = \alpha^2$ $\quad \gamma = \sin(\omega h)$
$\frac{s}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$	$b_1 = \frac{1}{\omega} e^{-\zeta\omega_0 h} \sin(\omega h)$ $\quad b_2 = -b_1$ $a_1 = -2e^{-\zeta\omega_0 h} \cos(\omega h)$ $\quad a_2 = e^{-2\zeta\omega_0 h}$ $\omega = \omega_0 \sqrt{1 - \zeta^2}$
$\frac{a^2}{s^2 + a^2}$	$b_1 = 1 - \cos ah$ $\quad b_2 = 1 - \cos ah$ $a_1 = -2 \cos ah$ $\quad a_2 = 1$
$\frac{s}{s^2 + a^2}$	$b_1 = \frac{1}{a} \sin ah$ $\quad b_2 = -\frac{1}{a} \sin ah$ $a_1 = -2 \cos ah$ $\quad a_2 = 1$
$\frac{a}{s^2(s+a)}$	$b_1 = \frac{1-\alpha}{a^2} + h \left(\frac{h}{2} - \frac{1}{a} \right)$ $\quad \alpha = e^{-ah}$ $b_2 = (1-\alpha)\left(\frac{h^2}{2} - \frac{2}{a^2} \right) + \frac{h}{a}(1+\alpha)$ $b_3 = -\left[\frac{1}{a^2}(\alpha - 1) + \alpha h \left(\frac{h}{2} + \frac{1}{a} \right) \right]$ $a_1 = -(\alpha + 2)$ $\quad a_2 = 2\alpha + 1$ $\quad a_3 = -\alpha$