Spring 2019

數位控制系統
Digital Control Systems

DCS-11
Discrete-Time Systems – State Space Model

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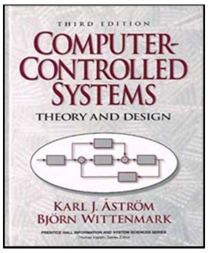
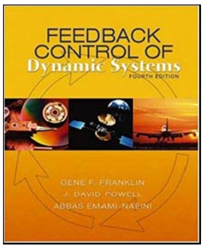
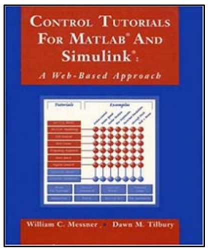
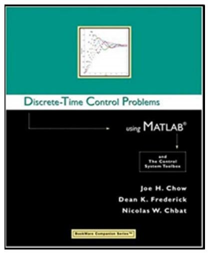
NTU-EE

Feb19 – Jun19





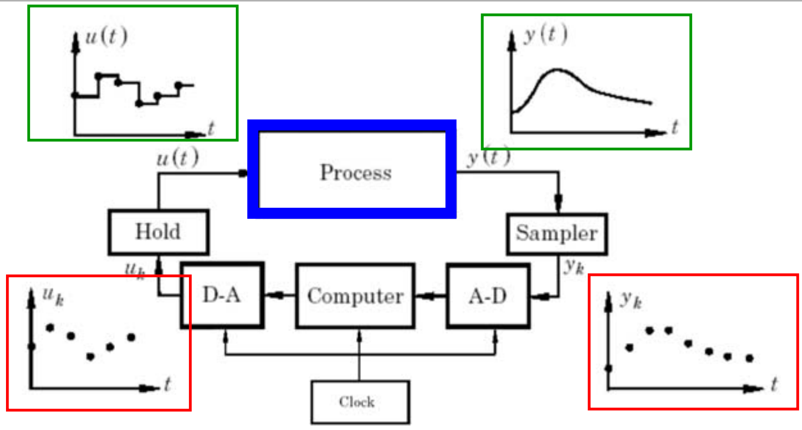




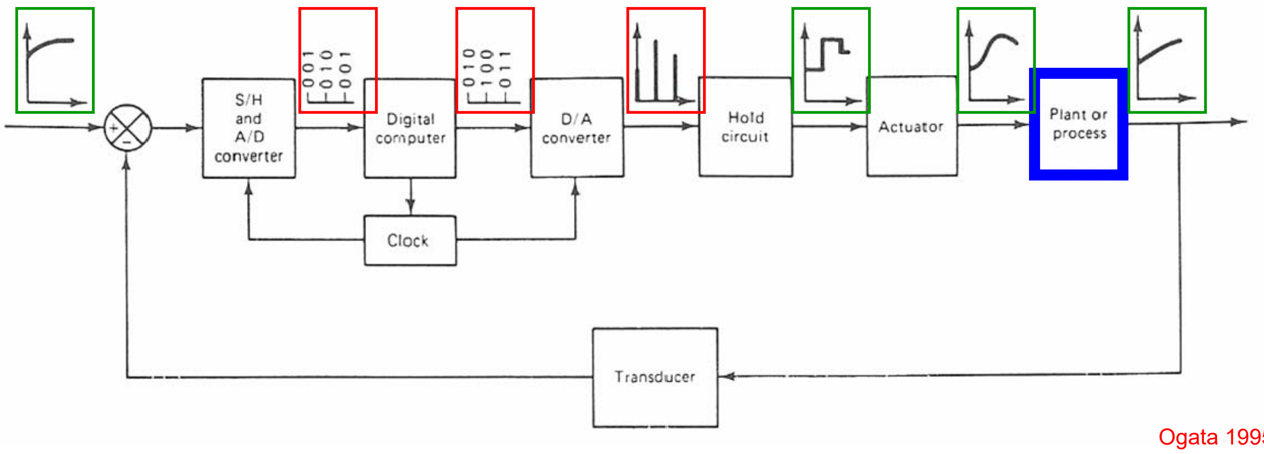
Review:

- System Modeling at Control Tutorials for Matlab & Simulink:
  - http://ctms.engin.umich.edu/CTMS/index.php?example=Introduction&section=SystemModeling
- System Models in LS:
  - http://cc.ee.ntu.edu.tw/~fengli/Teaching/LinearSystems/971lsNotePhoto080924.pdf
- Solving 1st-order Differential Equations:
  - https://case.ntu.edu.tw/CASTUDIO/Files/speech/Ref/CS0101S1B02_03.pdf
- Systems of Linear 1st-order Differential Equations:
  - http://case.ntu.edu.tw/CASTUDIO/Files/speech/Ref/CS0101S1B02_16.pdf
- Solving State-Space Equations:
  - http://cc.ee.ntu.edu.tw/~fengli/Teaching/LinearSystems/971lsNotePhoto081001.pdf

Introduction: Digital Signals and Systems

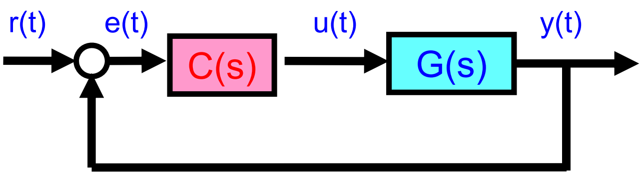


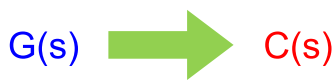
Astrom & Wittenmark 1997



Ogata 1995

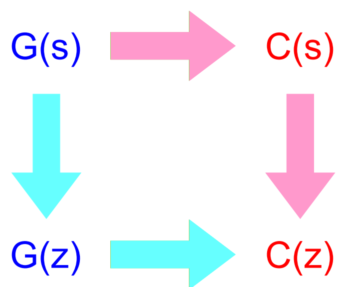
Introduction: CT and DT Plant-Controller

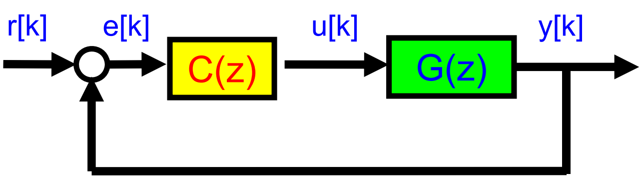


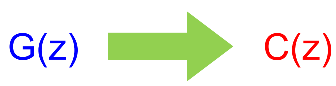


- Transform CT plant into DT plant
- By DT plant, design DT controller

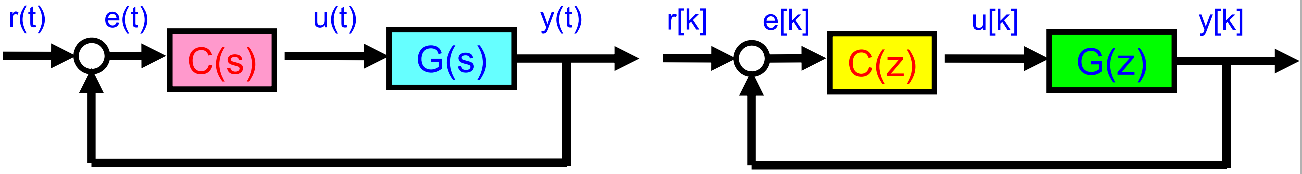
- By CT plant, design CT controller
- Transform CT controller into DT controller







Introduction: From CT Plant to DT Plant



- Plant (CT):
  - Input-Output Model:

$$u(t)$$
$$y(t)$$

$$G(s) = \frac{Y(s)}{U(s)}$$

  - State-Space Model:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\ y(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t) \end{aligned}$$

- Plant (DT):
  - Input-Output Model:

$$u[k]$$
$$y[k]$$

$$G(z) = \frac{Y(z)}{U(z)}$$

  - State-Space Model:

$$\begin{aligned} \mathbf{x}[k+1] &= \mathbf{F}\mathbf{x}[k] + \mathbf{H}u[k] \\ y[k] &= \mathbf{C}\mathbf{x}[k] + \mathbf{D}u[k] \end{aligned}$$

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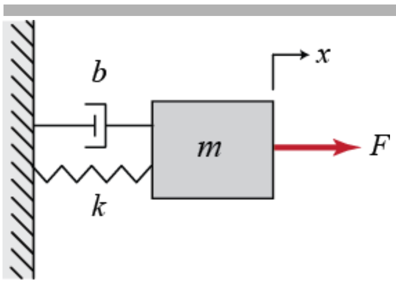
3

1

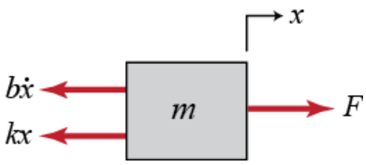
Introduction: Example - Mass-Spring-Damper System



$$F(t) - b\dot{x}(t) - kx(t) = m\ddot{x}(t)$$

$$\Rightarrow \quad F(s) - bsX(s) - kX(s) = ms^2X(s)$$

$$\Rightarrow \quad \frac{X(s)}{F(s)} = \frac{1}{ms^2+bs+k} \quad = \frac{Y(s)}{U(s)} = G(s)$$



$$y = x = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u$$

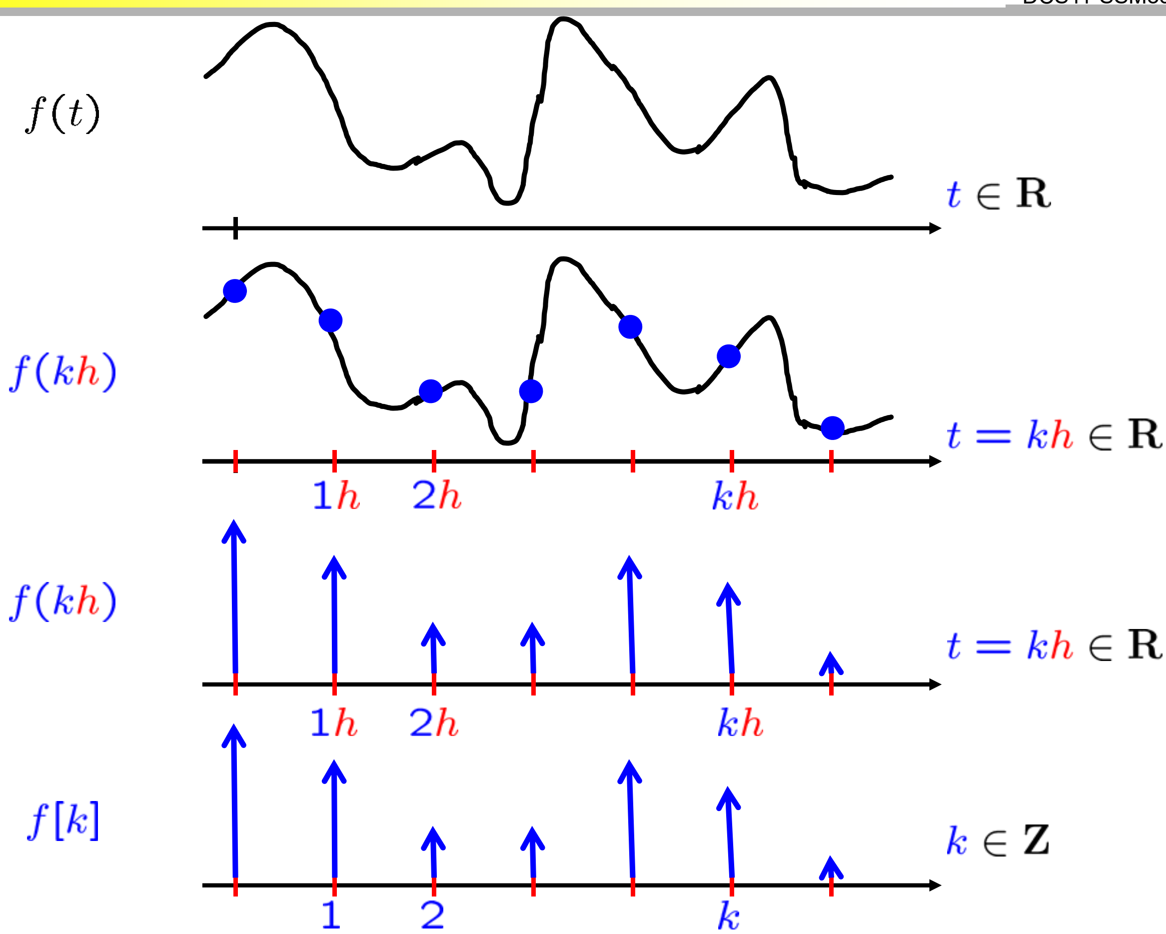
$$\mathbf{x} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ -\frac{k}{m}x - \frac{b}{m}\dot{x} + \frac{1}{m}F(t) \end{bmatrix}$$

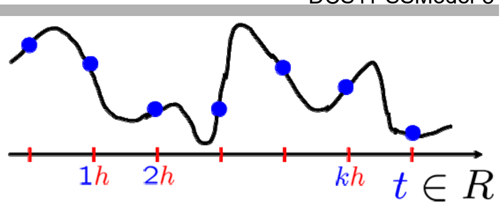
$$= \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} F(t)$$

$$= \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

Sampling a CT Signal



Sampling a CT Signal



$\mathbf{Z} = \{\cdots, -1, 0, 1, 2, \cdots\}$

- Sampling Period: h (sec)
- Sampling Rate: $f_s = \frac{1}{h}$ (Hz)
- Sampling Frequency: $w_s = \frac{2\pi}{h}$ (rad/s)
- Sampling Instants: t_k

$$\mathbf{T} = \{t_k = kh, k \in \mathbf{Z}\}$$

- CT Signals: $f(t) : f(t) \in \mathbf{R}, t \in \mathbf{R}$
- DT Signals: $f(t_k) : f(t_k) \in \mathbf{R}, t_k \in \mathbf{T}$

  $f(t_k) \in \mathbf{D}$
- DT Signals: $f[k] : f[k] \in \mathbf{R}, k \in \mathbf{Z}$

  $f[k] \in \mathbf{D}$

Sampling a CT State-Space System

- Consider the following LTI system:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)\\ y(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t)\end{aligned} \qquad \dot{\mathbf{x}}(t) = \frac{d}{dt}\mathbf{x}(t)$$

$$\begin{aligned}\mathbf{x}(t) &\in \mathbf{R}^{n}\\ \mathbf{u}(t) &\in \mathbf{R}^{r}\\ \mathbf{y}(t) &\in \mathbf{R}^{p}\end{aligned} \qquad \begin{aligned}\mathbf{A} &\in \mathbf{R}^{n\times n}\\ \mathbf{B} &\in \mathbf{R}^{n\times r}\\ \mathbf{C} &\in \mathbf{R}^{p\times n}\\ \mathbf{D} &\in \mathbf{R}^{p\times r}\end{aligned}$$

For SISO system,
$r = 1, p = 1$

See: Solution of 1st-order DE

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

$$\Rightarrow \quad \mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)d\tau$$

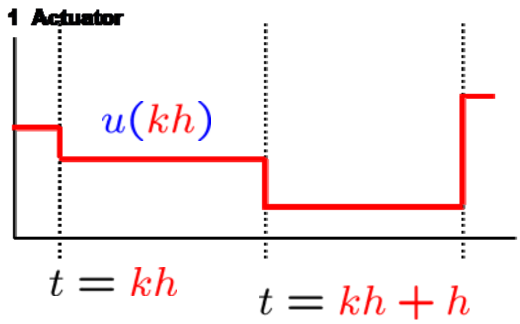
Let $t = t_{k+1} = kh + h$ & $t_0 = t_k = kh$

$$\Rightarrow \quad \mathbf{x}(kh+h) = e^{\mathbf{A}(h)}\mathbf{x}(kh) + \int_{kh}^{kh+h} e^{\mathbf{A}(kh+h-\tau)}\mathbf{B}u(\tau)d\tau$$

Sampling a CT State-Space System: Piecewise Constant Input

Let $u(\tau)$ be piecewise constant through h

$$u(\tau) = u(kh), \quad kh \le \tau < kh + h$$



Let $\eta = kh + h - \tau$

$$\Rightarrow \quad \mathbf{x}(kh+h) = e^{\mathbf{A}h}\mathbf{x}(kh) + \left(\int_0^h e^{\mathbf{A}\eta}d\eta\right)\mathbf{B}u(kh)$$

Let $\mathbf{F} = e^{\mathbf{A}h}$ & $\mathbf{H} = \left(\int_0^h e^{\mathbf{A}\eta}d\eta\right)\mathbf{B}$

$$\Rightarrow \quad \mathbf{x}((k+1)h) = \mathbf{F}\mathbf{x}(kh) + \mathbf{H}u(kh)$$

Then,

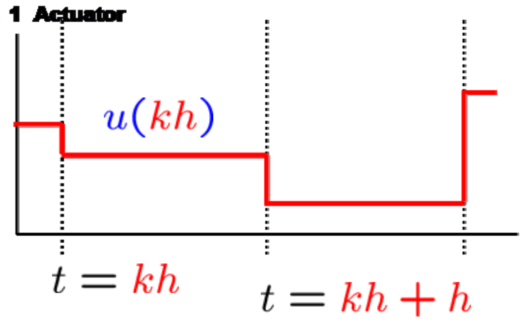
$$\begin{aligned}\mathbf{x}[k+1] &= \mathbf{F}\mathbf{x}[k] + \mathbf{H}u[k]\\ y[k] &= \mathbf{C}\mathbf{x}[k] + \mathbf{D}u[k]\end{aligned}$$

Franklin et al. 2002

Sampling a CT State-Space System: CT (A, B) to DT (F, H)

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)\\ y(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t)\end{aligned} \quad \xrightarrow{1} \quad \begin{aligned}\mathbf{x}[k+1] &= \mathbf{F}\mathbf{x}[k] + \mathbf{H}u[k]\\ y[k] &= \mathbf{C}\mathbf{x}[k] + \mathbf{D}u[k]\end{aligned}$$

- No approximation
- Control signal = Piecewise constant



$$\mathbf{F} = e^{\mathbf{A}h} \quad \& \quad \mathbf{H} = \left(\int_0^h e^{\mathbf{A}\eta} d\eta\right)\mathbf{B}$$

$$\Rightarrow \mathbf{F}(h) = e^{\mathbf{A}h} \quad \& \quad \mathbf{H}(h) = \left(\int_0^h e^{\mathbf{A}\eta} d\eta\right)\mathbf{B}$$

Sampling a CT State-Space System: CT (A, B) to DT (F, H)

$$\Rightarrow \mathbf{F}(h) = e^{\mathbf{A}h} \quad \& \quad \mathbf{H}(h) = \left(\int_0^h e^{\mathbf{A}\eta} d\eta\right)\mathbf{B}$$

$$\Rightarrow \frac{d}{dh}\mathbf{F}(h) = \frac{d}{dh}\left(e^{\mathbf{A}h}\right) = \mathbf{A}(e^{\mathbf{A}h}) = \mathbf{A}\mathbf{F}(h)$$

$$= (e^{\mathbf{A}h})\mathbf{A} = \mathbf{F}(h)\mathbf{A}$$

$$\Rightarrow \frac{d}{dh}\mathbf{H}(h) = \frac{d}{dh}\left(\int_0^h e^{\mathbf{A}\eta} d\eta\right)\mathbf{B} \qquad \mathbf{MN} \neq \mathbf{NM}$$

$$= \left(e^{\mathbf{A}h}\right)\mathbf{B} = \mathbf{F}(h)\mathbf{B} \neq \mathbf{B}\mathbf{F}(h)$$

$$\Rightarrow \frac{d}{dh}\begin{bmatrix}\mathbf{F}(h) & \mathbf{H}(h)\\ 0 & \mathbf{I}\end{bmatrix} = \begin{bmatrix}\mathbf{F}(h) & \mathbf{H}(h)\\ 0 & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ 0 & 0\end{bmatrix}$$

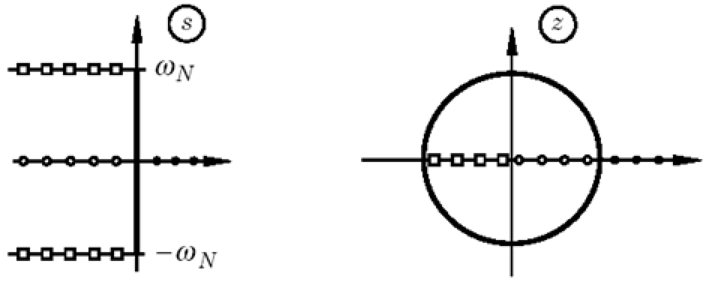
$$\Rightarrow \begin{bmatrix}\mathbf{F}(h) & \mathbf{H}(h)\\ 0 & \mathbf{I}\end{bmatrix} = \exp\left(\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ 0 & 0\end{bmatrix} h\right)$$

$$\Rightarrow \begin{bmatrix}\mathbf{A} & \mathbf{B}\\ 0 & 0\end{bmatrix} = \frac{1}{h}\ln\begin{bmatrix}\mathbf{F}(h) & \mathbf{H}(h)\\ 0 & \mathbf{I}\end{bmatrix}$$

Sampling a CT State-Space System: CT (A, B) to DT (F, H)

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\ y(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t) \end{aligned} \quad \xrightarrow{1} \quad \begin{aligned} \mathbf{x}[k+1] &= \mathbf{F}\mathbf{x}[k] + \mathbf{H}u[k] \\ y[k] &= \mathbf{C}\mathbf{x}[k] + \mathbf{D}u[k] \end{aligned}$$

$$\Rightarrow \quad \mathbf{F}(h) = e^{\mathbf{A}h} \quad \& \quad \mathbf{H}(h) = \left(\int_0^h e^{\mathbf{A}\eta} d\eta \right) \mathbf{B}$$



Sampling a CT State-Space System: Example

- Consider the following SS Model (Double Integrator):

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u &&= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x} &&= \mathbf{C}\,\mathbf{x} \end{aligned}$$

$$\begin{aligned} \mathbf{F} &= e^{\mathbf{A}h} = \mathbf{I} + \mathbf{A}h + \frac{\mathbf{A}^2 h^2}{2!} + \cdots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} h + \frac{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 h^2}{2!} + \cdots \\ &= \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Sampling a CT State-Space System: Example

- Consider the following SS Model (Double Integrator):

$$\begin{aligned}\dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u &&= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x} &&= \mathbf{C}\,\mathbf{x}\end{aligned}$$

$$\mathbf{H} = \left(\int_0^h e^{\mathbf{A}\eta} d\eta \right) \mathbf{B} \qquad e^{\mathbf{A}\tau} = \mathbf{I} + \mathbf{A}\tau + \frac{\mathbf{A}^2\tau^2}{2!} + \cdots$$

$$\sum_{k=0}^{\infty} \frac{\mathbf{A}^k h^{k+1}}{(k+1)!}$$

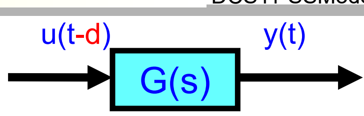
$$\mathbf{H} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k h^{k+1}}{(k+1)!} \mathbf{B} = \mathbf{I}h + \mathbf{A}\frac{h^2}{2!}\mathbf{B} + \mathbf{A}^2\frac{h^3}{3!}\mathbf{B} + \cdots$$

$$= \left(\begin{bmatrix} h & 0 \\ 0 & h \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \frac{h^2}{2} \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} h^2/2 \\ h \end{bmatrix}$$

- Homework 2-2

Sampling a CT State-Space System with Time Delay

- Consider the LTI system with delayed input:

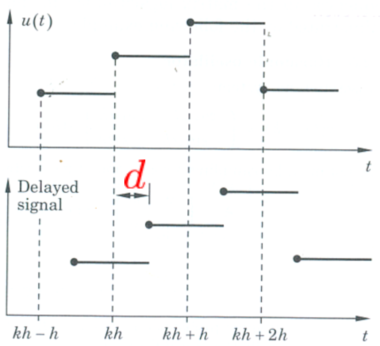


$0 \le d < h$

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t-d) \\ y(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t-d)\end{aligned}$$

$u(t)$

$u(t-d)$



$$\Rightarrow \mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau-d)d\tau$$

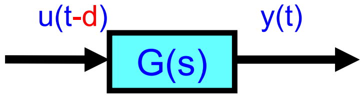
Let $t = t_{k+1} = kh + h$ & $t_0 = t_k = kh$

$$\Rightarrow \mathbf{x}(kh+h) = e^{\mathbf{A}(h)}\mathbf{x}(kh) + \int_{kh}^{kh+h} e^{\mathbf{A}(kh+h-\tau)}\mathbf{B}u(\tau-d)d\tau$$

$$\int_{kh}^{kh+h} \Rightarrow \int_{kh}^{kh+d} + \int_{kh+d}^{kh+h}$$

$$+ \int_{kh}^{kh+d} e^{\mathbf{A}(kh+h-\tau)}\mathbf{B}u(\tau-d)d\tau + \int_{kh+d}^{kh+h} e^{\mathbf{A}(kh+h-\tau)}\mathbf{B}u(\tau-d)d\tau$$

Sampling a CT State-Space System with Time Delay

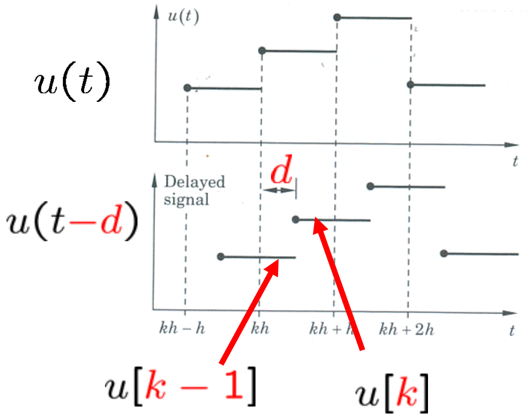


$$\Rightarrow \quad \mathbf{x}(kh+h)$$

$$= e^{\mathbf{A}(h)}\mathbf{x}(kh)$$

$$+ \int_{kh}^{kh+d} e^{\mathbf{A}(kh+h-\tau)}\mathbf{B}u(\tau-d)d\tau \qquad u((k-1)h) = u[k-1]$$

$$+ \int_{kh+d}^{kh+h} e^{\mathbf{A}(kh+h-\tau)}\mathbf{B}u(\tau-d)d\tau \qquad u(kh) = u[k]$$

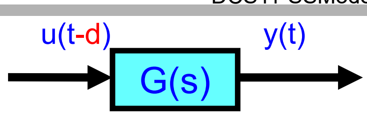


$$= \mathbf{F}\mathbf{x}(kh) \quad + \quad \mathbf{H}_1 u((k-1)h) \quad + \quad \mathbf{H}_0 u(kh)$$

$$\mathbf{H}_1 = \int_{kh}^{kh+d} e^{\mathbf{A}(kh+h-\tau)}\mathbf{B}d\tau = e^{\mathbf{A}(h-d)}\int_0^d e^{(\mathbf{A}\eta)}\mathbf{B}d\eta$$

$$\mathbf{H}_0 = \int_{kh+d}^{kh+h} e^{\mathbf{A}(kh+h-\tau)}\mathbf{B}d\tau = \int_0^{h-d} e^{(\mathbf{A}\eta)}\mathbf{B}d\eta$$

Sampling a CT State-Space System with Time Delay

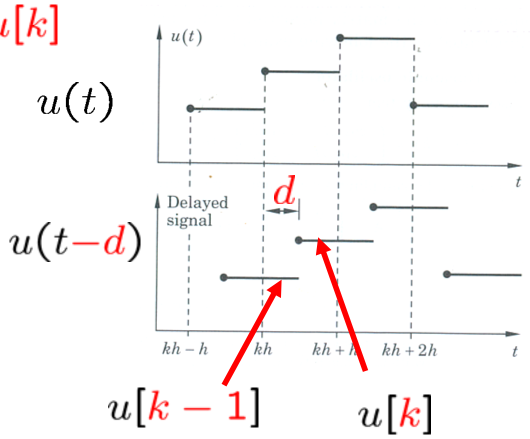


$$\Rightarrow \quad \mathbf{x}[k+1]$$

$$= \mathbf{F}\mathbf{x}[k] \quad + \quad \mathbf{H}_1 u[k-1] \quad + \quad \mathbf{H}_0 u[k]$$

$$\mathbf{H}_1 = e^{\mathbf{A}(h-d)}\int_0^d e^{(\mathbf{A}\eta)}\mathbf{B}d\eta$$

$$\mathbf{H}_0 = \int_0^{h-d} e^{(\mathbf{A}\eta)}\mathbf{B}d\eta$$

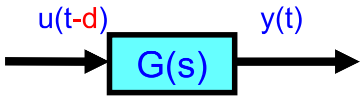


$$\begin{bmatrix} \mathbf{x}[k+1] \\ \mathbf{u}[k] \end{bmatrix} = \begin{bmatrix} \mathbf{F} & \mathbf{H}_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}[k] \\ \mathbf{u}[k-1] \end{bmatrix} + \begin{bmatrix} \mathbf{H}_0 \\ \mathbf{I} \end{bmatrix} \mathbf{u}[k]$$

- Homework 2-4

Sampling a CT State-Space System with Longer Time Delay

- Consider the LTI system with longer delayed input:



If $d \geq h$

Then, $d = (m-1)h + \eta$ $\qquad 0 \leq \eta < h$, $\quad m \in N$

$$\Rightarrow \quad \mathbf{x}(kh+h) = \mathbf{F}\mathbf{x}(kh) + \mathbf{H}_1 u((kh-(m-1)h)) + \mathbf{H}_0 u(kh-mh)$$

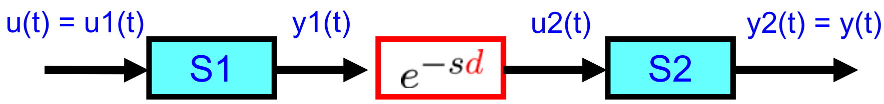
$$\mathbf{H}_1 = \int_{kh+(m-1)h}^{kh+(m-1)h+d} e^{\mathbf{A}(kh+h-\tau)}\mathbf{B}d\tau$$

$$\mathbf{H}_0 = \int_{kh+(m-1)h+d}^{kh+(m-1)h+h} e^{\mathbf{A}(kh+h-\tau)}\mathbf{B}d\tau$$

$$\begin{bmatrix} \mathbf{x}[k+1] \\ \mathbf{u}[k-(m-1)] \\ \mathbf{u}[k-m] \\ \vdots \\ \mathbf{u}[k] \end{bmatrix} = \begin{bmatrix} \mathbf{F} & \mathbf{H}_1 & \mathbf{H}_0 & 0 & \cdots & 0 \\ 0 & 0 & \mathbf{I} & 0 & \cdots & 0 \\ & & & & \ddots & \vdots \\ 0 & & \cdots & & & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}[k] \\ \mathbf{u}[k-m] \\ \mathbf{u}[k-m-1] \\ \vdots \\ \mathbf{u}[k-1] \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \mathbf{I} \end{bmatrix} \mathbf{u}[k]$$

Sampling a CT State-Space System with Inner Time Delay

- Consider the LTI system with inner time delay:



$$\textbf{S1:} \quad \begin{aligned} \dot{\mathbf{x}}_1(t) &= \mathbf{A}_1\mathbf{x}_1(t) + \mathbf{B}_1 u_1(t) \\ y_1(t) &= \mathbf{C}_1\mathbf{x}_1(t) + \mathbf{D}_1 u_1(t) \end{aligned}$$

$$\textbf{S2:} \quad \begin{aligned} \dot{\mathbf{x}}_2(t) &= \mathbf{A}_2\mathbf{x}_2(t) + \mathbf{B}_2 u_2(t) \\ y_2(t) &= \mathbf{C}_2\mathbf{x}_2(t) + \mathbf{D}_2 u_2(t) \end{aligned} \qquad u_2(t) = y_1(t-d)$$

- By using sampling interval, h, and $0 < d \leq h$
- Then, the sampled-data representation is as follows:

$$\mathbf{x}_1(kh+h) = \mathbf{F}_1(h)\mathbf{x}_1(kh) + \mathbf{H}_1(h)u(kh)$$

$$\mathbf{x}_2(kh+h) = \mathbf{F}_{21}\mathbf{x}_1(kh-h) + \mathbf{F}_2(h)\mathbf{x}_2(kh) + \mathbf{H}_{21}u(kh-h) + \mathbf{H}_2(h-d)u(kh)$$

Sampling a CT State-Space System with Inner Time Delay

- Where

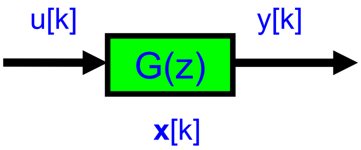
$$
\begin{aligned}
\mathbf{F}_i(\eta) &= e^{\mathbf{A}_i \eta}, \quad i = 1, 2 \\
\mathbf{F}^a_{21}(\eta) &= \int_0^\eta e^{\mathbf{A}_2 s}\mathbf{B}_2\mathbf{C}_1 e^{\mathbf{A}_1(\eta - s)} ds \\
\mathbf{H}_1(\eta) &= \int_0^\eta e^{\mathbf{A}_1 s}\mathbf{B}_1 ds \\
\mathbf{H}_2(\eta) &= \int_0^\eta e^{\mathbf{A}_2 s}\mathbf{B}_2\mathbf{C}_1\mathbf{H}_1(\tau - s) ds
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{F}_{21} &= \mathbf{F}^a_{21}(h)\mathbf{F}_1(h - d) \\
\mathbf{H}_{21} &= \mathbf{F}^a_{21}(h)\mathbf{H}_1(h - d) + \mathbf{F}^a_{21}(h - d)\mathbf{H}_1(d) \\
&\quad + \mathbf{F}_2(h - d)\mathbf{H}_2(d)
\end{aligned}
$$

- Reference:
  - Bjorn Wittenmark, "Sampling of a system with a time delay,“ IEEE Transactions on Automatic Control, Vol. 30, No. 5, pp. 507-510, May 1985.
  - https://ieeexplore.ieee.org/document/1103985

- Homework 2-1

Solution of DT State-Space System



$$
\begin{aligned}
\mathbf{x}[k+1] &= \mathbf{F}\mathbf{x}[k] + \mathbf{H}u[k] \\
y[k] &= \mathbf{C}\mathbf{x}[k] + \mathbf{D}u[k]
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{x}[k_0+1] &= \mathbf{F}\mathbf{x}[k_0] + \mathbf{H}u[k_0] \\
\mathbf{x}[k_0+2] &= \mathbf{F}\mathbf{x}[k_0+1] + \mathbf{H}u[k_0+1] \\
&= \mathbf{F}\left\{\mathbf{F}\mathbf{x}[k_0] + \mathbf{H}u[k_0]\right\} + \mathbf{H}u[k_0+1] \\
&= \mathbf{F}^2\mathbf{x}[k_0] + \mathbf{F}\mathbf{H}u[k_0] + \mathbf{H}u[k_0+1]
\end{aligned}
$$

$$
\begin{aligned}
&\mathbf{x}[k_0 + (k - k_0)] \\
\mathbf{x}[k] &= \mathbf{F}^{k-k_0}\mathbf{x}[k_0] + \mathbf{F}^{k-k_0-1}\mathbf{H}u[k_0] + \cdots + \mathbf{H}u[k-1] \\
&= \mathbf{F}^{k-k_0}\mathbf{x}[k_0] + \sum_{j=k_0}^{k-1} \mathbf{F}^{k-j-1}\mathbf{H}u[j] \\
y[k] &= \mathbf{C}\mathbf{F}^{k-k_0}\mathbf{x}[k_0] + \sum_{j=k_0}^{k-1} \mathbf{C}\mathbf{F}^{k-j-1}\mathbf{H}u[j] \quad + \mathbf{D}u[k]
\end{aligned}
$$