(4.2) $y'' - y = 0$

5 $y_1 = \cosh(x)$

let $y(x) = u(x)\cosh(x)$.

$\Rightarrow y' = u'\cosh(x) + u\sinh(x)$

$y'' = u''\cosh(x) + \underbrace{u'\sinh(x) + u'\sinh(x)}_{+} + u\cosh(x)$

$y'' - y = u''\cosh(x) + 2u'\sinh(x) + \cancel{u\cosh(x)} - \cancel{u\cosh(x)}$

$= u''\cosh(x) + 2u'\sinh(x) = 0$

$\Rightarrow u'' + 2u'\tanh(x) = 0$

$w \triangleq u' \Rightarrow w' + 2w\tanh(x) = 0$

$I.F. = e^{\int 2\tanh(x)dx} = e^{2\ln(\cosh(x))} = \cosh^2(x)$

$\frac{d}{dx}[\cosh^2(x)\, w] = 0$

$\Rightarrow \cosh^2(x)\, w = c \Rightarrow w = \frac{c}{\cosh^2(x)} = c\,\mathrm{sech}^2(x)$

$u = \int w = c\int \mathrm{sech}^2(x)dx = c\tanh(x)$

$\therefore\ y = u y_1 = c\tanh(x)\cosh(x) = c\sinh(x)$

$y_2 = \sinh(x)$

(4.2) $x^2y'' - xy' + 2y = 0$

13 $y_1 = x\sin(\ln x)$

$\frac{0}{0}x^2$, $y'' \underbrace{- \frac{1}{x}}_{P(x)} y' + \underbrace{\frac{2}{x^2}}_{Q(x)} y = 0 \quad x \in (0,\infty)$

$P(x) = -\frac{1}{x}$

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx$$

$$= x\sin(\ln x) \int \frac{e^{-\int \frac{-1}{x}dx}}{x^2\sin^2(\ln x)} dx$$

$$= x\sin(\ln x) \int \frac{x}{x^2\sin^2(\ln x)} dx$$

$$= x\sin(\ln x) \int \frac{\csc^2(\ln x)}{x} dx$$

$$\left[\int \csc^2 u\, du = -\cot u + c \right]$$

$$= x\sin(\ln x)\left[-\cot(\ln x)\right]$$

$$= -x\cos(\ln x)$$

$$\Rightarrow y_2(x) = x\cos(\ln x)$$

(4.2)

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$$y'' - 3y' + 2y = 5e^{3x}$$

$$y_1 = e^x$$

$$y = u y_1 = u e^x$$

$$y' = u' e^x + u e^x$$

$$y'' = u'' e^x + u' e^x + u' e^x + u e^x$$

$$y'' - 3y' + 2y = u'' e^x + 2u' e^x + u e^x$$

$$-3(u' e^x + u e^x)$$

$$+2(u e^x)$$

$$= u'' e^x - u' e^x = 5e^{3x}$$

$\div e^x \Rightarrow \quad u'' - u' = 5e^{2x}$

$w \triangleq u' \Rightarrow \quad w' - w = 5e^{2x}$

$$\text{I.F.} = e^{\int (-1)dx} = e^{-x}$$

$$\Rightarrow \frac{d}{dx}\left[e^{-x} w\right] = 5e^x$$

$$e^{-x} w = 5e^x + c_1$$

$$w = 5e^{2x} + c_1 e^x$$

$$u = \int w\, dx = \frac{5}{2} e^{2x} + c_1 e^x + c_2$$

$$y = u e^x = \underbrace{\frac{5}{2} e^{3x}}_{y_p} + \underbrace{c_1 e^{2x}}_{y_{c1}} + \underbrace{c_2 e^x}_{y_{c2}}$$