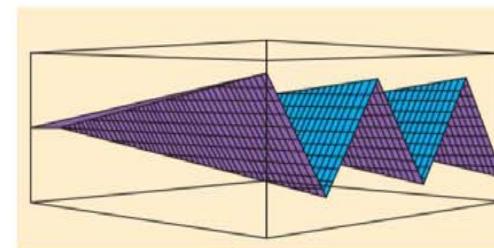


Fall 2019



微分方程 Differential Equations

Unit 12.1 Separable Partial Differential Equations

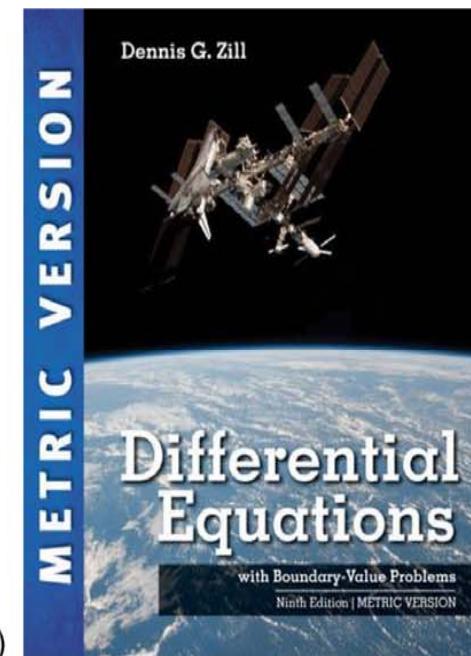
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NTU-EE

Sep19 – Jan20

$$\frac{\partial^2 u}{\partial \textcolor{red}{x}^2} = 4 \frac{\partial u}{\partial \textcolor{blue}{y}}$$

$$u(x, y) = \textcolor{red}{X}(x) Y(y)$$



- **12.1: Separable Partial Differential Equations**
- 12.2: Classical PDEs and BVPs
- 12.3: Heat Equation
- 12.4: Wave Equation
- 12.5: Laplace's Equation
- 12.6: Nonhomogeneous BVPs
- 12.7: Orthogonal Series Expansions
- 12.8: Higher-Dimensional Problems

- **Linear Second-Order PDE:**

$$A(x, y) \frac{\partial^2 u(x, y)}{\partial x^2} + B(x, y) \frac{\partial^2 u(x, y)}{\partial x \partial y} + C(x, y) \frac{\partial^2 u(x, y)}{\partial y^2} + D(x, y) \frac{\partial u(x, y)}{\partial x} + E(x, y) \frac{\partial u(x, y)}{\partial y} + F(x, y)u(x, y) = G(x, y)$$

$\left\{ \begin{array}{ll} G(x, y) = 0 & \text{Homogeneous} \\ G(x, y) \neq 0 & \text{Nonhomogeneous} \end{array} \right.$

- Finding **general solutions** is very difficult
Not all that useful in applications
- Finding **particular solutions** of
some of more important linear PDE
appearing in many applications

- Assume:

$$u(x, y) = X(x)Y(y)$$

$$\frac{\partial u}{\partial x} = \underline{X' Y} \quad \frac{\partial^2 u}{\partial x^2} = \underline{\underline{X'' Y}}$$

etc.

$$\frac{\partial u}{\partial y} = \underline{X Y'} \quad \frac{\partial^2 u}{\partial y^2} = \underline{\underline{X Y''}}$$

Example 2: Separation of Variables

$$\frac{\partial^2 u}{\partial x^2} = 4 \frac{\partial u}{\partial y}$$

If $u(x, y) = X(x) Y(y)$.

$$\Rightarrow \begin{cases} \frac{\partial^2 u}{\partial x^2} = X'' Y \\ \frac{\partial u}{\partial y} = X Y' \end{cases}$$

$$\Rightarrow X'' Y = 4 X Y'$$

$$\Rightarrow \frac{X''}{4X} = \frac{Y'}{Y} < 0$$

$\lambda = -\lambda$ (constant)

$\lambda = 0$

$\lambda > 0$

$\lambda < 0$

- Case 1: If $\lambda = 0$,

$$\Rightarrow \frac{X''}{4X} = \frac{Y'}{Y} = 0$$

$$\begin{aligned} X(x) &\neq 0 \\ Y(y) &\neq 0 \end{aligned}$$

$$\Rightarrow \left\{ \begin{array}{l} X'' = 0 \\ Y' = 0 \end{array} \right.$$

$$\Rightarrow \begin{cases} X(x) = C_1 + C_2 x \\ Y(y) = C_3 \end{cases}$$

$$\Rightarrow u(x, y) = X(x) Y(y)$$

$$= (C_1 + C_2 x)(C_3)$$

$$= A_1 + B_1 x$$

$$A_1 = C_1 C_3$$

$$B_1 = C_2 C_3$$

- Case 2: If $\lambda = -\alpha^2 < 0$,

$$\Rightarrow \frac{X''}{4X} = \frac{Y'}{Y} = +\alpha^2$$

$$\Rightarrow \begin{cases} X'' - 4\alpha^2 X = 0 \\ Y' = \alpha^2 Y \end{cases}$$

$$\Rightarrow \begin{cases} X(x) = C_4 \cosh(2\alpha x) + C_5 \sinh(2\alpha x) \\ Y(y) = C_6 e^{\alpha^2 y} \end{cases}$$

$$\Rightarrow u(x, y) = X(x) Y(y)$$

$$\begin{aligned} &= ((C_4 \cosh(2\alpha x) + C_5 \sinh(2\alpha x)) (C_6 e^{\alpha^2 y})) \\ &= A_2 e^{\alpha^2 y} \cosh(2\alpha x) + B_2 e^{\alpha^2 y} \sinh(2\alpha x) \end{aligned}$$

Example 2:

- Case 3: If $\lambda = \alpha^2 > 0$,

$$\Rightarrow \frac{X''}{4X} = \frac{Y'}{Y} = -\alpha^2$$

$$\Rightarrow \begin{cases} X'' + 4\alpha^2 X = 0 \\ Y' + \alpha^2 Y = 0 \end{cases}$$

$$\Rightarrow \begin{cases} X(x) = C_1 \cos(2\alpha x) + C_2 \sin(2\alpha x) \\ Y(y) = C_3 e^{-\alpha y} \end{cases}$$

$$\Rightarrow u(x, y) = X(x) Y(y)$$

$$= (C_1 \cos(2\alpha x) + C_2 \sin(2\alpha x)) (C_3 e^{-\alpha y})$$

$$= A_3 e^{-\alpha y} \cos(2\alpha x) + B_3 e^{-\alpha y} \sin(2\alpha x)$$

IF

$u_1(x, y), u_2(x, y), \dots, u_k(x, y)$ are
solutions of homogeneous linear PDE

THEN

$$u(x, y) = c_1 u_1(x, y) + c_2 u_2(x, y) + \dots + c_k u_k(x, y)$$

is also a solution,

where c_1, c_2, \dots, c_k are constants.

$$u(x, y) = \sum_{k=1}^{\infty} c_k u_k(x, y)$$

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + F u = 0$$

x *y*

A, B, C, D, E, F are real numbers

is said to be

hyperbolic

if $B^2 - 4AC > 0$ (Wave Equation)

parabolic

if $B^2 - 4AC = 0$ (heat Equation)

elliptic

if $B^2 - 4AC < 0$ (Laplace Equation)

Example 3: Classifying Linear 2nd-Order PDEs

(a) $3\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y}$

$\Rightarrow A = 3, B = 0, C = 0$

$\Rightarrow B^2 - 4AC = 0$

\Rightarrow parabolic

(b) $\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$

$\Rightarrow A = 1, B = 0, C = -1$

$\Rightarrow B^2 - 4AC > 0$

\Rightarrow hyperbolic

(c) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow A = 1, B = 0, C = 1$

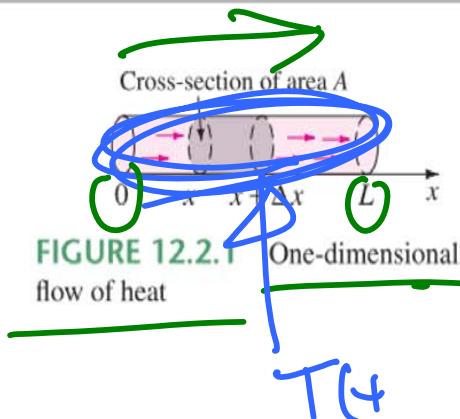
$\Rightarrow B^2 - 4AC < 0$

\Rightarrow elliptic

- One-Dimensional Heat Equation:

$$k \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad k > 0$$

$f(t, x)$



x, t

- One-Dimensional Wave Equation:

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

$c \infty^{-1}$

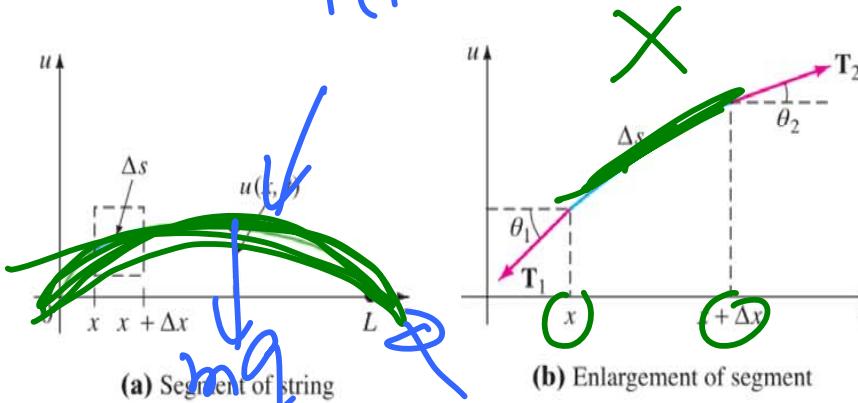


FIGURE 12.2.2 Flexible string anchored at $x = 0$ and $x = L$

- Two-Dimensional Form of Laplace Equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

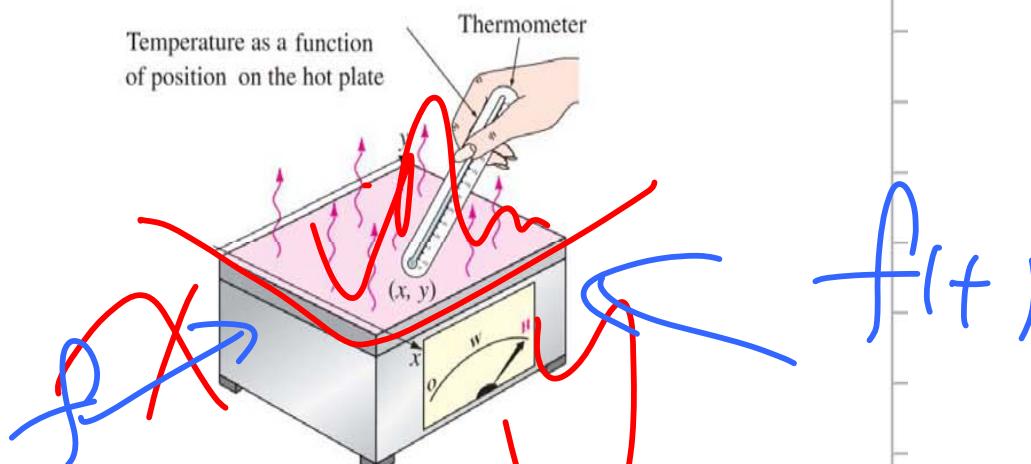


FIGURE 12.2.3 Steady-state temperatures in a rectangular plate

$f(+)$

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + F u$$

$$u(x, y) = X(x) Y(y)$$

$\frac{\partial u}{\partial x}$	$=$	$X' Y$	$\frac{\partial^2 u}{\partial x^2}$	$=$	$X'' Y$
$\frac{\partial u}{\partial y}$	$=$	$X Y'$	$\frac{\partial^2 u}{\partial y^2}$	$=$	$X Y''$

$$\left\{ \begin{array}{ll} \text{Hyperbolic} & \text{if } B^2 - 4AC > 0 \text{ (Wave Equation)} \\ \\ \text{Parabolic} & \text{if } B^2 - 4AC = 0 \text{ (Heat Equation)} \\ \\ \text{Elliptic} & \text{if } B^2 - 4AC < 0 \text{ (Laplace Equation)} \end{array} \right.$$