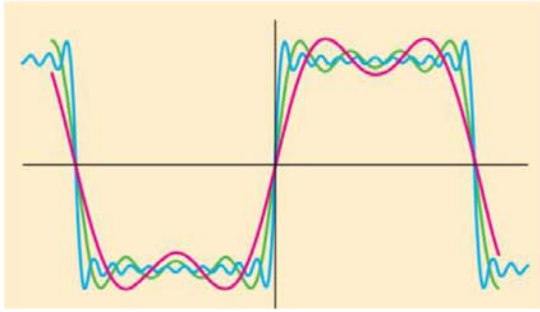

Fall 2019

微分方程
Differential Equations

Unit 11.2
Fourier Series

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NTU-EE

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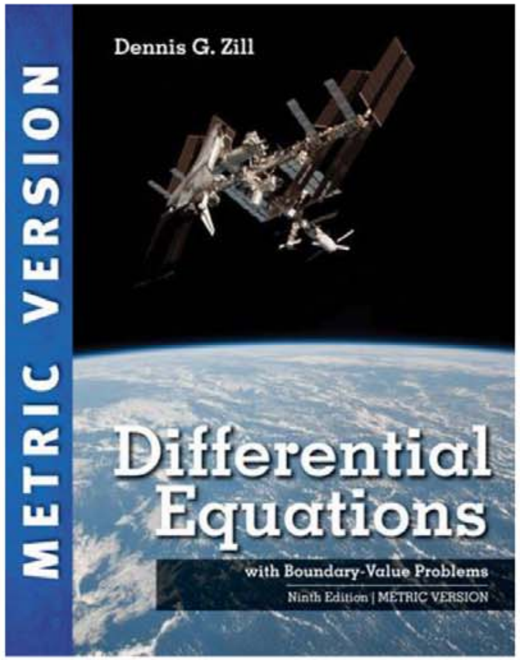
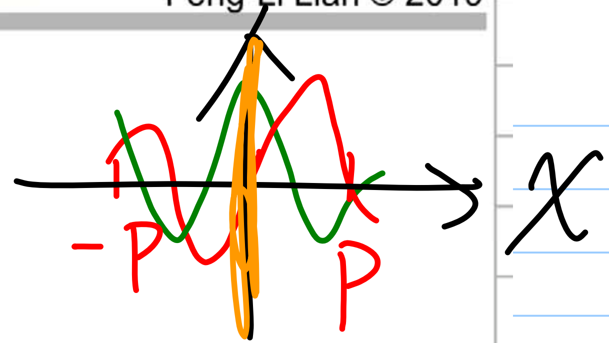
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cdot \cos\frac{n\pi}{p}x + b_n \cdot \sin\frac{n\pi}{p}x \right)$$



Figures and images used in these lecture notes are adopted from
Differential Equations with Boundary-Value Problems, 9th Ed., D.G. Zill, 2018 (Metric Version)

Outline

- 11.1: Orthogonal Functions
- **11.2: Fourier Series**
- 11.3: Fourier Cosine and Sine Series
- 11.4: Sturm-Liouville Problem (BVP)
- 11.5: Bessel and Legendre Series



- Fact:

$$\left\{ 1,\ \cos\frac{\pi}{p}x,\ \cos\frac{2\pi}{p}x,\ \cos\frac{3\pi}{p}x,\ \cdots \right.$$

$$\left. \sin\frac{\pi}{p}x,\ \sin\frac{2\pi}{p}x,\ \sin\frac{3\pi}{p}x,\ \cdots \right\} \text{ on } I = [-p, p]$$

is a complete orthogonal set of trigonometric functions

- A Trigonometric Series for $f(x)$ on $I = [-p, p]$

$$\Rightarrow f(x) = \frac{a_0}{2} \cdot 1 + a_1 \cos\frac{\pi}{p}x + a_2 \cos\frac{2\pi}{p}x + \cdots + a_n \cdot \cos\frac{n\pi}{p}x \cdots$$

$$+ b_1 \sin\frac{\pi}{p}x + b_2 \cdot \sin\frac{2\pi}{p}x + \cdots + b_n \cdot \sin\frac{n\pi}{p}x \cdots$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cdot \cos\frac{n\pi}{p}x + b_n \cdot \sin\frac{n\pi}{p}x \right)$$

- What are a_n, b_n:

- $n = 0$

$$\left(f(x), 1 \right) = \int_{-p}^{p} f(x) \cdot 1 \, dx$$

$$= \int_{-p}^{p} \left[\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right) \right] \cdot 1 \, dx$$

$$= \int_{-p}^{p} \frac{a_0}{2} \cdot 1 \, dx + \int_{-p}^{p} \left(a_1 \cancel{\cos \frac{\pi}{p} x} \right) \cdot 1 \, dx + \int_{-p}^{p} b_1 \cancel{\sin \frac{\pi}{p} x} \cdot 1 \, dx \cdots$$

$0 \qquad 0$

$$= \frac{a_0}{2} \left(p - (-p) \right) = p \, a_0 \qquad \cos \frac{n\pi}{p} \Big|_{-p}^{p} \qquad \sin \frac{n\pi}{p} \Big|_{-p}^{p}$$

$$a_0 = \frac{1}{p} \left(f(x), 1 \right) = \frac{1}{p} \int_{-p}^{p} f(x) \, dx$$

- $n = m$

$$\left(f(x),\ \cos\frac{m\pi}{p}x \right) = \int_{-p}^{p} f(x) \cdot \cos\frac{m\pi}{p}x \, dx$$

$$= \int_{-p}^{p} \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x \right) \right) \cos\frac{m\pi}{p}x \, dx$$

$$= \int_{-p}^{p} \frac{a_0}{2} \cos\frac{m\pi}{p}x \, dx \to 0 \ + \sum_{n=1}^{\infty} \int_{-p}^{p} a_n \cos\frac{n\pi x}{p} \cos\frac{m\pi}{p}x \, dx \quad m=n$$

$$n=m \quad \int_{-p}^{p} b_n \sin\frac{n\pi}{p}x \cos\frac{m\pi}{p}x \, dx \to 0$$

$$= \int_{-p}^{p} a_m \cos\frac{m\pi x}{p} \cos\frac{m\pi x}{p} \, dx$$

$$a_m = \frac{\int_{-p}^{p} f(x) \cos\frac{m\pi x}{p} dx}{\int_{-p}^{p} \cos\frac{m\pi x}{p} \cos\frac{m\pi x}{p} dx = p} = \frac{1}{p} \int_{-p}^{p} f(x) \cos\frac{m\pi x}{p} dx$$

11.2: Def 11.2.1: Fourier Series

$f(x)$ on $I = [-p, p]$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x \right)$$

1

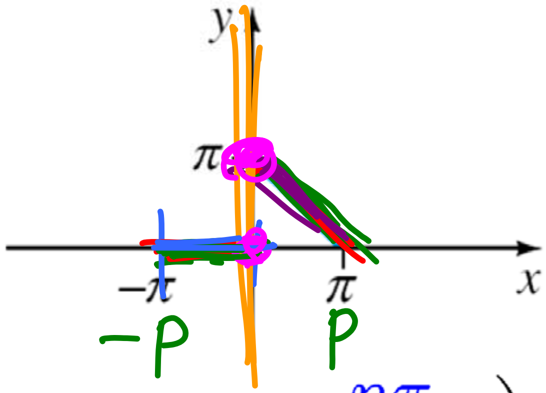
$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,1\,dx$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cdot\cos\frac{n\pi}{p}x\,dx$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\cdot\sin\frac{n\pi}{p}x\,dx$$

11.2: Example 1: Expansion in a Fourier Series

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases}$$



$a_0 \to \frac{\pi}{4}$

$p = \pi \quad \Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cdot \cos \frac{n\pi}{\pi} x + b_n \cdot \sin \frac{n\pi}{\pi} x \right)$

$a_0 = \frac{1}{p} \int_{-p}^{p} f(x)\,dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi} \int_0^{\pi} (\pi - x)\,dx$

$= \left(\frac{\pi^2}{2}\right) \to \frac{\pi}{4}$

$a_n = \frac{1}{p} \int_{-p}^{p} f(x) \cos \frac{n\cancel{\pi}x}{\cancel{p}}\,dx \quad = \frac{1}{p} \int_0^{\pi} (\pi - x) \cos nx\,dx$

$= \frac{1}{p} \int_0^{\pi} \pi \cos nx\,dx - \frac{1}{p} \int_0^{\pi} x \cos nx\,dx$

$u \quad v \quad = uv - \int u'v$

$= \frac{1-(-1)^n}{n^2 \pi} \qquad \cos(n\pi)$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\,\sin\frac{n\cancel{\pi} x}{\cancel{p}=\pi}\,dx$$

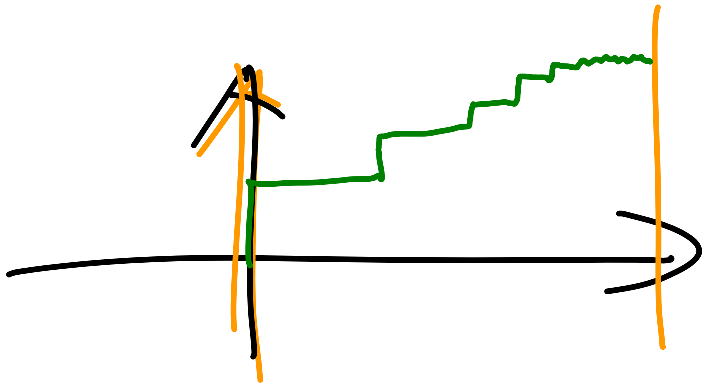
$$= \frac{1}{p}\int_0^p (\pi - x)\,\sin nx\,dx = \frac{1}{p}\int_0^p \pi\sin nx\,dx - \frac{1}{p}\int_0^p x\sin nx\,dx$$

$$\int u v' = uv - \int u' v$$

$$= \frac{1}{n}$$

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2\pi}\cos nx + \frac{1}{n}\sin nx\right]$$

$$= \frac{\pi}{4} + \left(\frac{2}{\pi}\cos x + \sin x\right) + \left(0 + \frac{1}{2}\sin x\right) + \cdots$$



11.2: Theorem 11.1: Conditions for Convergence

- f, f' : piecewise continuous on $I = (-p, p)$

  $\rightarrow$ may have discontinuities at a finite number of points

THEN,

$\rightarrow$ at a point of continuity

the fourier series of $f(x)$ converges to $f(x)$ at the point

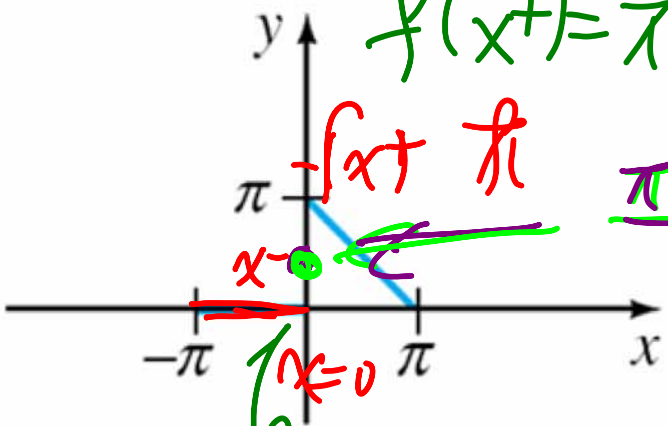
$\rightarrow$ at a point of discontinuity

the fourier series of $f(x)$ converges to the average

$$\frac{f(x^+) + f(x^-)}{2}$$

11.2: Example 2

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases}$$



$f(x^+) = \pi$

$f(x^-) = 0$

$\frac{\pi + 0}{2} = \frac{\pi}{2}$

$$\Rightarrow \frac{f(x^+) + f(x^-)}{2} = \frac{\pi + 0}{2} = \frac{\pi}{2}$$

$$\Rightarrow f(x)\Big|_{x=0} = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2\pi}\cos nx + \frac{1}{n}\sin nx\right)\Bigg|_{x=0}$$

$$= \frac{\pi}{4} + \boxed{\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2\pi}} \rightarrow \frac{\pi}{2} = \frac{\pi}{4}$$

$\rightarrow \frac{\pi}{4}$

$\frac{2}{\pi}\left(1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49}\right)$

$$\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2\pi} = \frac{\pi}{4}$$

$$\left(\frac{2}{\pi} + 0 + \frac{2}{9\pi} + 0 + \frac{2}{25\pi} + \cdots\right.$$

on $I = [-p, p]$

$$\left\{1, \quad \cos\frac{\pi}{p}x, \quad \cos\frac{2\pi}{p}x, \quad \cos\frac{3\pi}{p}x, \quad \cdots \cos\frac{n\pi}{p}x, \quad \cdots \right.$$
$$\left. \sin\frac{\pi}{p}x, \quad \sin\frac{2\pi}{p}x, \quad \sin\frac{3\pi}{p}x, \quad \cdots \quad \sin\frac{n\pi}{p}x, \quad \cdots \right\}$$

$\sin \frac{2\pi t}{2P}$

$\Rightarrow$ Period $= \frac{2\pi}{\frac{n\pi}{p}} = \frac{2p}{n}$ $\quad n = 1, 2, 3$

$P = 24 \Rightarrow$ Period $= 2 \times 24 = 48, 24, 16$

$\Rightarrow$ Fundamental Period $= 2P = 48$

- $f(x)$ on $I = [-p, p]$ $\to 2p$

How about on

$[-5p, -3p], \quad [-3p, -p], \quad [p, 3p], \quad [3p, 5p] \quad \cdots$

$2p$

$$\Rightarrow f(x+2p) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cdot \cos \frac{n\pi}{p}(x+2p) + b_n \cdot \sin \frac{n\pi}{p}(x+2p) \right)$$

$\cos \frac{n\pi}{p} x \qquad \sin \frac{n\pi x}{p}$

$= \boxed{f(x)}$

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cdot \cos\frac{n\pi}{p}x + b_n \cdot \sin\frac{n\pi}{p}x \right)$$ $2N+1$

$N \quad N$

$$S_0(x) = \frac{a_0}{2}$$

- In Example 1:

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left(\frac{1-(-1)^n}{n^2\,\pi}\cos nx + \frac{1}{n}\sin nx \right)$$

$$S_0(x) = \frac{\pi}{4}$$

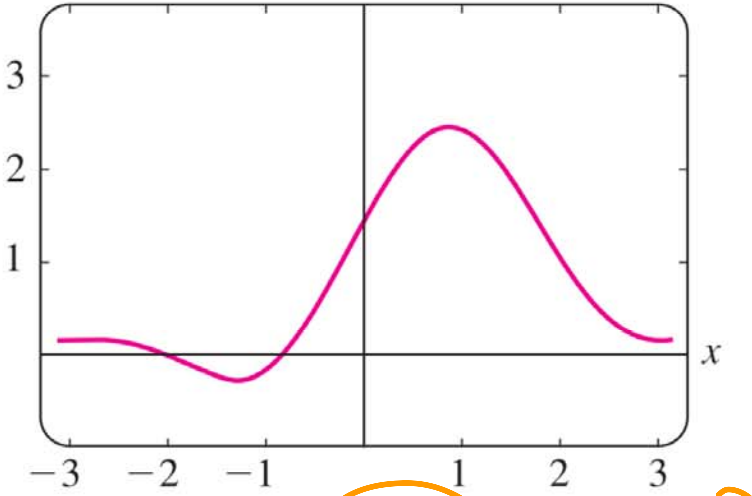
$$S_1(x) = \frac{\pi}{4} + \left(\frac{2}{\pi}\cdot\cos x + \sin x\right)$$ $1+1+1 = 2N+1 = 3$

$$S_2(x) = \frac{\pi}{4} + \left(\frac{2}{\pi}\cdot\cos x + \sin x\right) + \left(\frac{1}{2}\sin 2x\right)$$ $2N+1 = 5$

$5 \quad 2$

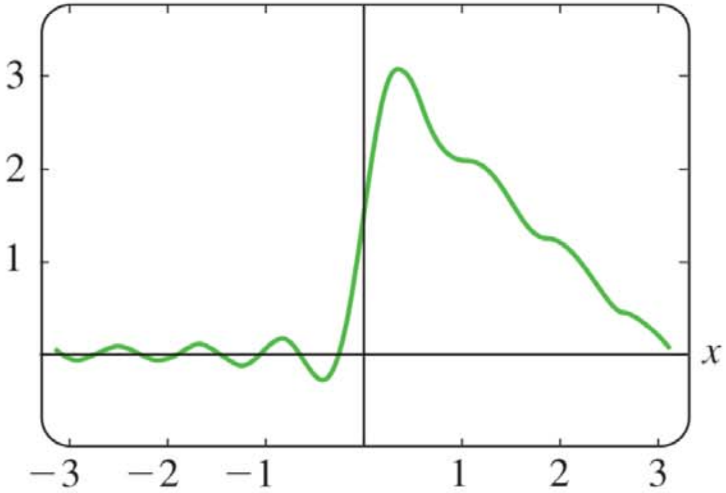
11.2: Sequence of Partial Sums

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cdot \cos\frac{n\pi}{p}x + b_n \cdot \sin\frac{n\pi}{p}x \right)$$



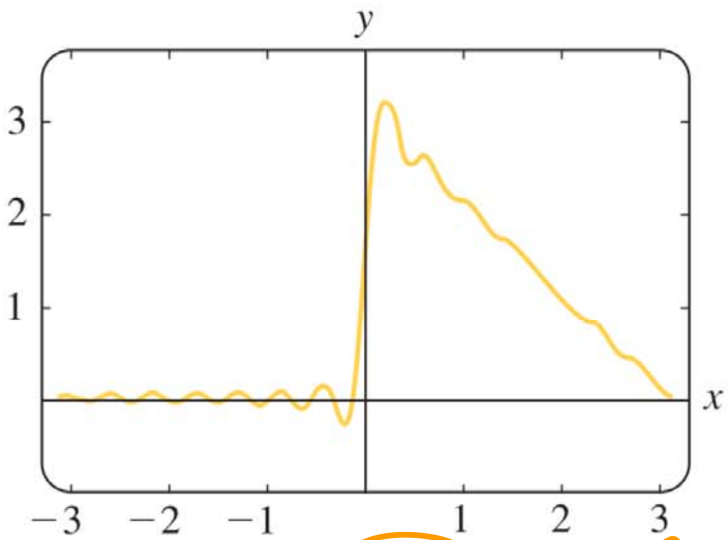
(a) $S_3(x)$

7.



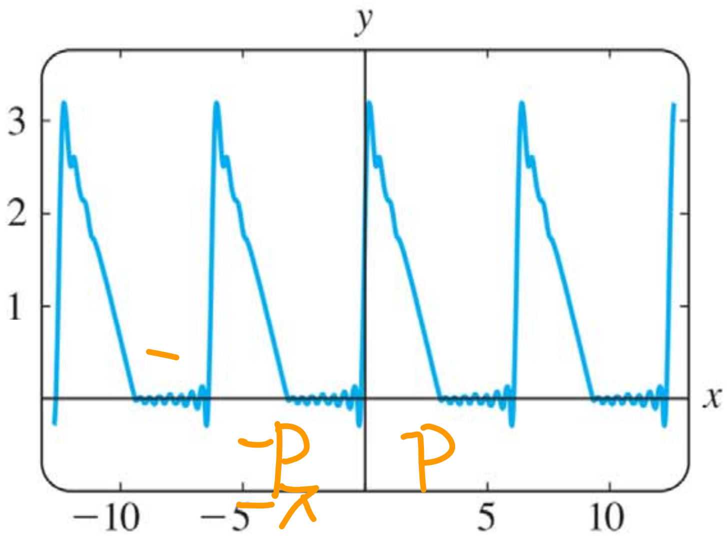
(b) $S_8(x)$

17



(c) $S_{15}(x)$

31



(d) $S_{15}(x)$

- Fourier Series $\quad f(x)$ on $I = [-p, p]$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cdot \cos\frac{n\pi}{p}x + b_n \cdot \sin\frac{n\pi}{p}x \right)$$

$$a_0 = \frac{1}{p} \int_{-p}^{p} f(x)\, dx$$

$$a_n = \frac{1}{p} \int_{-p}^{p} f(x) \cdot \cos\frac{n\pi}{p}x\, dx$$

$$b_n = \frac{1}{p} \int_{-p}^{p} f(x) \cdot \sin\frac{n\pi}{p}x\, dx$$

- Sequence of Partial Sums

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cdot \cos\frac{n\pi}{p}x + b_n \cdot \sin\frac{n\pi}{p}x \right)$$

$$S_0(x) = \frac{a_0}{2}$$