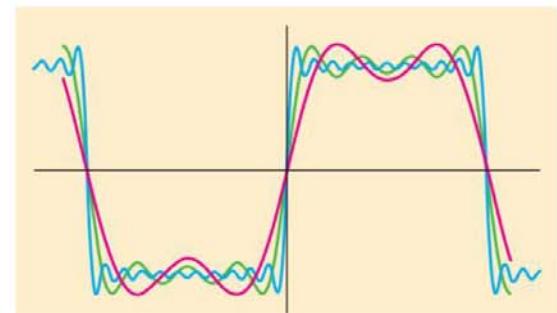


Fall 2019



微分方程 Differential Equations

Unit 11.1 Orthogonal Functions

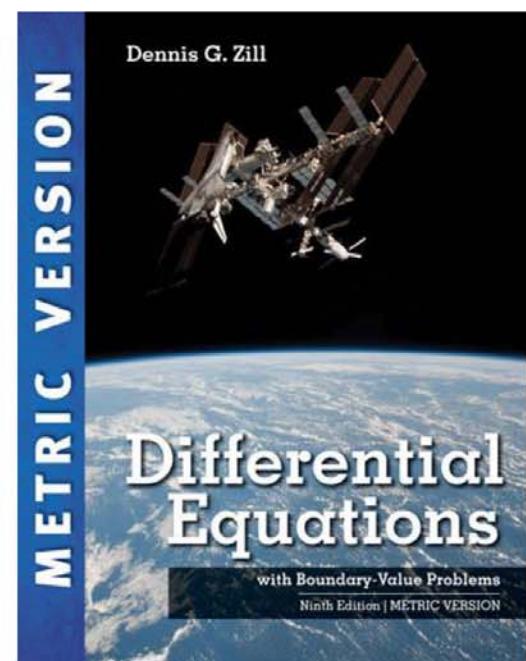
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Sep19 – Jan20

$$\left(f_1, f_2 \right) = \int_a^b f_1(x) \cdot f_2(x) dx = 0$$

Figures and images used in these lecture notes are adopted from
Differential Equations with Boundary-Value Problems, 9th Ed., D.G. Zill, 2018 (Metric Version)



- **11.1: Orthogonal Functions**
- 11.2: Fourier Series
- 11.3: Fourier Cosine and Sine Series
- 11.4: Sturm-Liouville Problem (BVP)
- 11.5: Bessel and Legendre Series

- In R^3 vector space

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad x_i, y_i \in R$$

- Inner Product (Dot Product) of x and y in R^3 :

$$\begin{aligned} (\cdot, \cdot) : \overline{R^3} \times \overline{R^3} &\rightarrow R \\ (x, y) &= x^T y = [x_1 \ x_2 \ x_3] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = x_1 y_1 + x_2 y_2 + x_3 y_3 \\ &= y_1 x_1 + y_2 x_2 + y_3 x_3 = [y_1 \ y_2 \ y_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = y^T x \\ &= (y, x) \end{aligned}$$

$$\underline{x}, \underline{y}, z \in R^3, \quad k \in R$$

$$(1) \quad (\underline{\underline{x}}, \underline{\underline{y}}) = (\underline{\underline{y}}, \underline{\underline{x}})$$

$$(2) \quad (\underline{k}\underline{x}, \underline{y}) = (\underline{k}\underline{x}, \underline{y}) = (\underline{x}, \underline{k}\underline{y})$$

$$(3) \quad (\underline{\underline{x}}, \underline{\underline{x}}) = 0 \quad \text{only if} \quad \underline{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1^2 + x_2^2 + x_3^2 = 0$

$$(\underline{x}, \underline{x}) > 0 \quad \text{only if} \quad \underline{x} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(4) \quad (\underline{\underline{x}} + \underline{\underline{y}}, z) = (\underline{\underline{x}}, z) + (\underline{\underline{y}}, z)$$

f_1, f_2 , : two functions on an interval $[a, b]$

$$\left(\underline{\underline{f_1}}, \underline{\underline{f_2}} \right) \triangleq \underline{\underline{\int_a^b f_1(x) \cdot f_2(x) dx}}$$

$$(1) \quad (\underline{\underline{f_1}}, \underline{\underline{f_2}}) = (\underline{\underline{f_2}}, \underline{\underline{f_1}})$$

$$(2) \quad (\underline{k} \underline{\underline{f_1}}, \underline{\underline{f_2}}) = \underline{k} (\underline{\underline{f_1}}, \underline{\underline{f_2}}) = (\underline{\underline{f_1}}, \underline{k} \underline{\underline{f_2}})$$

$$(3) \quad (\underline{\underline{f_1}}, \underline{\underline{f_1}}) = 0 \quad \text{only if } \underline{\underline{f_1}} \equiv 0, \quad \forall x \in [a, b]$$

$$(\underline{\underline{f_1}}, \underline{\underline{f_1}}) > 0 \quad \text{only if } \underline{\underline{f_1}} \not\equiv 0, \quad \forall x \in [a, b]$$

$$(4) \quad (\underline{\underline{f_1}} \oplus \underline{\underline{f_2}}, \underline{\underline{f_3}}) = (\underline{\underline{f_1}}, \underline{\underline{f_3}}) + (\underline{\underline{f_2}}, \underline{\underline{f_3}})$$

f_1, f_2 , : two functions on an interval $[a, b]$

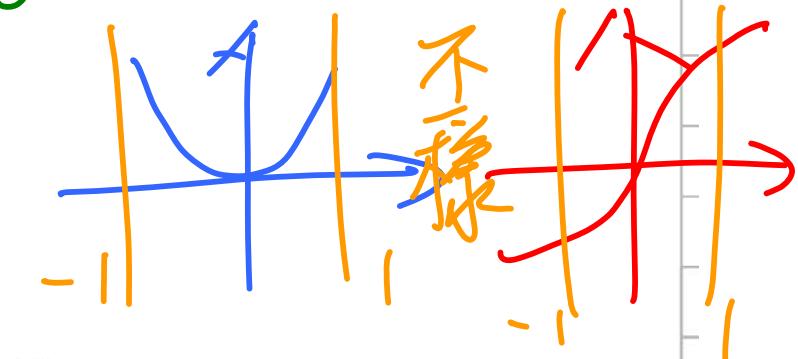
IF $\left(f_1, f_2 \right) = \int_a^b f_1(x) \cdot f_2(x) dx = 0$

THEN f_1, f_2 is said to be **orthogonal** on $[a, b]$

$$(a) \quad f_1(x) = x^2, \quad f_2(x) = x^3, \quad \text{on } [-1, 1]$$

$$\int_{-1}^1 x^2 \cdot x^3 dx = \int_{-1}^1 x^5 dx = \frac{1}{6} x^6 \Big|_{-1}^1 = \frac{1}{6} (1^6 - (-1)^6) = 0$$

x^2, x^3 orthogonal

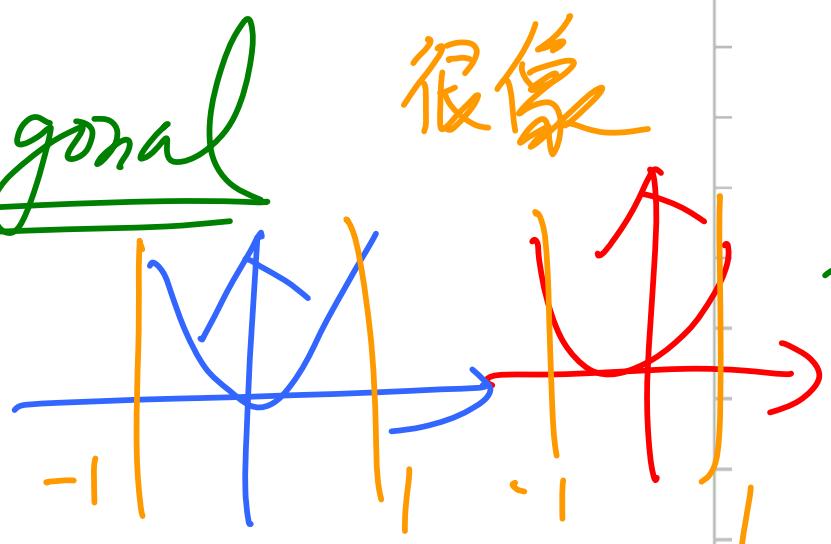


$$(b) \quad f_1(x) = x^2, \quad f_2(x) = x^4, \quad \text{on } [-1, 1]$$

$$\int_{-1}^1 x^2 \cdot x^4 dx = \int_{-1}^1 x^6 dx = \frac{1}{7} x^7 \Big|_{-1}^1 = \frac{1}{7} [1^7 - (-1)^7] = \frac{2}{7}$$

x^2, x^4 not orthogonal

很像



$$\left\{ \underline{\phi_0(x)}, \underline{\phi_1(x)}, \underline{\phi_2(x)}, \dots \right\} :$$

a set of real-valued functions on $[a, b]$

IF $\left(\underline{\phi_m(x)}, \underline{\phi_n(x)} \right) = \int_a^b \underline{\phi_m(x)} \cdot \underline{\phi_n(x)} dx = 0,$ $m \neq n$

THEN it is an orthogonal set on $[a, b]$

$$\left\{ \underline{1}, \underline{\cos(x)}, \underline{\cos(2x)}, \dots, \underline{\cos(nx)}, \dots \right\} \quad \text{on } I = [-\pi, \pi]$$

$n \neq 0$

$$\int_{-\pi}^{\pi} 1 \cdot \cos(nx) dx = \int_{-\pi}^{\pi} \cos nx dx$$

$$= \frac{1}{n} \sin nx \Big|_{-\pi}^{\pi} = \frac{1}{n} [0 - 0] = 0$$

$m \neq n$

$$\int_{-\pi}^{\pi} \cos mx \cos nx dx = \int_{-\pi}^{\pi} \left[\frac{1}{2} [\cos(m+n)x + \cos(m-n)x] \right] dx$$

$$= \frac{1}{2} \frac{1}{m+n} \sin(m+n)x \Big|_{-\pi}^{\pi} + \frac{1}{2} \frac{1}{m-n} \sin(m-n)x \Big|_{-\pi}^{\pi}$$

$$= \frac{1}{2(m+n)} [0 - 0] + \frac{1}{2(m-n)} [0 - 0] = 0$$

\Rightarrow orthogonal set *

- Norm of $\begin{cases} \text{a vector} & \text{in } R^n \\ \text{a function} & \text{on } [a, b] \end{cases}$

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \in R^3 \quad f(x) \text{ a function on } [a, b]$$

$$\|u\| \triangleq \sqrt{u_1^2 + u_2^2 + u_3^2} = \sqrt{(u, u)}$$

$$\|f(x)\| \triangleq \sqrt{(f, f)} = \sqrt{\int_a^b f^2(x) dx}$$

$\{\phi_0(x), \phi_1(x), \phi_2(x), \dots\}$: on $[a, b]$ orthonormal set

(1) $\{\phi_0(x), \phi_1(x), \phi_2(x), \dots\}$: orthogonal set

(2) $\|\phi_n(x)\| = 1$, $n = 0, 1, 2, \dots$

i.e., $(\phi_i(x), \phi_j(x)) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

$$\{1, \cos(x), \cos(2x), \dots, \cos(nx), \dots\} \quad \text{on } I = [-\pi, \pi]$$

$$\|1\| = \sqrt{\int_{-\pi}^{\pi} 1 \cdot 1 dx} = \sqrt{x \Big|_{-\pi}^{\pi}} = \sqrt{2\pi}$$

$$\begin{aligned}\|\cos nx\| &= \sqrt{\int_{-\pi}^{\pi} \cos nx \cos nx dx} = \sqrt{\int_{-\pi}^{\pi} \frac{1}{2} (1 + \cos 2nx) dx} \\ &= \sqrt{\frac{1}{2} (2\pi)} = \sqrt{\pi}\end{aligned}$$

$$\Rightarrow \left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos(x)}{\sqrt{\pi}}, \frac{\cos(2x)}{\sqrt{\pi}}, \dots, \frac{\cos(nx)}{\sqrt{\pi}}, \dots \right\}$$

orthonormal set

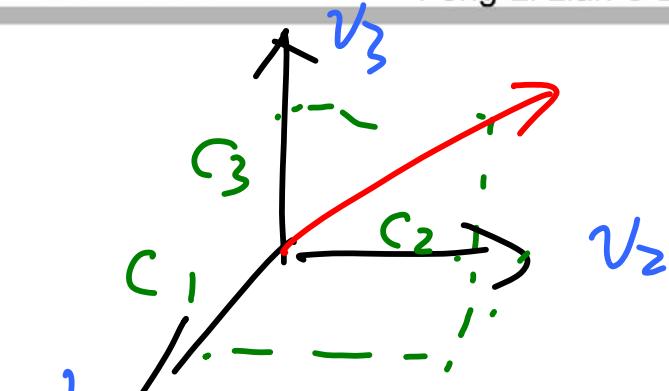
ortho normal set

11.1: Vector Decomposition & Orthogonal Series Expansion

u in R^3

$$u = c_1 v_1 + c_2 v_2 + c_3 v_3$$

$$\begin{aligned} (u, v_1) &= (c_1 v_1 + c_2 v_2 + c_3 v_3, v_1) \\ &= c_1 (v_1, v_1) + c_2 (v_2, v_1) + c_3 (v_3, v_1) \\ &= c_1 (v_1, v_1) \end{aligned}$$



$$c_1 = \frac{(u, v_1)}{(v_1, v_1)} = \frac{(u, v_1)}{\|v_1\|^2} = \frac{(u, v_1)}{1}$$

$$c_2 = \frac{(u, v_2)}{(v_2, v_2)} = \frac{(u, v_2)}{\|v_2\|^2} = \frac{(u, v_2)}{1}$$

$y = f(x)$ on $[a, b]$

$$f(x) = C_0 \phi_0(x) + C_1 \phi_1(x) + C_2 \phi_2(x) + \dots$$

orthonormal

$$\int_a^b f(x) \phi_m(x) dx = \int_a^b (C_0 \phi_0 + C_1 \phi_1 + C_2 \phi_2 + \dots) \phi_m dx$$

$$= C_0 \int_a^b \phi_0 \phi_m dx + C_1 \int_a^b \phi_1 \phi_m dx + \dots$$

$$+ C_m \int_a^b \phi_m \phi_m dx + \dots$$

$$C_m = \frac{\int_a^b f(x) \phi_m(x) dx}{\int_a^b \phi_m \phi_m dx} = \frac{(f, \phi_m)}{(\phi_m, \phi_m)}$$

$$= \frac{1}{(\phi_m, \phi_m)}$$

$$\underline{\underline{f(x)}} = \sum_{n=0}^{\infty} c_n \phi_n(x)$$

$$c_n = \frac{\int_a^b f(x) \phi_n(x) dx}{\int_a^b \phi_n^2(x) dx}$$

$$= \frac{(f(x), \phi_n(x))}{(\phi_n(x), \phi_n(x))} \xrightarrow{\text{green arrow}} \|\phi_n(x)\|^2 = 1$$

$$\left\{ \phi_0(x), \phi_1(x), \phi_2(x), \dots \right\} :$$

orthogonal with respect to a weight function

$w(x)$

IF

$$\int_a^b w(x) \cdot \phi_m(x) \cdot \phi_n(x) dx = 0, \quad m \neq n$$

$$\underline{\underline{f(x)}} = \underline{\underline{c_0 \phi_0(x)}} + \underline{\underline{c_1 \phi_1(x)}} + \underline{\underline{c_2 \phi_2(x)}} + \dots$$

$$c_n = \frac{\int_a^b f(x) \cdot w(x) \cdot \phi_n(x) dx}{\int_a^b w(x) \cdot \phi_n^2(x) dx}$$

$$S = \{\phi_0(x), \phi_1(x), \phi_2(x), \dots\} : m$$

IF the ONLY function orthogonal to $\phi_i(x)$ is ZERO function,

THEN S is a complete set

$$\{1, \cos(x), \cos(2x), \dots\}$$

not complete

$$\int_{-\pi}^{\pi} \sin x \cos 3x dx = \dots = 0$$

$$\{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \dots\}$$

complete

f_1, f_2 , : two functions on an interval $[a, b]$

- Inner Product of Functions

$$(f_1, f_2) \triangleq \int_a^b f_1(x) \cdot f_2(x) dx$$

- Orthogonal Functions

$$(f_1, f_2) = \int_a^b f_1(x) \cdot f_2(x) dx = 0$$

- Orthogonal Set

$$\{\phi_0(x), \phi_1(x), \phi_2(x), \dots\} :$$

IF $(\phi_m(x), \phi_n(x)) = \int_a^b \phi_m(x) \cdot \phi_n(x) dx = 0, m \neq n$

- Orthonormal Set

$$\{\underline{\phi_0(x)}, \underline{\phi_1(x)}, \underline{\phi_2(x)}, \dots\} :$$

i.e., $(\underline{\phi_i(x)}, \underline{\phi_j(x)}) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

- Vector Decomposition & Orthogonal Series Expansion

$y = f(x)$ on $[a, b]$

$$\underline{f(x)} = \sum_{n=0}^{\infty} c_n \phi_n(x)$$

$$c_n = \frac{\int_a^b f(x) \cdot \phi_n(x) dx}{\int_a^b \phi_n^2(x) dx}$$

$$= \frac{(f(x), \phi_n(x))}{(\phi_n(x), \phi_n(x))} \|\phi_n(x)\|^2$$

- Orthogonal Set/Weight Functions

$\{\phi_0(x), \phi_1(x), \phi_2(x), \dots\} :$

orthogonal with respect to a weight function $w(x)$

IF $\int_a^b w(x) \cdot \phi_m(x) \cdot \phi_n(x) dx = 0, m \neq n$

$$f(x) = c_0 \phi_0(x) + c_1 \phi_1(x) + c_2 \phi_2(x) + \dots$$

$$c_n = \frac{\int_a^b f(x) \cdot w(x) \cdot \phi_n(x) dx}{\int_a^b w(x) \cdot \phi_n^2(x) dx}$$