

Fall 2019

微分方程 Differential Equations

Unit 07.6

Systems of Linear Differential Equations

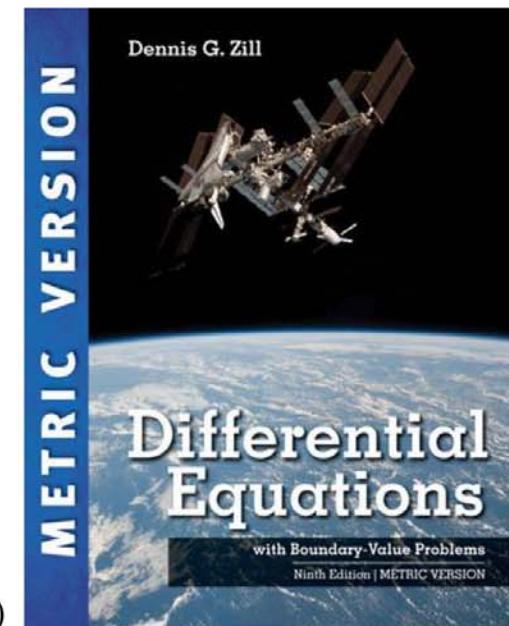
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NTU-EE

Sep19 – Jan20

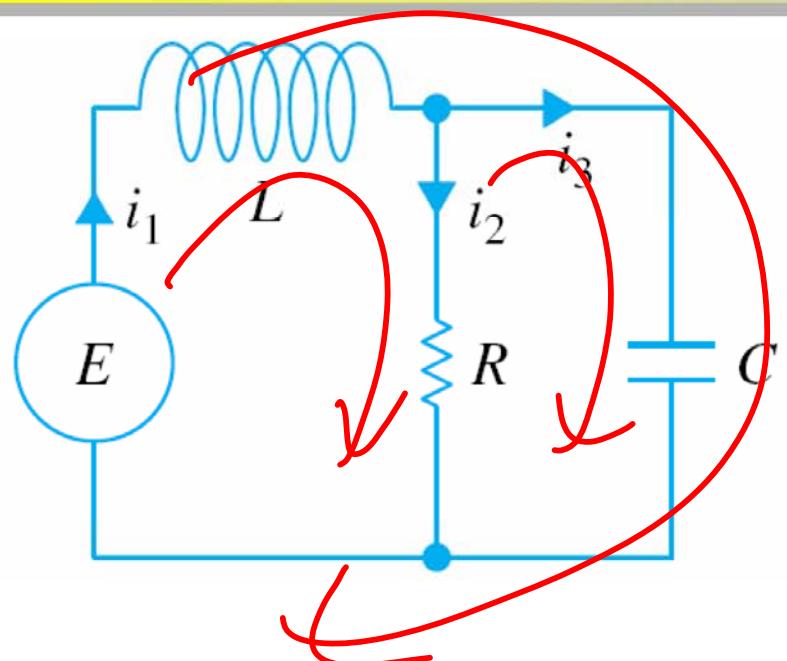
$$\begin{cases} L \frac{di_1}{dt} + R i_2 = E(t) \\ RC \frac{di_2}{dt} + i_2 - i_1 = 0 \end{cases}$$

Figures and images used in these lecture notes are adopted from
Differential Equations with Boundary-Value Problems, 9th Ed., D.G. Zill, 2018 (Metric Version)



- 7.1: Definition of Laplace Transform
- 7.2: Inverse Transforms and Transforms of Derivatives
 - 7.2.1: Inverse Transforms
 - 7.2.2: Transforms of Derivatives
- 7.3: Operational Properties I
 - 7.3.1: Translation on the s-Axis
 - 7.3.2: Translation on the t-Axis
- 7.4: Operational Properties II
 - 7.4.1: Derivatives of a Transform
 - 7.4.2: Transforms of Integrals
 - 7.4.3: Transform of a Periodic Function
- 7.5: The Dirac Delta Function
- **7.6: Systems of Linear Differential Equations**

7.6: Systems of Linear Equations: Networks



$$\Rightarrow \begin{cases} L \frac{di_1}{dt} + R i_2 &= E(t) \\ RC \frac{di_2}{dt} + i_2 - i_1 &= 0 \end{cases}$$

$i_1(0) = 0, \quad i_2(0) = 0$

By Laplace Transform

$$\begin{cases} -i_1(0) - i_2(0) \\ sL I_1(s) + RI_2(s) = E(s) \\ RC s I_2(s) + I_2(s) - I_1(s) = 0 \end{cases}$$

$$I_1(s) = \frac{s}{\begin{vmatrix} s & 50 \\ -1 & 5 \times 10^{-3}s + 1 \end{vmatrix}} \cdot \begin{vmatrix} 60 & 50 \\ 0 & 5 \times 10^{-3}s + 1 \end{vmatrix}$$

$$I_2(s) = \frac{-1}{\begin{vmatrix} s & 50 \\ -1 & 5 \times 10^{-3}s + 1 \end{vmatrix}} \cdot \begin{vmatrix} s & -60 \\ -1 & 0 \end{vmatrix}$$

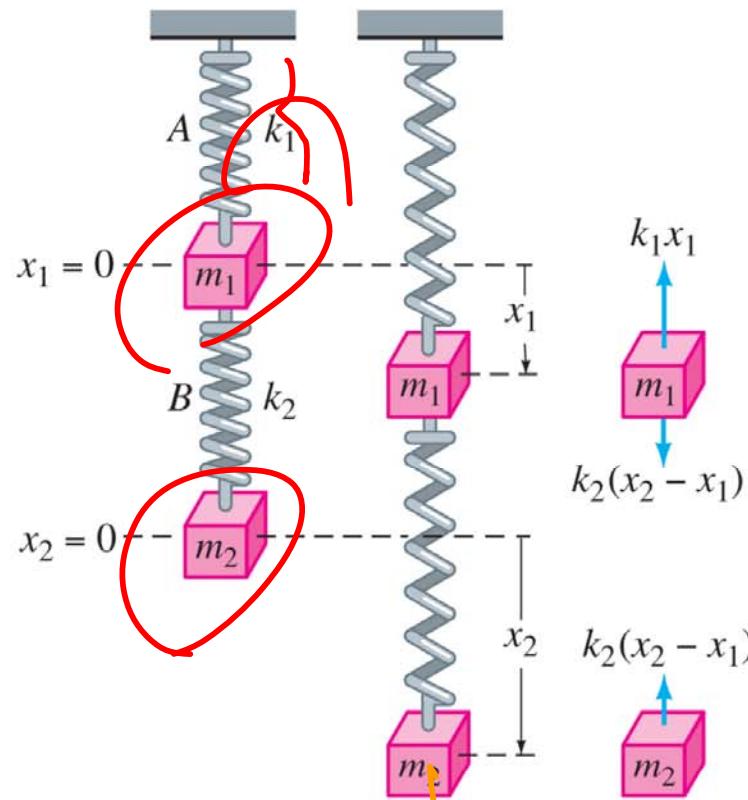
$$I_1(S) = \frac{60S+12000}{(S+100)^2} = \frac{\frac{6}{S}}{(S+100)} + \frac{-\frac{6}{5}}{(S+100)} + \frac{-60}{(S+100)^2}$$

$$I_2(S) = \frac{12000}{(S+100)^2} = \frac{\frac{6}{S}}{(S+100)} + \frac{-\frac{6}{5}}{(S+100)} + \frac{-120}{(S+100)^2}$$

$$i_1(t) = \frac{6}{5} \cdot -\frac{6}{5} e^{-100t} - 60 + \bar{e}^{100t}$$

$$i_2(t) = \frac{6}{5} \cdot -\frac{6}{5} e^{-100t} - 120 + \bar{e}^{100t}$$

7.6: Systems of Linear Equations: Spring/Mass & Pendulum



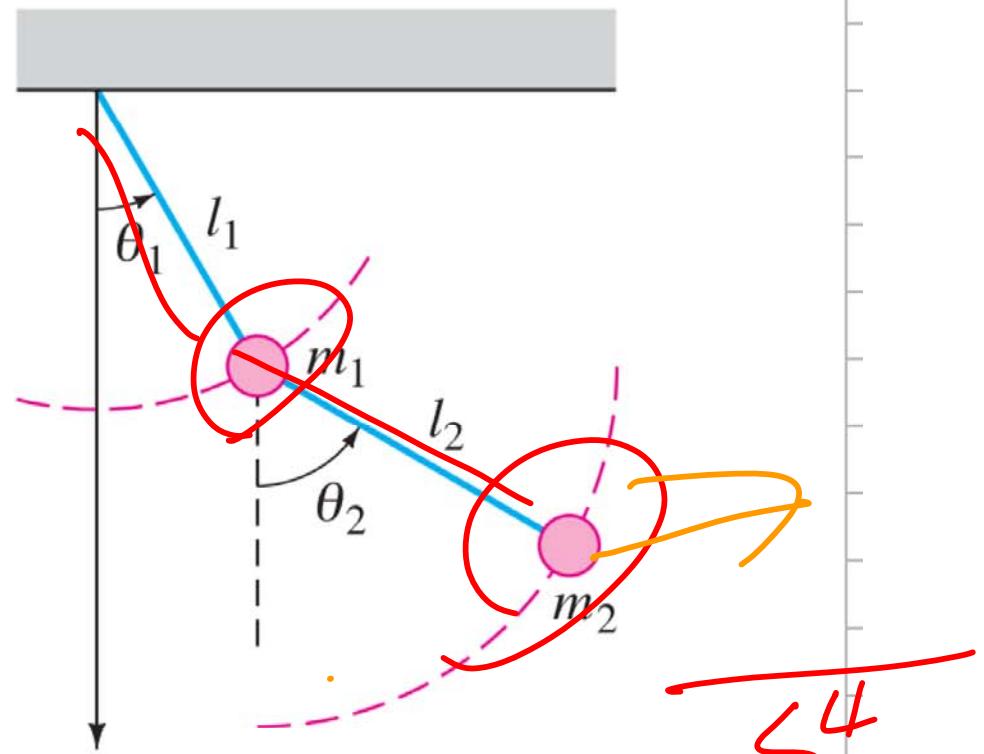
(a) equilibrium (b) motion (c) forces

$$m_1 \frac{d^2x_1}{dt^2} = -k_1 x_1 + k_2(x_2 - x_1).$$

$$m_2 \frac{d^2x_2}{dt^2} = -k_2(x_2 - x_1).$$

$$m_1 x_1'' = -k_1 x_1 + k_2(x_2 - x_1)$$

$$m_2 x_2'' = -k_2(x_2 - x_1).$$



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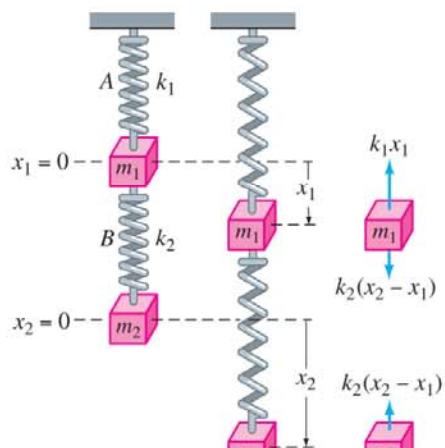
$$(m_1 + m_2)l_1^2 \theta_1'' + m_2 l_1 l_2 \theta_2'' \cos(\theta_1 - \theta_2) + m_2 l_1 l_2 (\theta_1')^2 \sin(\theta_1 - \theta_2) + (m_1 + m_2)l_1 g \sin \theta_1 = 0.$$

$$m_2 l_2^2 \theta_2'' + m_2 l_1 l_2 \theta_1'' \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 (\theta_1')^2 \sin(\theta_1 - \theta_2) + m_2 l_2 g \sin \theta_2 = 0.$$

$$(m_1 + m_2)l_1^2 \theta_1'' + m_2 l_1 l_2 \theta_2'' + (m_1 + m_2)l_1 g \theta_1 = 0$$

$$m_2 l_2^2 \theta_2'' + m_2 l_1 l_2 \theta_1'' + m_2 l_2 g \theta_2 = 0.$$

Summary - 7.6: Systems of Linear Equations



(a) equilibrium (b) motion (c) forces

$$m_1 x_1'' = -k_1 x_1 + k_2(x_2 - x_1)$$

$$m_2 x_2'' = -k_2(x_2 - x_1).$$

Solve

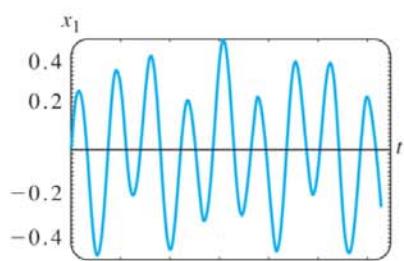
$$x_1'' + 10x_1 - 4x_2 = 0$$

$$-4x_1 + x_2'' + 4x_2 = 0$$

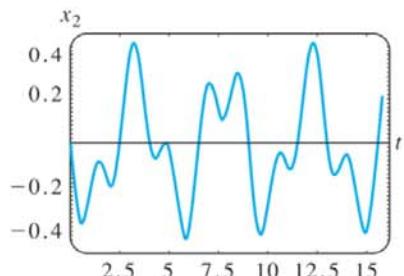
subject to $x_1(0) = 0, x_1'(0) = 1, x_2(0) = 0, x_2'(0) = -1.$

$$s^2 X_1(s) - sx_1(0) - x_1'(0) + 10X_1(s) - 4X_2(s) = 0$$

$$-4X_1(s) + s^2 X_2(s) - sx_2(0) - x_2'(0) + 4X_2(s) = 0,$$



(a) plot of $x_1(t)$ vs. t



(b) plot of $x_2(t)$ vs. t

$$X_1(s) = \frac{s^2}{(s^2 + 2)(s^2 + 12)} = -\frac{1/5}{s^2 + 2} + \frac{6/5}{s^2 + 12},$$

$$x_1(t) = -\frac{1}{5\sqrt{2}} \mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2 + 2}\right\} + \frac{6}{5\sqrt{12}} \mathcal{L}^{-1}\left\{\frac{\sqrt{12}}{s^2 + 12}\right\}$$

$$= -\frac{\sqrt{2}}{10} \sin \sqrt{2}t + \frac{\sqrt{3}}{5} \sin 2\sqrt{3}t.$$

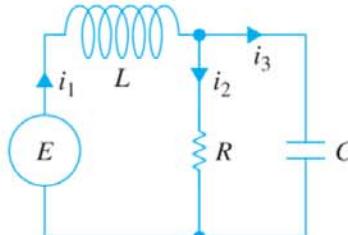
$$X_2(s) = -\frac{s^2 + 6}{(s^2 + 2)(s^2 + 12)} = -\frac{2/5}{s^2 + 2} - \frac{3/5}{s^2 + 12}$$

$$x_2(t) = -\frac{2}{5\sqrt{2}} \mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2 + 2}\right\} - \frac{3}{5\sqrt{12}} \mathcal{L}^{-1}\left\{\frac{\sqrt{12}}{s^2 + 12}\right\}$$

$$= -\frac{\sqrt{2}}{5} \sin \sqrt{2}t - \frac{\sqrt{3}}{10} \sin 2\sqrt{3}t.$$

$$L \frac{di_1}{dt} + Ri_2 = E(t)$$

$$RC \frac{di_2}{dt} + i_2 - i_1 = 0,$$



$$\frac{di_1}{dt} + 50i_2 = 60$$

$$50(10^{-4}) \frac{di_2}{dt} + i_2 - i_1 = 0$$

$$i_1(0) = 0, i_2(0) = 0.$$

$$sI_1(s) + 50I_2(s) = \frac{60}{s}$$

$$-200I_1(s) + (s + 200)I_2(s) = 0,$$

$$I_1(s) = \frac{60s + 12,000}{s(s + 100)^2} = \frac{6/5}{s} - \frac{6/5}{s + 100} - \frac{60}{(s + 100)^2}$$

$$I_2(s) = \frac{12,000}{s(s + 100)^2} = \frac{6/5}{s} - \frac{6/5}{s + 100} - \frac{120}{(s + 100)^2}.$$

$$i_1(t) = \frac{6}{5} - \frac{6}{5} e^{-100t} - 60te^{-100t}$$

$$i_2(t) = \frac{6}{5} - \frac{6}{5} e^{-100t} - 120te^{-100t}.$$