

Fall 2019

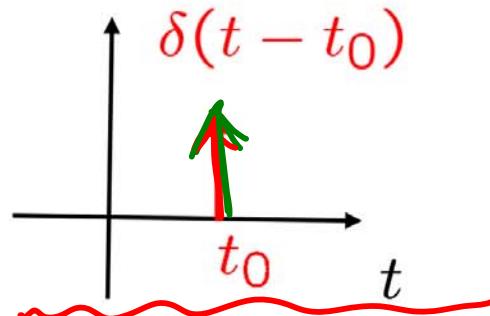
# 微分方程 Differential Equations

## Unit 07.5 The Dirac Delta Function

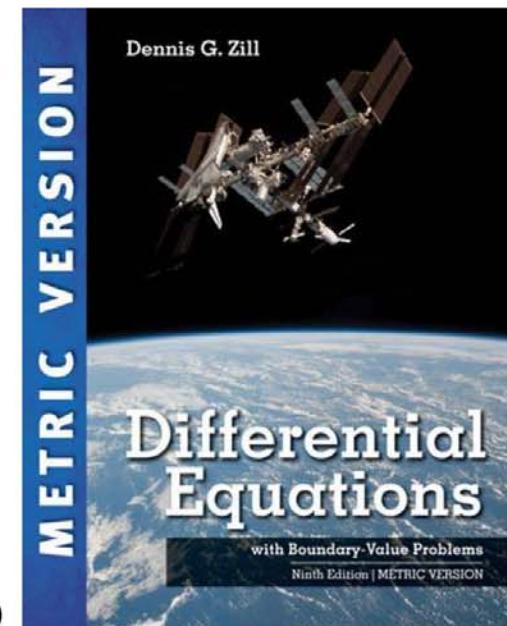
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Sep19 – Jan20



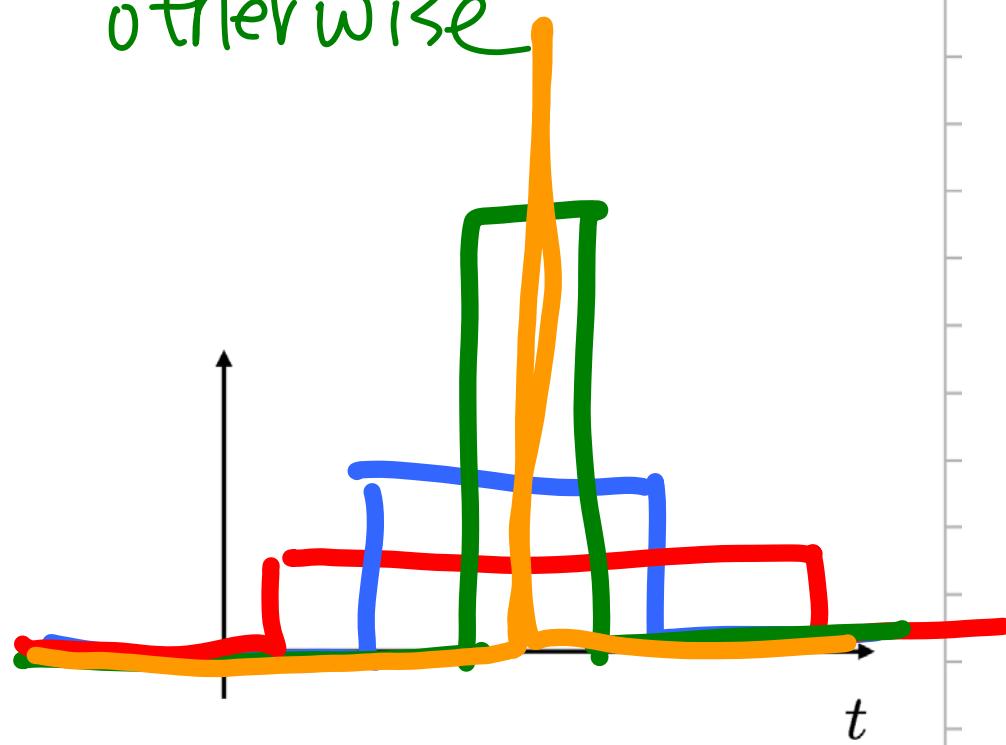
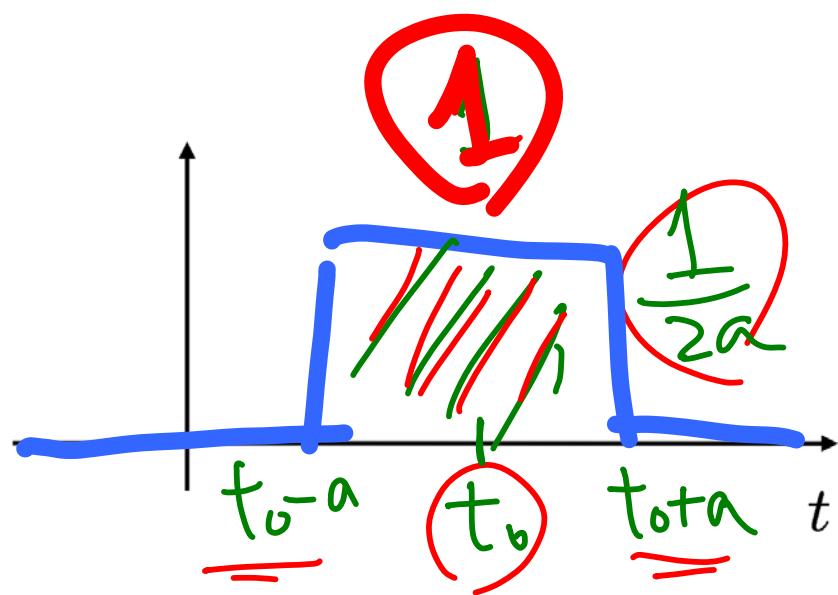
Figures and images used in these lecture notes are adopted from  
**Differential Equations with Boundary-Value Problems**, 9th Ed., D.G. Zill, 2018 (Metric Version)



- 7.1: Definition of Laplace Transform
- 7.2: Inverse Transforms and Transforms of Derivatives
  - 7.2.1: Inverse Transforms
  - 7.2.2: Transforms of Derivatives
- 7.3: Operational Properties I
  - 7.3.1: Translation on the s-Axis
  - 7.3.2: Translation on the t-Axis
- 7.4: Operational Properties II
  - 7.4.1: Derivatives of a Transform
  - 7.4.2: Transforms of Integrals
  - 7.4.3: Transform of a Periodic Function
- **7.5: The Dirac Delta Function**
- 7.6: Systems of Linear Differential Equations

- Unit Impulse:

$$\delta_a(t - t_0) = \begin{cases} \frac{1}{2a}, & t_0 - a \leq t < t_0 + a \\ 0, & \text{otherwise} \end{cases}$$

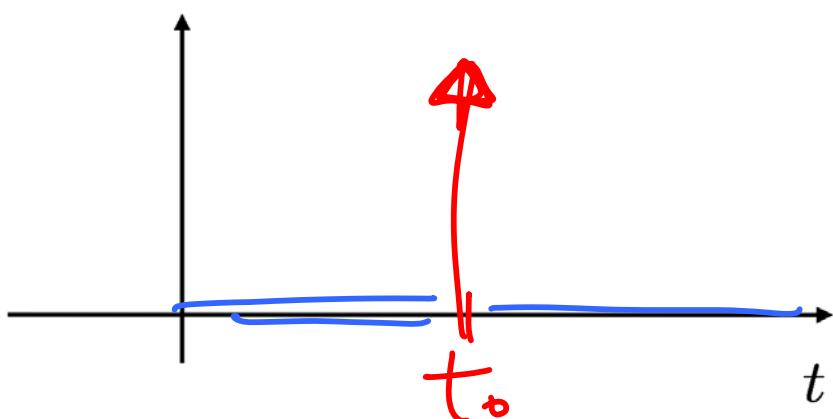


$$\Rightarrow \int_0^\infty \delta_a(t - t_0) dt = 1$$

- Dirac Delta Function:

$$\underline{\delta(t - t_0)} = \lim_{\alpha \rightarrow 0} \underline{\delta_\alpha(t - t_0)}$$

$$\Rightarrow \underline{\delta(t - t_0)} = \begin{cases} \infty, & t = \underline{t_0} \\ 0, & t \neq \underline{t_0} \end{cases}$$



$$\Rightarrow \boxed{\int_0^\infty \delta(t - t_0) dt = 1}$$

## 7.5: Theorem 7.5.1

$$\Rightarrow \mathcal{L}\{\delta_a(t - t_0)\} = ?$$

$$\underline{\delta_a(t - t_0)} = \frac{1}{2a} [U(t - (t_0 + a)) - U(t - (t_0 + a))]$$

$$\begin{aligned} \Rightarrow \mathcal{L}\{\delta_a(t - t_0)\} &= \frac{1}{2a} \left[ \frac{1}{s} e^{-(t_0+a)s} - \frac{1}{s} e^{-(t_0+a)s} \right] \\ &= \frac{e^{as} - e^{-as}}{2as} e^{-t_0 s} \end{aligned}$$

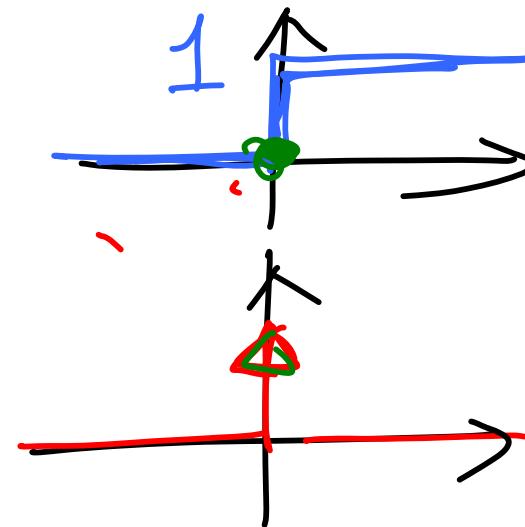
$$\begin{aligned} \Rightarrow \lim_{a \rightarrow 0} \mathcal{L}\{\delta_a(t - t_0)\} &= e^{-t_0 s} \lim_{a \rightarrow 0} \frac{e^{as} - e^{-as}}{2as} \xrightarrow{0} \frac{0}{0} \\ &= e^{-t_0 s} \lim_{a \rightarrow 0} \frac{se^{as} + se^{-as}}{2s} \xrightarrow{0} \frac{2s}{2s} = 1 \end{aligned}$$

$$\Rightarrow \mathcal{L}\{\delta(t - t_0)\} = e^{-t_0 s} \quad \underline{\mathcal{L}\{\delta(t)\}} = e^{-0s} = 1$$

- Theorem 7.5.1

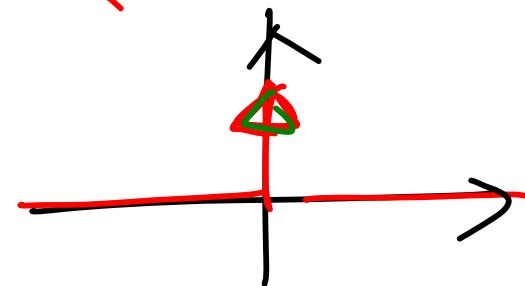
## Summary - 7.5: Dirac Delta Function

$$\Rightarrow \mathcal{L} \left\{ \underline{\mathcal{U}(t)} \right\} = \frac{1}{s}$$



$$\Rightarrow \mathcal{L} \left\{ \frac{d}{dt} \mathcal{U}(t) \right\} = s \mathcal{L} \left\{ \mathcal{U}(t) \right\} - \mathcal{U}(0)$$

$$= s \cdot \frac{1}{s} - 0 = 1$$



$$\Rightarrow \mathcal{L} \left\{ \delta(t) \right\} = 1$$

$$\Rightarrow \delta(t) = \frac{d}{dt} \mathcal{U}(t) = \mathcal{U}'(t)$$



$$\mathcal{L}(\gamma'') = \mathcal{L}(4\delta(t-2\pi))$$

$$s^2 Y(s) + s Y(s)$$

$$-sy(0) - y'(0)$$

$$= 4 e^{-2\pi s}$$

$$e^{-as} \leftrightarrow u(t-a)$$

$$(s^2+1)Y(s) = 4 e^{-2\pi s}$$

$$Y(s) = \frac{4 e^{-2\pi s}}{s^2+1} = \frac{1}{s+i} + \frac{1}{s-i}$$

$$\gamma(t) = \frac{4 \sin t}{4 \sin(t-\pi) \cup (t-\pi)}$$