

Fall 2019

# 微分方程 Differential Equations

## Unit 07.4 Operational Properties II

$$\mathcal{L} \left\{ t^n f(t) \right\}$$

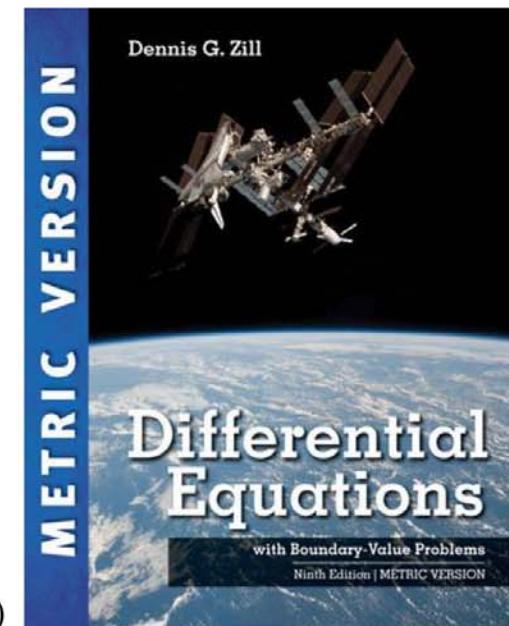
$$\mathcal{L} \left\{ f(t) * g(t) \right\}$$

$$\mathcal{L} \left\{ f(t + T) \right\}$$

Feng-Li Lian

NTU-EE

Sep19 – Jan20



Figures and images used in these lecture notes are adopted from  
**Differential Equations with Boundary-Value Problems**, 9th Ed., D.G. Zill, 2018 (Metric Version)



- Theorem 7.4.1: Derivatives of Transforms

IF

$$\mathcal{L}\left\{ \underline{\underline{f(t)}} \right\} = \underline{\underline{F(s)}}$$

THEN

$$\mathcal{L}\left\{ \underline{\underline{t^n f(t)}} \right\} = \underline{\underline{(-1)^n \frac{d}{ds} F(s)}}$$

• Proof:

$$\mathcal{L}(F(s)) = \mathcal{L}\left(\int_0^\infty e^{-st} f(t) dt\right) \quad \checkmark$$

$$= \int_0^\infty \frac{d}{ds} \left( e^{-st} f(t) \right) dt$$

$$= \int_0^\infty \left( \frac{d}{ds} e^{-st} \right) f(t) dt$$

$$= \int_0^\infty e^{-st} (-t) f(t) dt$$

$$= \mathcal{L}\left\{ \underline{\underline{(-t)f(t)}} \right\}$$

$$\mathcal{L} \{ t f(t) \} = (-1) \frac{d}{ds} F(s)$$

$$\frac{d^2}{ds^2} F(s) = \int_0^\infty \left[ \frac{d^2}{dt^2} (e^{-st}) \right] f(t) dt$$

$$(-t)^2 \left( \frac{1}{(-t)^2} e^{-st} \right) f(t)$$

$$= \int_0^\infty e^{-st} \left[ \left( \frac{1}{(-t)^2} f(t) \right) \right] dt$$

$$= \mathcal{L} \left\{ \frac{1}{(-t)^2} f(t) \right\}$$

$$\frac{d^n}{ds^n} F(s) = \mathcal{L} \left\{ \frac{1}{(-t)^n} f(t) \right\}$$

$$\mathcal{L}\left\{ \underline{\underline{t}} \underline{\underline{e^{-t}}} \underline{\underline{\cos t}} \right\} = \int_0^{\infty} e^{-st} (t e^{-t} \cos t) dt$$

- method 1

$$\mathcal{L}\left\{ \underline{\underline{\cos t}} \right\} = \frac{s}{s^2 + 1} \quad | \xrightarrow{s+1}$$

$$\mathcal{L}\left\{ \underline{\underline{e^{-t}}} \underline{\underline{\cos t}} \right\} = \frac{(s+1)}{(s+1)^2 + 1}$$

$$\begin{aligned} \mathcal{L}\left\{ \underline{\underline{t}} \underline{\underline{e^{-t}}} \underline{\underline{\cos t}} \right\} &= (-1) \frac{d}{ds} \frac{s+1}{(s+1)^2 + 1} = (-1) \frac{| (s+1)^2 + 1 - (s+1)(2(s+1)) }{((s+1)^2 + 1)^2} \\ &= \frac{(s+1)^2 - 1}{((s+1)^2 + 1)^2} \end{aligned}$$

$$\mathcal{L} \left\{ t e^{-t} \cos t \right\} =$$

- method 2

$$\begin{aligned}\mathcal{L} \left\{ \underline{\cos t} \right\} &= \frac{s}{s^2 + 1} \\ \mathcal{L} \left\{ \underline{t} \underline{\cos t} \right\} &= (-1) \frac{d}{ds} \frac{s}{s^2 + 1} = (-1) \frac{1 \cdot (s^2 + 1) - s(2s)}{(s^2 + 1)^2} \\ &= \frac{s^2 - 1}{(s^2 + 1)^2}\end{aligned}$$

$$\mathcal{L} \left\{ e^{-t} \left| \begin{array}{c} t \cos t \end{array} \right. \right\} = \frac{s \rightarrow s+1}{\frac{(s+1)^2 - 1}{((s+1)^2 + 1)^2}}$$

播

- $f(t)$  and  $g(t)$ : piecewise continuous on  $[0, \infty)$

$$\Rightarrow [f(t)] * [g(t)] = \int_0^t f(\tau) \cdot g(t-\tau) d\tau$$

$$g(t) * f(t) = \int_0^t g(\tau) f(t-\tau) d\tau$$

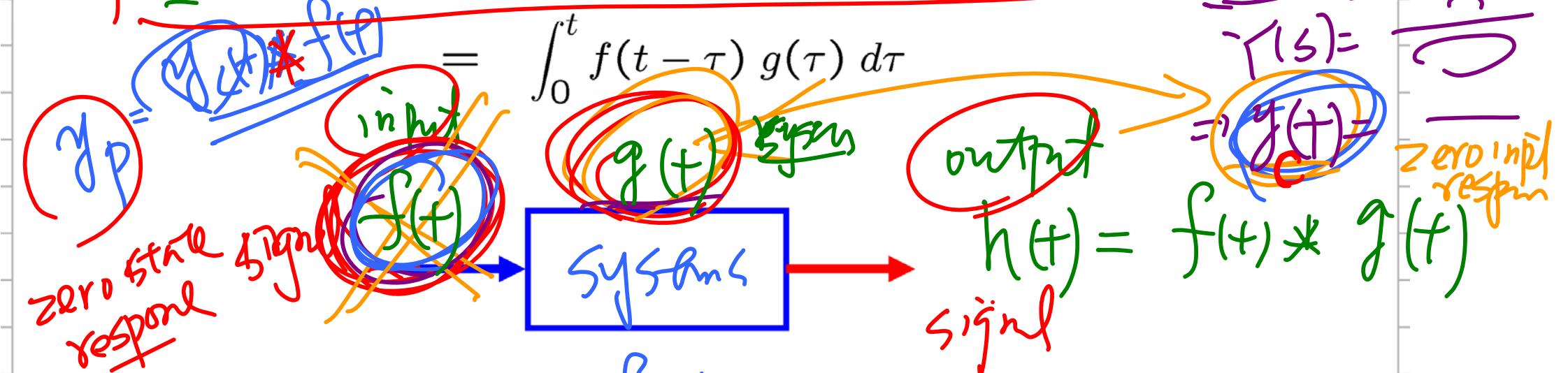
$$u = t - \tau \quad \tau: 0 \rightarrow t \\ du = -d\tau \quad u: t \rightarrow 0$$

$$= \int_t^0 g(t-u) f(u) (-du) \\ = (-1) \int_0^t g(t-u) f(u) du \\ = f(t) * g(t)$$

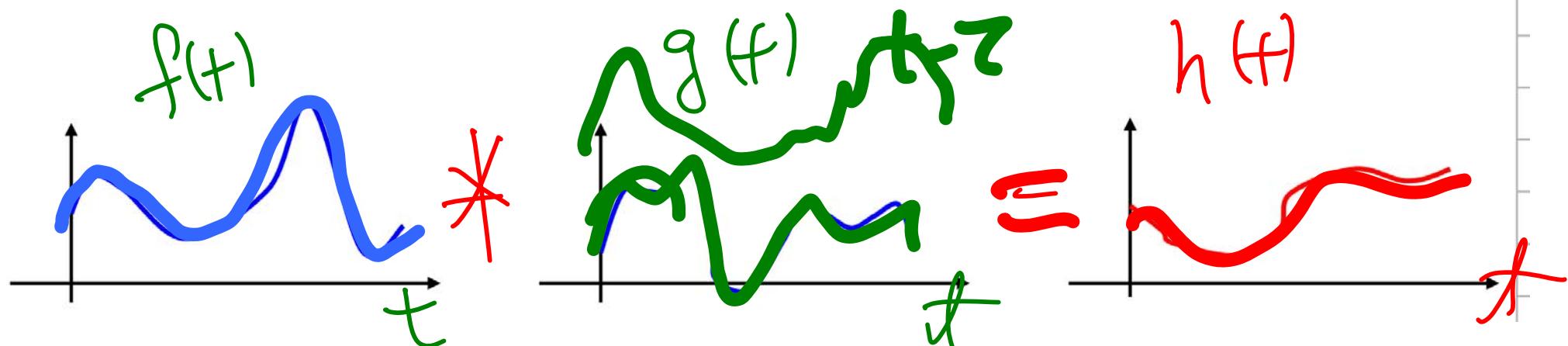
## 7.4.2: Transforms of Integrals: Convolution

$$\boxed{f(t) * g(t)} = \int_0^t f(\tau) g(t-\tau) d\tau = \boxed{h(t)}$$

$$ij + aij + bij = 0$$



$$g(t) \xrightarrow{\text{systems}} f(t) \quad h(t) = g(t) * f(t)$$



- Theorem 7.4.2: Convolution Theorem

IF  $f(t)$  and  $g(t)$ : piecewise continuous on  $(0, \infty)$   
of exponential order

THEN

$$\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(t)\} \cdot \mathcal{L}\{g(t)\}$$

- Proof:

$$\begin{aligned} F(s) &= \mathcal{L}\{f(t)\} \triangleq \int_0^\infty e^{-st} f(\tau) d\tau \\ G(s) &= \mathcal{L}\{g(t)\} \triangleq \int_0^\infty e^{-su} g(u) du \end{aligned}$$

$$\begin{aligned} F(s) \cdot G(s) &= \left( \int_0^\infty e^{-st} f(\tau) d\tau \right) \left( \int_0^\infty e^{-su} g(u) du \right) \\ &= \int_0^\infty \int_0^\infty e^{-st} f(\tau) e^{-su} g(u) d\tau du \\ &= \int_0^\infty \int_0^\infty e^{-s(\tau+u)} f(\tau) g(u) d\tau du \end{aligned}$$

$$\begin{aligned}
 & \int_0^\infty f(z) \left[ \int_0^\infty e^{-s(t+u)} g(u) du \right] dz \\
 &= \int_0^\infty f(z) \int_z^\infty e^{-st} g(t-z) dt dz \\
 &= \int_0^\infty e^{-st} \left[ \int_0^t f(z) g(t-z) dz \right] dt \\
 &= \int_0^\infty e^{-st} (f(t) * g(t)) dt \\
 &= \mathcal{L}(f(t) * g(t))
 \end{aligned}$$

### 7.4.2: Transform of an Integral

$$\bullet f(t) * u(t) = ?$$

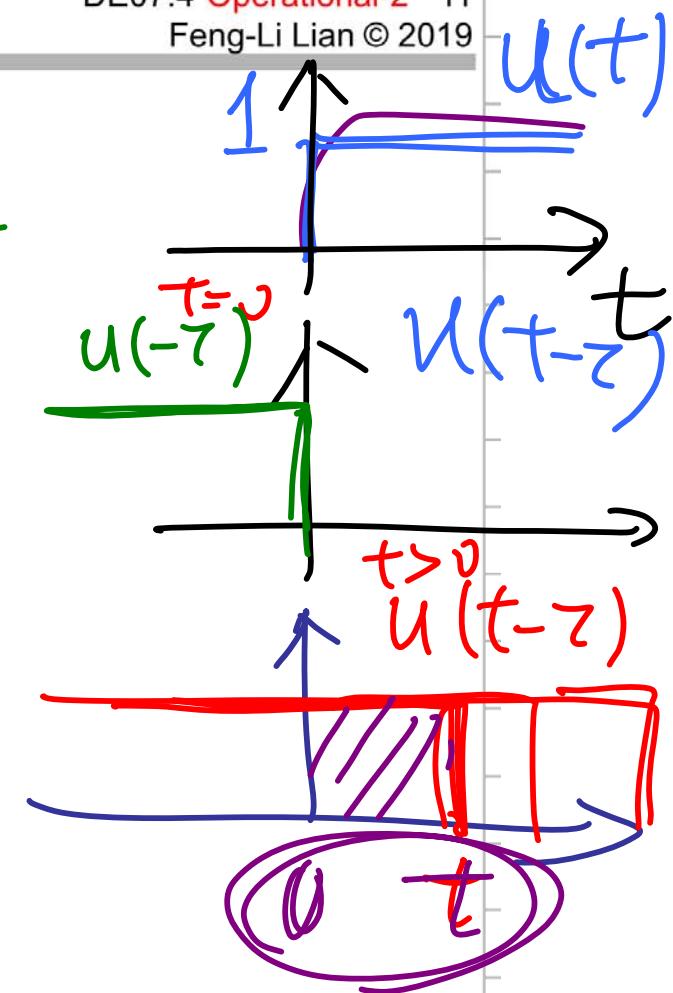
$$\int_0^t f(z) \cdot u(t-z) dz$$

$$= \int_0^t f(z) \cdot 1 dz$$

$$= \int_0^t f(z) dz$$

$$\mathcal{L}\{f(t) * u(t)\} = \mathcal{L}\{f(t)\} \cdot \mathcal{L}\{u(t)\}$$

$$= (\text{FS}) \frac{1}{s} L$$



$$\underline{f(t)} \xleftrightarrow{\mathcal{L}} \underline{F(s)}$$

$$\underline{\frac{d}{dt} f(t)} \xleftrightarrow{\mathcal{L}} \underline{s F(s)} - \underline{f(0)}$$

$$\underline{\frac{d^n}{dt^n} f(t)} \xleftrightarrow{\mathcal{L}} \underline{s^n F(s)} - \underline{s^{n-1} f(0)} - \underline{s^{n-2} f'(0)} - \dots - \underline{f^{(n)}(0)}$$

T.C.

$$t: 0 \rightarrow \infty$$

$$\int_0^t \underline{f(\tau) d\tau} \xleftrightarrow{\mathcal{L}} \underline{\frac{1}{s} F(s)}$$

$$\int_0^t \left( \int_0^{\tau_{n-1}} \left( \dots \left( \int_0^{\tau_1} \underline{f(\tau) d\tau} \right) d\tau_1 \right) \dots \right) d\tau_{n-1} \xleftrightarrow{\mathcal{L}} \underline{\frac{1}{s^n} F(s)}$$

$$\underline{(-t)f(t)} \xleftrightarrow{\mathcal{L}} \underline{\frac{d}{ds} F(s)}$$

$$\underline{(-t)^2} \Rightarrow \underline{t^2 f(t)} \xleftrightarrow{\mathcal{L}} \underline{\frac{d^2}{ds^2} F(s)}$$

$$\underline{(-t)^n} \xleftrightarrow{\mathcal{L}} \underline{(-1)^n t^n f(t)} \quad \underline{\frac{d^n}{ds^n} F(s)}$$

$$= \underline{\frac{st}{e^{-st}}}$$

# Volterra Integral Equation

$$\underline{\underline{f(t)}} = g(t) + \int_0^t \underline{\underline{f(z)}} h(t-z) dz$$

Ex6.  $\underline{\underline{f(t)}} = (3t^2 - e^{-t})$

$\Rightarrow \int_0^t f(z) (e^{t-z}) dz$

$F(s) = \mathcal{L}\{f(t)\} \quad g(t)$

$\rightarrow F(s) = \left( 3 \frac{z^2}{s^3} - \frac{1}{s+1} \right) - F(s) \frac{1}{s-1}$

$$F(s) = \frac{6}{s^3} - \frac{6}{s^4} + \frac{1}{s} - \frac{2}{s+1}$$

$$\rightarrow f(t) = \mathcal{L}^{-1}\{F(s)\} = \boxed{3t^2 - t^3 + 1 - e^{-t}, t \geq 0}$$

# Integro-differential Equation $i(0) = 0$

$$\mathcal{L}\left(\frac{di}{dt}\right) + R i(t) = \frac{1}{C} \int_0^t i(z) dz - \mathcal{L}[E(t)]$$

$\geq 10$        $\frac{120t - 120e^{-10t}}{10}$

$$0.1s \bar{I}(s) + 2 \bar{I}(s) + 10 \frac{1}{s} \bar{I}(s) = 120 \left[ \frac{1}{s^2} e^{-s} - \frac{1}{s} e^{-s} \right]$$

~~$-i(0)$~~

$$\begin{aligned} \bar{I}(s) &= 1200 \left[ \frac{1}{s(s+10)^2} + \dots \right] \\ &= 1200 \left[ \frac{\frac{1}{100}}{s} - \frac{\frac{1}{100}}{s+10} - \frac{\frac{1}{10}}{(s+10)^2} + \dots \right] \end{aligned}$$

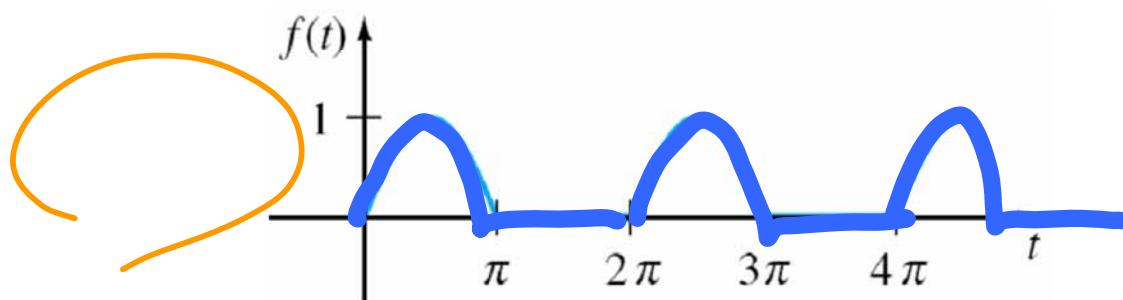
$$i(t) = \mathcal{L}^{-1}[\bar{I}(s)] = 12 - 12e^{-10t} - 120te^{-10t} \dots$$

- Definition: Periodic Function

$f(t)$  is a **periodic** function with a period  $\underline{T} > 0$

if 
$$f(t + T) = f(t)$$

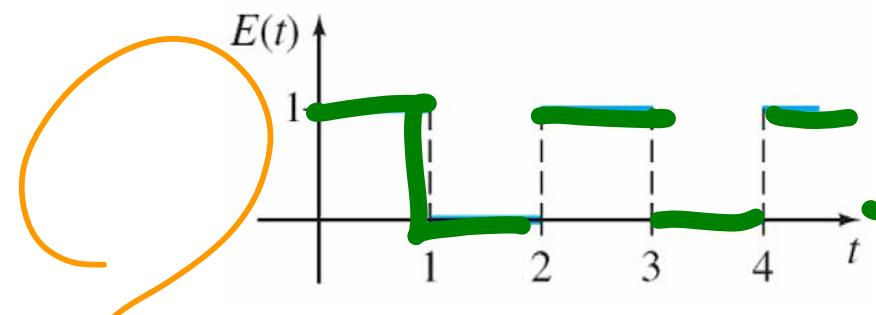
$\forall t \rightarrow \infty \rightarrow \infty$



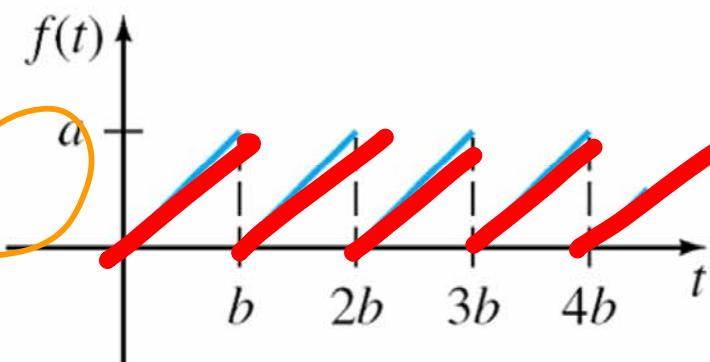
$$\begin{aligned} T &= 2\pi \\ &4\pi \\ &8\pi \end{aligned}$$

$$T = 2$$

half-wave rectification of  $\sin t$



square wave



sawtooth

$$\begin{aligned} T &= b \\ &> b \\ &3b \end{aligned}$$

• Theorem 7.4.3:

IF  $\underline{\underline{f(t)}}$  :   
 { (1) piecewise continuous or  $\bar{T}(0, \infty)$   
 (2) of exponential order  
 (3) periodic with  $\bar{T}$

THEN

$$\mathcal{L}\{\underline{\underline{f(t)}}\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt$$


## • Proof:

$$\begin{aligned}
 F(s) &= \mathcal{L}\{\underline{\underline{f(t)}}\} \triangleq \int_0^\infty e^{-st} \underline{\underline{f(t)}} dt \\
 &= \int_0^T e^{-st} f(t) dt + \int_T^\infty e^{-st} f(t) dt
 \end{aligned}$$

(1) (2)

$$\textcircled{2} \quad t = u + T$$

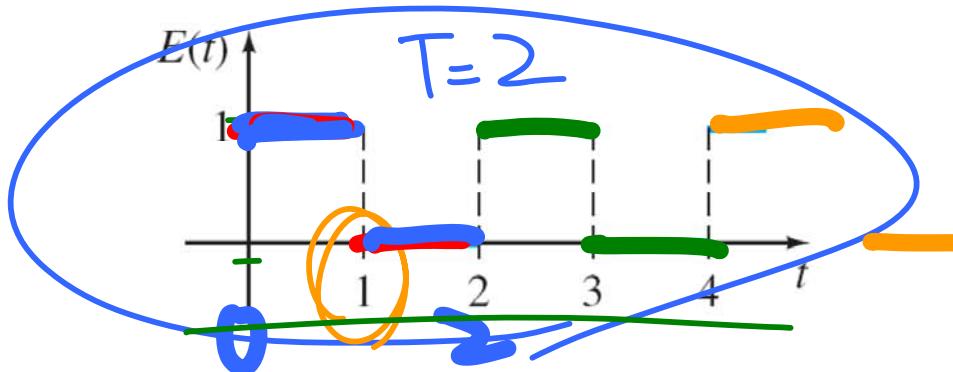
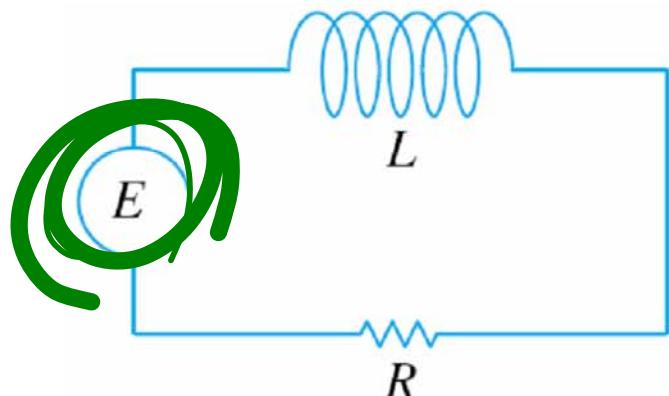
$$= \int_0^\infty e^{-su} f(u+T) du$$

$$= e^{-ST} \left[ \int_0^\infty e^{-su} f(u+T) du \right] = f(u)$$

$$= e^{-ST} F(s)$$

$$F(s) = \int_0^T e^{-st} f(t) dt + e^{-ST} F(s)$$

$$F(s)(1 - e^{-ST}) = \int_0^T e^{-st} f(t) dt$$



$$\mathcal{L} \left\{ L \frac{d}{dt} i(t) + R i(t) \right\} = \mathcal{L} \left\{ E(t) \right\}$$

$$\Rightarrow \boxed{L s I(s) + R I(s)} = \frac{1}{T - e^{-s \cdot 2}} \int_0^T e^{-st} E(t) dt$$

$$\Rightarrow \mathcal{L} \{ E(t) \} =$$

$$= \frac{1}{T - e^{-s \cdot 2}} \int_0^T e^{-st} \cdot 1 dt$$

$$= \frac{1}{T - e^{-s \cdot 2}} \frac{1}{(-s)} e^{-st} \Big|_0^1$$

$$= \frac{1}{T - e^{-s \cdot 2}} \frac{1}{(-s)} (e^{-s} - 1) = \frac{1}{s} \frac{1}{1 - e^{-s}}$$

$$(Ls + R) I(s) = \frac{1}{s} \frac{1}{T + e^{-s}}$$



$$\underline{I(s)} = \frac{1}{[s+R]} \frac{1}{s} \frac{1}{1+e^{-s}}$$

$$\frac{1}{1+e^{-s}} = 1 - e^{-s} + e^{-2s} - e^{-3s} + \dots$$

$$|s - s'| < 1$$

$$\underline{I(s)} = \frac{1}{[s+R]} \left( 1 - e^{-s} + e^{-2s} - e^{-3s} + \dots \right)$$

$t \rightarrow t-1$   
 $t \rightarrow t-2$

$$\frac{1}{R} \left[ 1 - e^{-\frac{R}{L}t} \right] u(t)$$

$e^{-as}$   
 $t \rightarrow t-a$

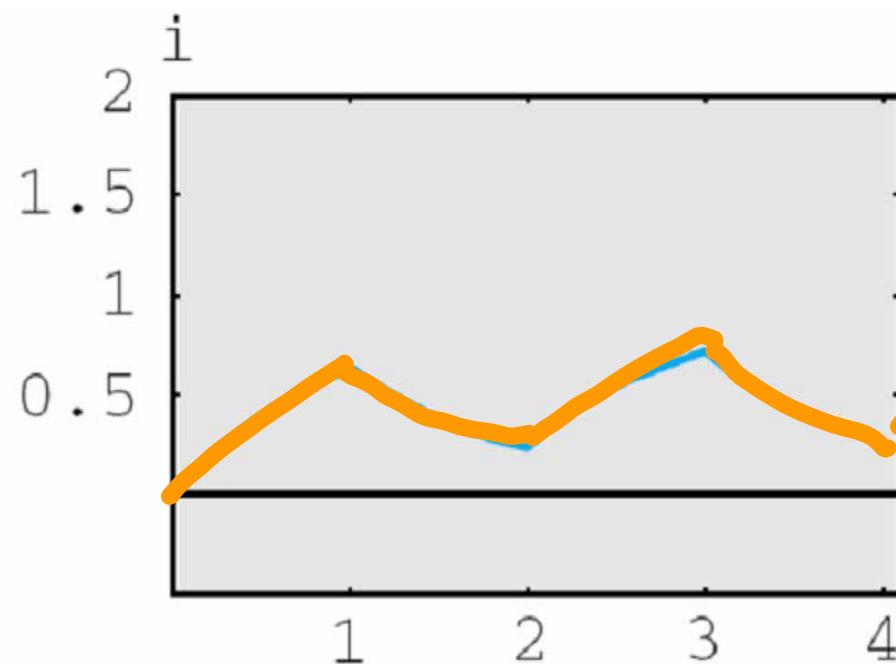
$$i(t) = \frac{1}{R} \left[ 1 - e^{-\frac{R}{L}t} \right] u(t)$$

$$- \frac{1}{R} \left[ 1 - e^{-\frac{R}{L}(t-1)} \right] u(t-1)$$

$$+ \frac{1}{R} \left[ 1 - e^{-\frac{R}{L}(t-2)} \right] u(t-2) + \dots$$

- IF  $R = 1, L = 1, 0 \leq t < 4$

$$i(t) = \begin{cases} 1 - e^{-(t)}, & 0 \leq t < 1 \\ -e^{-(t)} + e^{-(t-1)}, & 1 \leq t < 2 \\ 1 - e^{-(t)} + e^{-(t-1)} - e^{-(t-2)}, & 2 \leq t < 3 \\ -e^{-(t)} + e^{-(t-1)} - e^{-(t-2)} + e^{-(t-3)}, & 3 \leq t < 4 \end{cases}$$



$$\mathcal{L} \{ f(t) \} \triangleq \int_0^\infty e^{-st} f(t) dt = F(s)$$

$$\mathcal{L} \{ f(t) \} = \underline{\underline{F(s)}}$$

$$\mathcal{L} \{ t^n f(t) \} = (-1)^n \underline{\underline{\frac{d^n}{ds^n} F(s)}}$$

$$\mathcal{L} \{ f(t) * g(t) \} = \underline{\underline{\mathcal{L} \{ f(t) \}} \mathcal{L} \{ g(t) \}} = \underline{\underline{F(s) G(s)}}$$

$$\underline{\underline{f(t+T)}} = \underline{\underline{f(t)}}$$

$$\mathcal{L} \{ f(t) \} = \frac{1}{1 - e^{sT}} \boxed{\int_0^T e^{-st} f(t) dt}$$