

Fall 2019

微分方程 Differential Equations

Unit 07.3 Operational Properties I

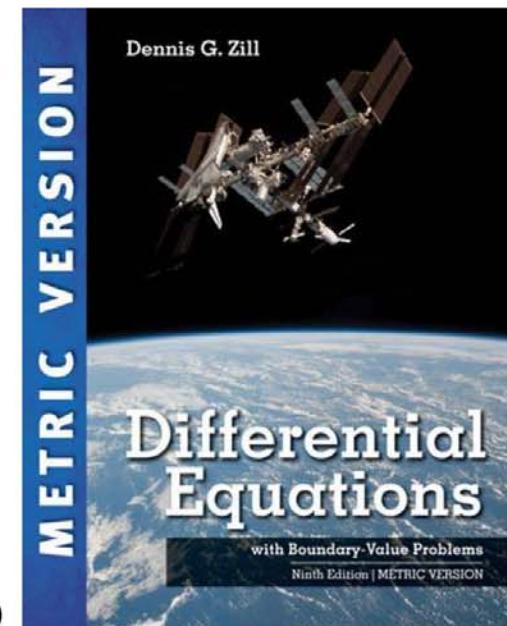
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NTU-EE

$$\mathcal{L} \left\{ e^{at} f(t) \right\} = F(s - a) \quad \text{Sep19 – Jan20}$$

$$\mathcal{L} \left\{ f(t-a) U(t-a) \right\} = e^{-as} F(s)$$

Figures and images used in these lecture notes are adopted from
Differential Equations with Boundary-Value Problems, 9th Ed., D.G. Zill, 2018 (Metric Version)



- 7.1: Definition of Laplace Transform
- 7.2: Inverse Transforms and Transforms of Derivatives
 - 7.2.1: Inverse Transforms
 - 7.2.2: Transforms of Derivatives
- **7.3: Operational Properties I**
 - 7.3.1: Translation on the s-Axis
 - 7.3.2: Translation on the t-Axis
- 7.4: Operational Properties II
 - 7.4.1: Derivatives of a Transform
 - 7.4.2: Transforms of Integrals
 - 7.4.3: Transform of a Periodic Function
- 7.5: The Dirac Delta Function
- 7.6: Systems of Linear Differential Equations

$$\underline{\mathcal{L} \{ f(t) \}} \triangleq \int_0^{\infty} e^{-st} \underline{f(t)} dt = \underline{F(s)}$$

$$\underline{\mathcal{L} \{ f(t) \}} = \underline{F(s)}$$

$$\underline{f(t)} \xleftrightarrow{\mathcal{L}} \underline{F(s)}$$

$$\frac{1}{s}$$

$$e^{at} \underline{f(t)} \xleftrightarrow{\mathcal{L}} F(s-a)$$

$$f(t-a) \xleftrightarrow{\mathcal{L}} e^{as} F(s)$$

$\mathcal{U}(t-a)$

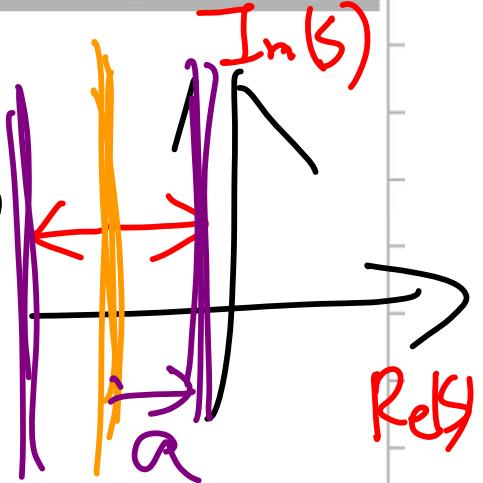
- Theorem 7.3.1: First Translation Theorem

IF

$$\mathcal{L}\{f(t)\} = F(s)$$

AND,

$$a \in \mathbb{R}$$



THEN

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

Proof:

$$= \int_0^\infty e^{-st} (e^{at} f(t)) dt$$

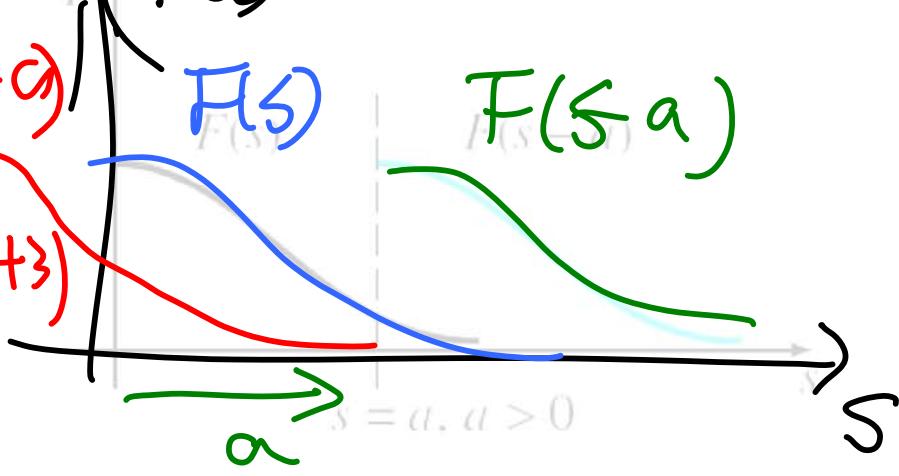
$$= \int_0^\infty e^{-\frac{(s-a)t}{s}} f(t) dt$$

$$\bar{s} = s-a$$

$$= \int_0^\infty e^{-\bar{s}t} f(t) dt$$

$$= F(\bar{s})$$

$$= F(s-a)$$



$$\mathcal{L} \left\{ \underline{\underline{t^3}} \right\} = \frac{3!}{\underline{\underline{s^4}}} \rightarrow s \rightarrow s-a$$

$$\Rightarrow \mathcal{L} \left\{ \underline{\underline{e^{at} t^3}} \right\} = \frac{3!}{\underline{\underline{(s-a)^4}}}$$

$$\mathcal{L} \left\{ \underline{\underline{\cos 4t}} \right\} = \frac{s}{\underline{\underline{s^2 + 4^2}}}$$

$$\Rightarrow \mathcal{L} \left\{ \underline{\underline{e^{at} \cos 4t}} \right\} = \frac{(s-a)}{\underline{\underline{(s-a)^2 + 4^2}}}$$

$$\mathcal{L} \left\{ \underline{\underline{\sin 4t}} \right\} = \frac{4}{\underline{\underline{s^2 + 4^2}}} \quad s \rightarrow s-a$$

$$\Rightarrow \mathcal{L} \left\{ \underline{\underline{e^{at} \sin 4t}} \right\} = \frac{4}{\underline{\underline{(s-a)^2 + 4^2}}}$$

7.3.1: Examples

$$\begin{aligned}
 & \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 6s + 11} \right\} \times \cdot \frac{A}{s-(a+jb)} + \frac{B}{s-(a-jb)} \\
 & \Rightarrow A e^{(a+jb)t} + B e^{(a-jb)t} \\
 & = \boxed{e^{at} (\bar{A} \cos bt + \bar{B} \sin bt)}
 \end{aligned}$$

$\frac{s}{(s+a)^2+b^2}$
 $(s+3)$
 $(s+3)^2+(\sqrt{2})^2$
 $\sqrt{2}$
 $(s+3)^2+(\sqrt{2})^2$

$$\begin{aligned}
 & = \boxed{e^{-3t} (\cos 3\sqrt{2}t) - \frac{3}{\sqrt{2}} e^{-3t} \sin(\sqrt{2}t)}
 \end{aligned}$$

$$\begin{aligned}
 \frac{s}{s^2+w^2} & \hookrightarrow \cos wt \\
 \frac{w}{s^2+w^2} & \hookrightarrow \sin wt \\
 s \rightarrow s+a & \\
 e^{-at} &
 \end{aligned}$$

7.3.2: Translation on the t-Axis

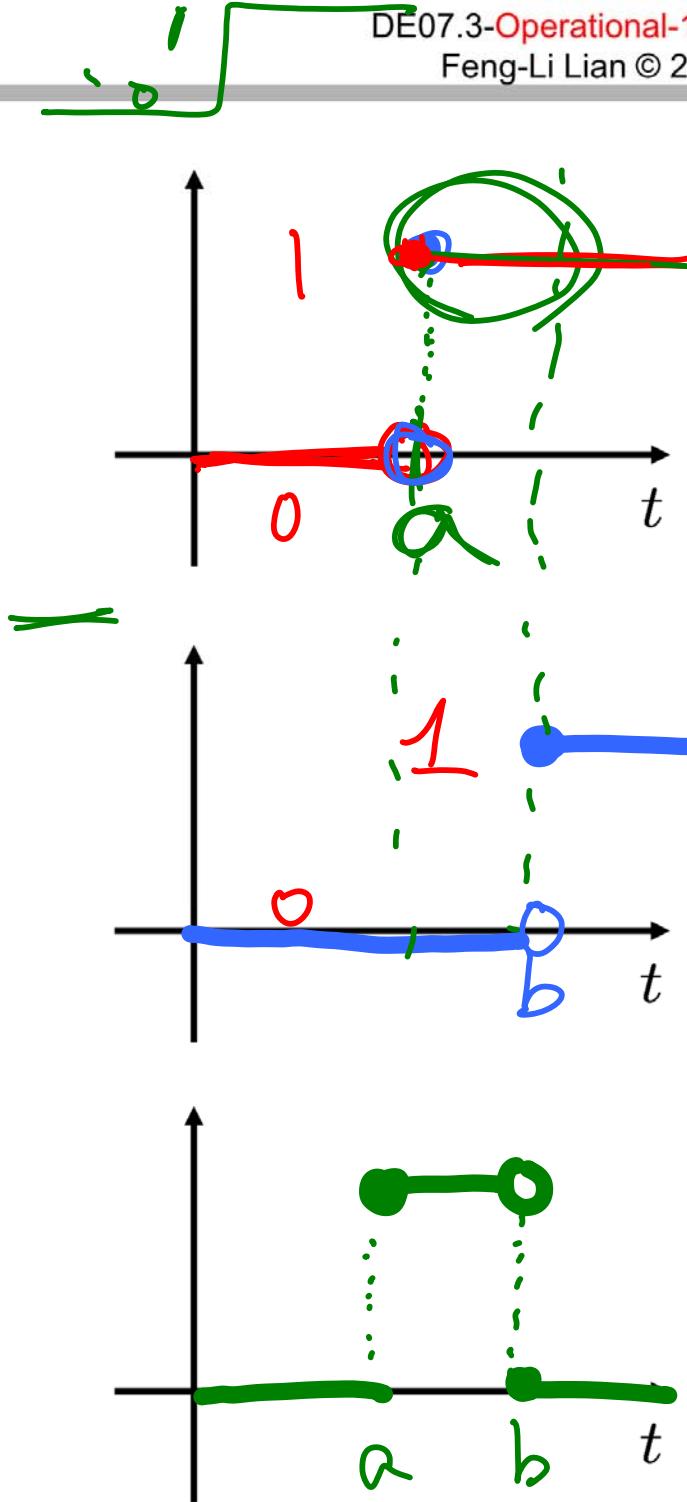
- Definition 7.3.1: Unit Step Function

$$u(t-a) = \begin{cases} 0, & 0 \leq t < a \\ 1, & a \leq t \end{cases}$$

$t-a \geq 0$
 $t-a < 0$

$$\Rightarrow u(t-a) - u(t-b)$$

$a < b$



7.3.2: Translation on the t-Axis

- $f(t) = t$

$$t \geq 0$$

- $g(t) = f(t)u(t-a) = t u(t-a)$

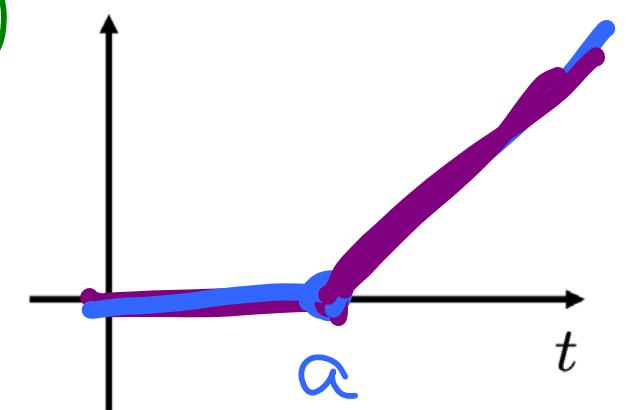
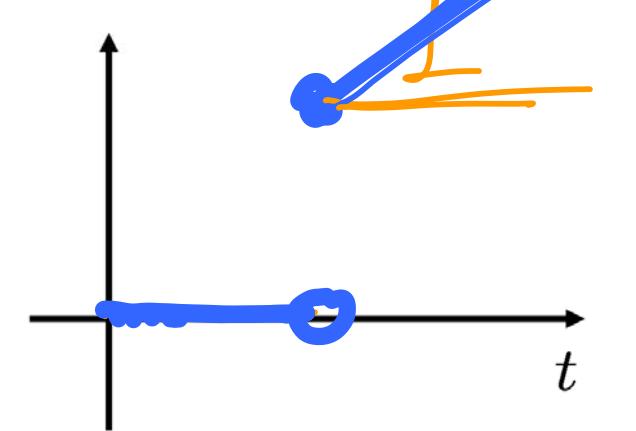
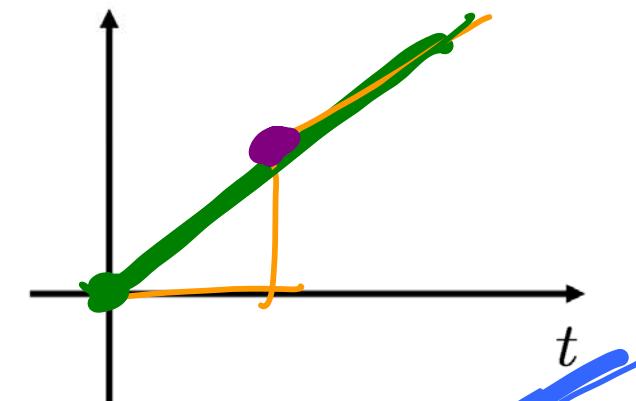
$$= \begin{cases} t & t \geq a \\ 0 & 0 \leq t < a \end{cases}$$

truncation

- $h(t) = f(t-a)u(t-a) = (t-a)u(t-a)$

$$= \begin{cases} 0 & 0 \leq t < a \\ (t-a) & t \geq a \end{cases}$$

translation



- Theorem 7.3.2: 2nd Translation Theorem

IF

$$\underline{\mathcal{L}\{f(t)\}} = \underline{F(s)} \quad \text{AND,} \quad \underline{a > 0}$$

THEN

$$\mathcal{L}\{f(t-a) u(t-a)\} = e^{-as} \underline{F(s)}$$

Proof:

$$g(t) = f(t-a) u(t-a) = \begin{cases} 0, & t < a \\ f(t-a), & t \geq a \end{cases}$$

$$\mathcal{L}\{g(t)\} = \int_0^\infty e^{-st} g(t) dt$$

$$= \int_0^a e^{-st} \cdot 0 dt + \int_a^\infty e^{-st} f(t-a) dt$$

$$t \cancel{st}$$

$$= \int_0^\infty e^{-s(\bar{t}+a)} f(\bar{t}) d\bar{t}$$

$$\bar{t} = t-a$$

$$d\bar{t} = dt$$

$$= \overline{e}^{sa} \frac{\int_0^\infty e^{-st} f(t) dt}{\overline{e}^{sa} F(s)}$$

$$\bullet \mathcal{L} \{ u(t-a) \} = \frac{1}{s} e^{-as}$$

$$\bullet \mathcal{L} \{ g(t) u(t-a) \} = \cancel{\mathcal{L} \{ g(t) \}} e^{-as}$$

$$= \mathcal{L} \{ g(t-a+a) u(t-a) \} = \int_0^{\infty} e^{-st} g(t) u(t-a) dt$$

$$= \mathcal{L} \{ g(\underline{t+a}-a) u(\underline{t-a}) \}$$

$$= \boxed{\mathcal{L} \{ g(\underline{t+a}) \} e^{-as}}$$

$$\bullet \mathcal{L}^{-1} \{ e^{-as} F(s) \} =$$

$$\begin{aligned} &= e^{-as} \int_0^{\infty} e^{-st} \cancel{g(\bar{t}+a)} d\bar{t} \\ &= \boxed{e^{-as} \mathcal{L} \{ g(\bar{t}+a) \}} \end{aligned}$$

$$t = \bar{t} + a$$

$$\bar{t} = t - a$$

$$\bar{t}$$

$$\mathcal{L} \left\{ (t-2)^3 u(t-2) \right\} = \frac{3!}{s^4}$$

$\mathcal{L}[t^3] = \frac{3!}{s^4}$

~~$(s-2)^4$~~

$= \frac{3!}{s^4} e^{-2s}$

~~e^{2s}~~

$$\mathcal{L} \left\{ 2 - 3u(t-2) + u(t-3) \right\} =$$

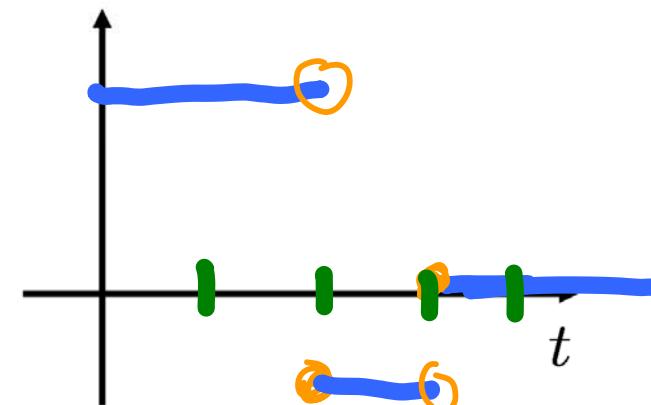
$\frac{1}{2 \cdot \frac{1}{s}} - 3 \frac{1}{\frac{1}{s}} e^{-2s} + \frac{1}{\frac{1}{s}} e^{-3s}$

$$\int_0^\infty e^{-st} (2 - 3u(t-2) + u(t-3)) dt$$

$$= \int_0^\infty e^{-st} 2 dt - 3 \underbrace{\int_2^\infty e^{-st} dt}_{= 0} + \int_3^\infty e^{-st} dt$$

$$\frac{1}{-s} e^{-st} \Big|_2^\infty$$

$$= \int_0^2 e^{-st} 2 dt - 1 \int_2^3 e^{-st} dt + \int_3^\infty e^{-st} 0 dt$$



7.3.2: Examples

$$\mathcal{L}^{-1} \left\{ \frac{e^{-\frac{\pi}{2}s}}{s^2 + 9} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{3} \frac{3}{s^2 + 3^2} e^{-\frac{\pi}{2}s} \right\} \leftrightarrow \frac{1}{3} \sin 3(t - \frac{\pi}{2}) u(t - \frac{\pi}{2})$$

$$\mathcal{L}^{-1} \left\{ \frac{3}{s^2 + 3^2} \right\} = \sin 3t$$

$$\mathcal{L}^{-1} \left\{ g(t) u(t-a) \right\} = e^{-as} \mathcal{L}^{-1} \left\{ g(t+a) \right\}$$

7.3.2: Examples

$$\mathcal{L} \left\{ (2t - 3) u(t-1) \right\} = \mathcal{L} \left\{ 2t u(t-1) \right\} - 3 \mathcal{L} \left\{ u(t-1) \right\}$$

g(t)

$$\textcircled{1} = \mathcal{L} \left\{ (2(t-1) - 1) u(t-1) \right\}.$$

$$= 2 \mathcal{L} \left\{ (t-1) u(t-1) \right\} - \mathcal{L} \left\{ u(t-1) \right\}$$

$$= 2 \frac{1}{s^2} e^{-s} - \frac{1}{s} e^{-s}$$

$$= \boxed{e^{-s} \left[\frac{2}{s^2} - \frac{1}{s} \right]}$$

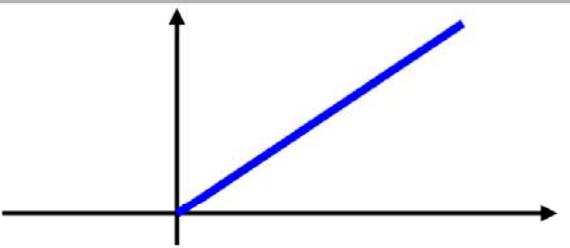
$$\textcircled{2} = e^{-as} \mathcal{L} \{ g(t+a) \} = e^{-s} \mathcal{L} \{ g(t+1) \}$$

$$= e^{-s} \mathcal{L} \{ 2(t+1) - 3 \} = \boxed{e^{-s} \mathcal{L} \{ 2t - 1 \}}$$

$$= \boxed{e^{-s} \left(2 \frac{1}{s^2} - \frac{1}{s} \right)}$$

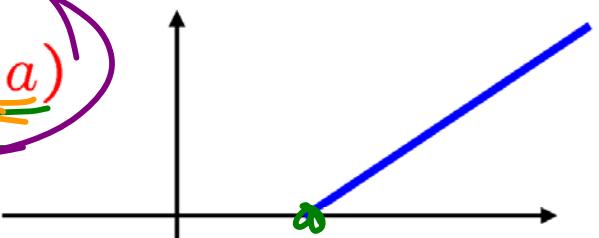
7.3.2: More Comparisons

$$f(t) = t, \quad t \geq 0$$



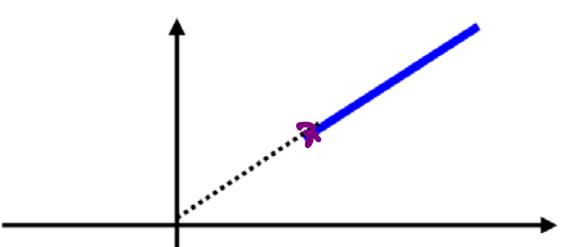
$$F(s)$$

$$f(t-a)U(t-a)$$



$$e^{-as} F(s)$$

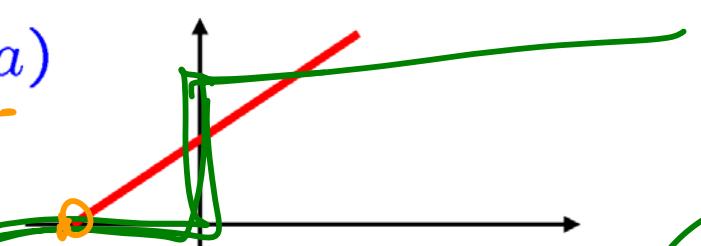
$$f(t)U(t-a)$$



$$e^{-as} L\{f(t+a)\}$$

$$g(t) = f(t+a)$$

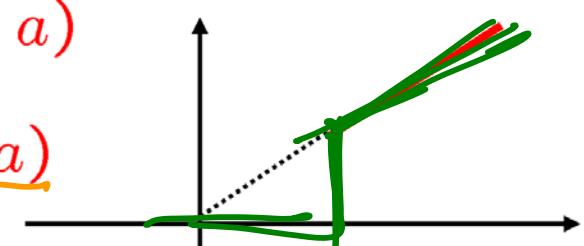
$$\text{---} \quad \text{---}$$



$$e^{-as} G(s)$$

$$g(t-a)U(t-a)$$

$$= f(t)U(t-a)$$



$$= e^{-as} L\{g(t)\}$$

$$= e^{-as} L\{f(t+a)\}$$

Summary - 7.3: Translation on the s-Axis and the t-Axis

$$\mathcal{L} \left\{ \cos(4t) \right\} = \frac{s}{s^2 + 4^2}$$

$$\mathcal{L} \left\{ e^{3t} \cos(4t) \dots \right\} = \frac{(s - 3)}{(s - 3)^2 + 4^2}$$

$$t \rightarrow (t - \frac{\pi}{2})$$

$$\mathcal{L} \left\{ \cos(4(t - \frac{\pi}{2})) \mathcal{U}(t - \frac{\pi}{2}) \right\} = \frac{s}{s^2 + 4^2} \left(e^{-\frac{\pi}{2}s} \right)$$

$$\boxed{\mathcal{L} \left\{ f(t) \right\} \triangleq \int_0^\infty e^{-st} f(t) dt = F(s)}$$

$$\mathcal{L} \left\{ f(t) \right\} = F(s)$$

$$\begin{aligned} &+ \mathcal{L} \left\{ e^{at} f(t) \right\} = F(s-a) \\ \mathcal{L} \left\{ f(t-a) \mathcal{U}(t-a) \right\} &= e^{-as} F(s) \end{aligned}$$