Fall 2019

微分方程
Differential Equations

Unit 07.2
Inverse Transforms
and Transforms of Derivatives

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NTU-EE

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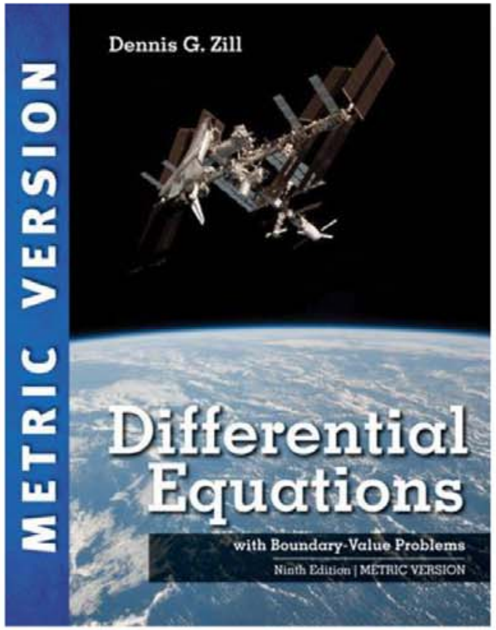


$$\mathcal{L}\{f(t)\} = F(s)$$

$$\mathcal{L}^{-1}\{F(s)\} = f(t)$$

Figures and images used in these lecture notes are adopted from
Differential Equations with Boundary-Value Problems, 9th Ed., D.G. Zill, 2018 (Metric Version)

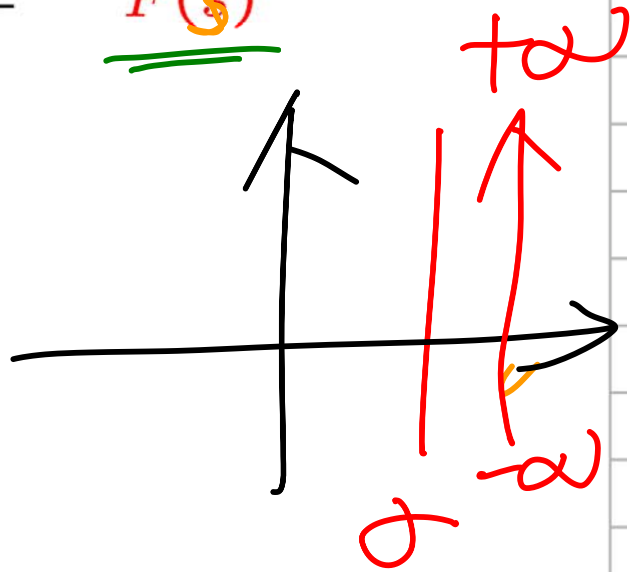
Outline

$$\mathcal{L}\left\{f(t)\right\} \triangleq \int_0^{\infty} e^{-st} f(t)\, dt = F(s)$$

- Given $F(s)$ $\Rightarrow$ Find $f(t)$

$$\Rightarrow \mathcal{L}^{-1}\{ F(s) \} = f(t)$$

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st} ds$$



7.2: Theorem 7.2.1: Some Inverse Transforms

$$(a)\quad \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$(b)\quad \mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n, \quad n = 1, 2, 3, \cdots$$

$$(c)\quad \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

$$(d)\quad \mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt \qquad (e)\quad \mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$$

$$(f)\quad \mathcal{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\} = \sinh kt \qquad (g)\quad \mathcal{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\} = \cosh kt$$

See Appendix III for a list of Laplace transforms

7.2: Example

$$\mathcal{L}^{-1}\left\{\frac{3s-2}{s^3(s^2+4)}\right\} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \boxed{\frac{Ds+E}{s^2+4}}$$

$$\frac{D}{s-j2} \quad \frac{E}{s+j2}$$

$\times\, s^3(s^2+4)$

$$3s-2 = A s^2(s^2+4) + B s (s^2+4) + C(s^2+4) + (Ds+E)s^3$$

$$s=0 \Rightarrow -2 = C(4) \Rightarrow \boxed{C = \frac{-2}{4}} = -\frac{1}{2}$$

$$A = \frac{1}{8}, \quad B = \frac{3}{4}, \quad C = -\frac{1}{2}, \quad D = -\frac{1}{8}, \quad E = -\frac{3}{4}$$

$$= \frac{\frac{1}{8}}{s} + \frac{\frac{3}{4}}{s^2} + \frac{-\frac{1}{2}}{s^3} + \frac{-\frac{1}{8}s - \frac{3}{4}}{s^2+4}$$

$$= \frac{1}{8}\boxed{\left(\frac{1}{s}\right)} + \frac{3}{4}\boxed{\left(\frac{1}{s^2}\right)} - \frac{1}{2}\boxed{\frac{1}{s^3}} - \frac{1}{8}\boxed{\frac{s}{s^2+2^2}} - \frac{3}{4}\frac{1}{2}\boxed{\frac{2}{s^2+2^2}}$$

$$= \frac{1}{8}\, 1 + \frac{3}{4}\, t - \frac{1}{2}\,\frac{1}{2!}\, t^2 - \frac{1}{8}\cos 2t - \frac{3}{8}\sin 2t$$

7.2: Theorem 7.2.2: Transform of a Derivative

- If $f, f', \cdots, f^{(n-1)}$ are continuous on $[0, \infty)$

  are of exponential order C

- and, if $f^{(n)}$ is piecewise continuous on $[0, \infty)$

$$\mathcal{L}\left\{f(t)\right\} = F(s)$$

- then $\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n F(s)$ (n)

$$- s^{n-1} f(0) \quad n-1+0 = n-1$$

$$- s^{n-2} f'(0) \quad n-2+1 = n-1 \quad - \cdots$$

$$- 1 \, f^{(n-1)}(0)$$

y' $\quad \mathcal{L}\{y\} = Y(s)$

$\mathcal{L}\{y''\} = s^2 Y(s) - s\,y(0) - y'(0)$

7.2: Theorem 7.2.2: Transform of a Derivative

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\, dt$$

$v \quad u' = vu - \int v'u$

$$= e^{-st} f(t) \Big|_0^\infty - \int_0^\infty (-s) e^{-st} f(t)\, dt$$

$$= e^{-s\infty} f(\infty) - e^{0} f(0) + s \int_0^\infty e^{-st} f(t)\, dt$$

$$-f(0) + s \boxed{\int_0^\infty e^{-st} f(t)\, dt}$$

$F(s)$

$$= sF(s) - f(0)$$

$$\mathcal{L}\{f^{(2)}(t)\} =$$

$$= \int_0^\infty e^{-st} f^{(2)}(t)\, dt$$

$u \quad v' = uv - \int u'v$

$$= e^{-st} f'(t) \Big|_0^\infty - \int_0^\infty (-s) e^{-st} f'(t) dt$$

$$= -f'(0) + s \int_0^\infty e^{-st} f'(t) dt = -f'(0) + s\left[sF - f(0)\right]$$

$$= -f'(0) + s^2 F - s f(0)$$

$$= \boxed{s^2 F - s f(0) - f'(0)}$$

7.2: Example 5: Solving a 2nd-Order IVP

$$\mathcal{L}\{y'' - 3y' + 2y\} = \mathcal{L}\{e^{-4t}\} \qquad e^{mt}$$

$$\begin{cases} y(0) = 1 \\ y'(0) = 5 \end{cases}$$

$$\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \frac{1}{s+4} \Rightarrow s = -4$$

$$= s^2 Y(s) - s\,y(0) - y'(0)$$

$$-3\left(sY(s) - y(0)\right)$$

$$+2\,Y(s)$$

$$= \left(\boxed{s^2 - 3s + 2}\right) Y(s) - s\,y(0) - y'(0) + 3y(0)$$

with $y(0) = 1$, $y'(0) = 5$, $y(0) = 1$:

$$(s^2 - 3s + 2)\boxed{Y(s)} - s - 5 + 3 = \frac{1}{s+4}$$

7.2: Example 5:

$$(s^2-3s+2)Y(s) = \frac{1}{s+4} + (s+2)$$

$$Y(s) = \frac{\dfrac{1+(s+2)(s+4)}{(s+4)}}{(s^2-3s+2)}$$

$$= \frac{1+(s+2)(s+4)}{(s-2)(s-1)(s+4)} \quad \begin{matrix} 2 \\ 3 \end{matrix}$$

$$= \left(\frac{\frac{25}{6}}{(s-2)} + \frac{-\frac{16}{5}}{(s-1)} + \frac{\frac{1}{30}}{(s+4)} \right. e^{3s}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \boxed{\frac{25}{6} e^{2t} - \frac{16}{5} e^{t} + \frac{1}{30} e^{-4t}}$$

Summary - 7.2: Inverse Laplace T. and T. of Derivatives

$$\mathcal{L}\left\{f(t)\right\} \triangleq \int_0^\infty e^{-st}\, f(t)\ dt = F(s)$$

$$\mathcal{L}^{-1}\left\{F(s)\right\} = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st}\, F(s)\ ds = f(t)$$

$$(a)\quad \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$(b)\quad \mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n, \quad n = 1, 2, 3, \cdots$$

$$(c)\quad \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

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$$(f)\quad \mathcal{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\} = \sinh kt \qquad (g)\quad \mathcal{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\} = \cosh kt$$

Summary - 7.2: Inverse Laplace T. and T. of Derivatives

$$\mathcal{L}\left\{f(t)\right\} \triangleq \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

$$\mathcal{L}\left\{f'(t)\right\} = sF(s) - f(0)$$

$$\mathcal{L}\left\{f''(t)\right\} = s^2 F(s) - sf(0) - f'(0)$$

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

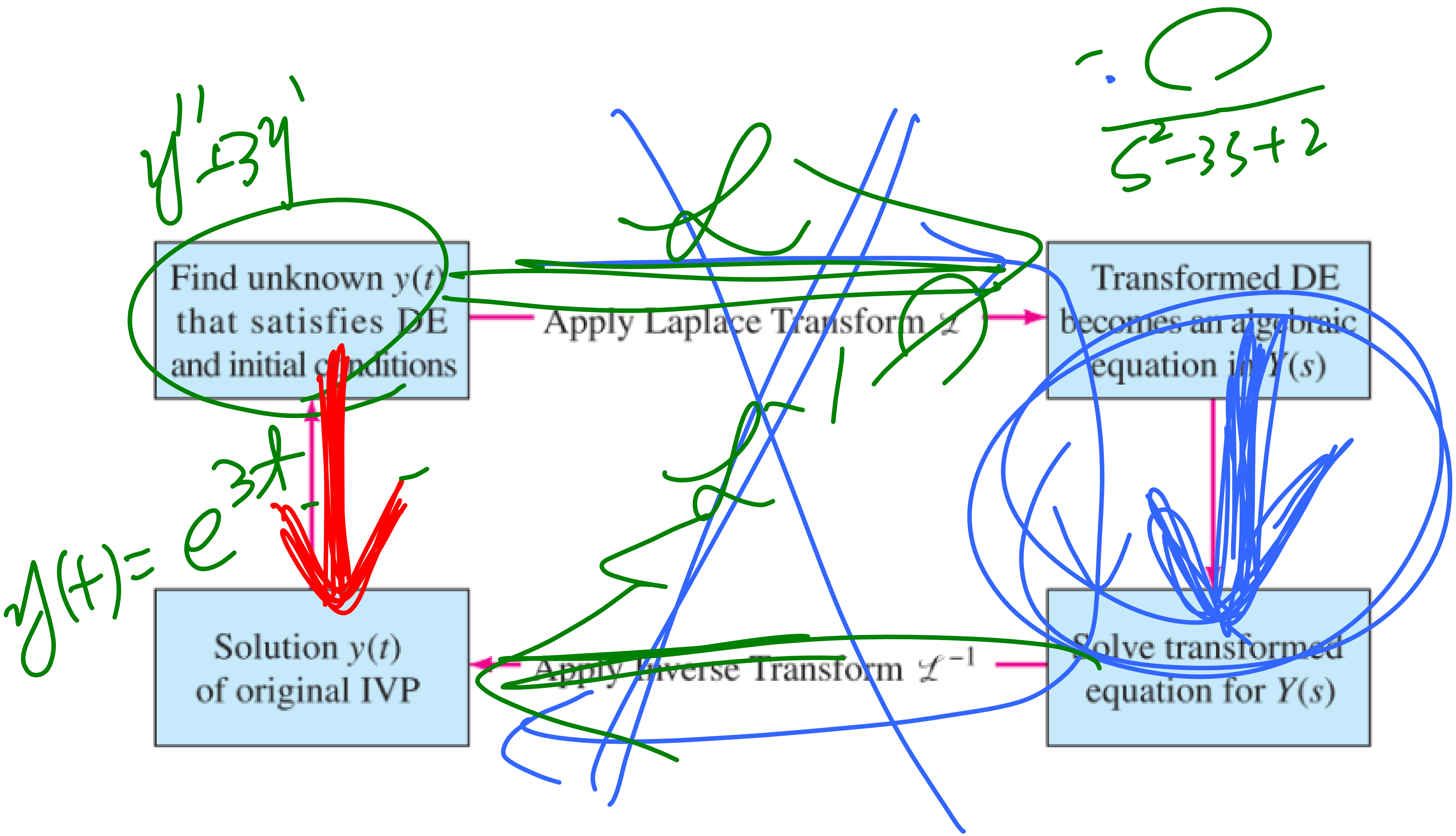
$$y'' - 3y' + 2y = e^{-4t}$$

$$\mathcal{L}\left\{y'' - 3y' + 2y\right\} = \left(s^2 - 3s + 2\right) Y(s) - sy(0) - y'(0) + 3y(0)$$

$$\mathcal{L}\left\{e^{-4t}\right\} = \frac{1}{s+4}$$

$$Y(s) = \ldots \longrightarrow y(t) = \ldots$$

Summary - 7.2: Inverse Laplace T. and T. of Derivatives



7.2: Example 5: Solving a 2nd-Order IVP

$$y'' - 3y' + 2y = e^{-4t} \qquad \begin{cases} y(0) = 1 \\ y'(0) = 5 \end{cases}$$

- Method in 4.2:

$$y_1(t) = e^t$$

$$y(t) = u(t)\, y_1(t) = u\, e^t$$

$$y'(t) = u'\, y_1 + u\, y_1' = (u' + u)\, e^t$$

$$y''(t) = u''\, y_1 + 2\, u'\, y_1' + u\, y_1'' = (u'' + 2u' + u)\, e^t$$

$$\Rightarrow (u'' + 2u' + u)\, e^t \quad -3(u' + u)\, e^t \quad +2u\, e^t = 0$$

$$\Rightarrow (u'' - u')\, e^t = 0$$

$$\Rightarrow (u'' - u') = 0$$

$$\rightarrow \text{let } w = u' \qquad \Rightarrow e^{-t}\, w = c_1 \qquad \Rightarrow u = c_1\, e^t + c_2$$

$$\Rightarrow (w' - w) = 0 \qquad \Rightarrow w = c_1\, e^t \qquad \Rightarrow y = u\, y_1$$

$$\Rightarrow \text{I.F.} = e^{-t} \qquad \Rightarrow u' = w = c_1\, e^t \qquad = c_1\, e^{2t} + c_2\, e^t$$

$$\Rightarrow \frac{d}{dt}\left[e^{-t}\, w\right] = 0 \qquad \Rightarrow u = \int c_1\, e^t dt$$

$$y'' - 3y' + 2y = e^{-4t} \qquad \begin{cases} y(0) = 1 \\ y'(0) = 5 \end{cases}$$

- Method in 4.3 & 4.4:

$$y(t) = e^{mt} \qquad \Rightarrow \quad y_p(t) = Ae^{-4t}$$

$$y'(t) = me^{mt} \qquad y_p'(t) = A(-4)e^{-4t}$$

$$y''(t) = m^2 e^{mt} \qquad y_p''(t) = A(-4)^2 e^{-4t}$$

$$y'' - 3y' + 2y = 0 \qquad y_p'' - 3y_p' + 2y_p = e^{-4t}$$

$$\Rightarrow \quad m^2 - 3m + 2 = 0 \qquad (16A + 12A + 2A)e^{-4t} = e^{-4t}$$

$$\Rightarrow \quad m_{1,2} = 1, 2 \qquad \Rightarrow \quad A = \frac{1}{30}$$

$$\Rightarrow \quad y_c(t) = c_1 e^{1t} + c_2 e^{2t} \qquad \Rightarrow \quad y_p(t) = \frac{1}{30} e^{-4t}$$

$$\Rightarrow \quad y(t) = y_c(t) + y_p(t) = c_1 e^{1t} + c_2 e^{2t} + \frac{1}{30} e^{-4t}$$

$$\begin{cases} y(0) = 1 \\ y'(0) = 5 \end{cases} \quad = \frac{-16}{5} e^{1t} + \frac{25}{6} e^{2t} + \frac{1}{30} e^{-4t}$$

7.2: Example 5: Solving a 2nd-Order IVP

$$y'' - 3y' + 2y = e^{-4t} \qquad \begin{cases} y(0) = 1 \\ y'(0) = 5 \end{cases}$$

- Method in 7.2:

$$\mathcal{L}\left\{y'' - 3y' + 2y\right\} = \mathcal{L}\left\{e^{-4t}\right\}$$

$$\Rightarrow (s^2 - 3s + 2)Y(s) - y(0)(s-3) - y'(0) = \frac{1}{s+4}$$

$$\Rightarrow (s^2 - 3s + 2)Y(s) = y(0)(s-3) + y'(0) + \frac{1}{s+4}$$

$$\Rightarrow Y(s) = \frac{1}{(s^2 - 3s + 2)}\left(y(0)(s-3) + y'(0)\right)$$

- zero-input response

$$+ \frac{1}{(s^2 - 3s + 2)}\left(\frac{1}{s+4}\right)$$

- zero-state response

$$= \left(\frac{-16}{5}\right)\frac{1}{s-1} + \left(\frac{25}{6}\right)\frac{1}{s-2} + \left(\frac{1}{30}\right)\frac{1}{s+4}$$

$$\Rightarrow y(t) = \left(\frac{-16}{5}\right) e^{1t} + \left(\frac{25}{6}\right) e^{2t} + \left(\frac{1}{30}\right) e^{-4t}$$