

Fall 2019

微分方程 Differential Equations

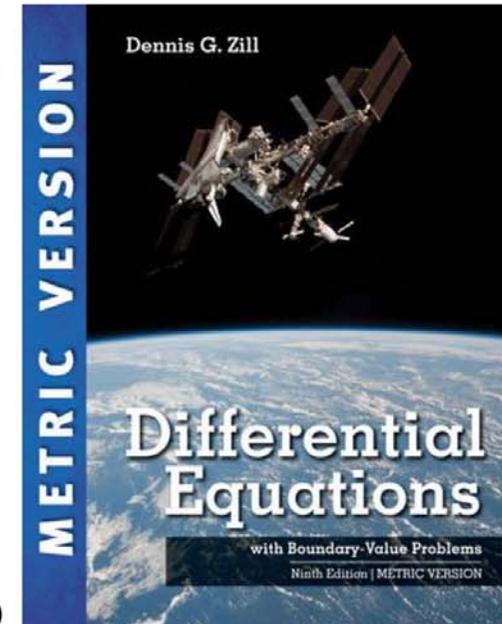
Unit 07.1 Definition of Laplace Transform

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$$\mathcal{L}\{f(t)\} = F(s)$$
$$\triangleq \int_0^{\infty} e^{-st} f(t) dt$$



- **7.1: Definition of Laplace Transform**
- 7.2: Inverse Transforms and Transforms of Derivatives
 - 7.2.1: Inverse Transforms
 - 7.2.2: Transforms of Derivatives
- 7.3: Operational Properties I
 - 7.3.1: Translation on the s-Axis
 - 7.3.2: Translation on the t-Axis
- 7.4: Operational Properties II
 - 7.4.1: Derivatives of a Transform
 - 7.4.2: Transforms of Integrals
 - 7.4.3: Transform of a Periodic Function
- 7.5: The Dirac Delta Function
- 7.6: Systems of Linear Differential Equations

- electrical

$$L \frac{dq^2(t)}{dt^2} + R \frac{dq(t)}{dt} + \frac{1}{C} q(t) = E(t)$$

- mechanical

$$m \frac{dx^2(t)}{dt^2} + \beta \frac{dx(t)}{dt} + k x(t) = f(t)$$

$$\Rightarrow a \frac{dx^2(t)}{dt^2} + b \frac{dx(t)}{dt} + c x(t) = g(t)$$

$$X(t) = e^{mt}$$

$$\Rightarrow a m^2 + b m + c = 0$$

$f(t)$	$\xrightarrow{\frac{d}{dt}}$	$f'(t)$	differentiation
$\omega \sin t$		$-\sin(\omega t)$	operator
$f(t)$	$\xrightarrow{\int dt}$	$\int f(t) dt$	indefinite
$\omega \sin t$		$-\cos(\omega t)$	integral
$f(t)$	$\xrightarrow{\int_a^b dt}$	$\int_a^b f(t) dt$	operator
			definite
			integral
$f(t)$	$\xrightarrow{\int_a^b \boxed{k(s,t)} dt}$	$\int_a^b \underline{k(s,t)} f(t) dt$	operator
	e^{-st}	$k(s,t)$	definite
		$= F(s)$	integral
			operator

- f : a function defined for $t \geq 0$

$$f(t) \quad | \quad t \geq 0$$

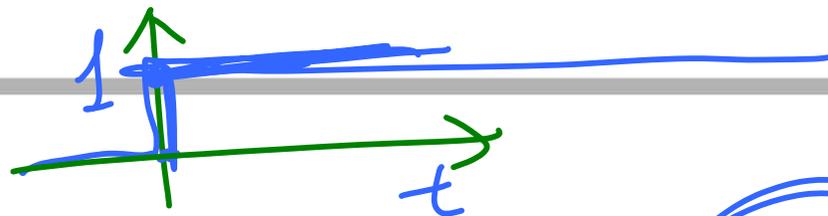
or, $f: [0, \infty) \rightarrow \mathbb{R}$

$$\mathcal{L}\{f(t)\} \triangleq \int_0^{\infty} e^{-st} f(t) dt = F(s)$$

is said to be the Laplace Transform of $f(t)$,

provided that the integral converges

7.1: Example 1:



$f(t) = 1$

$\Rightarrow \mathcal{L}\{f(t)\} = \mathcal{L}\{1\} =$

$\int_0^{\infty} e^{-st} \cdot 1 \, dt \quad \mathcal{L}\{1\}$

$= \int_0^{\infty} e^{-st} \, dt$

$e^{+\infty}$

$= \left[-\frac{e^{-st}}{s} \right]_0^{\infty}$

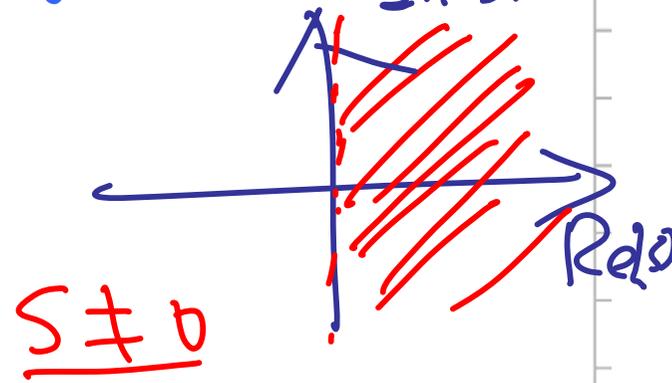
$= \left[-\frac{e^{-\infty}}{s} - \left(-\frac{e^0}{s} \right) \right]$

$= \left[-\frac{0}{s} - \left(-\frac{1}{s} \right) \right]$

$= \left[0 + \frac{1}{s} \right]$

$= \frac{1}{s}$

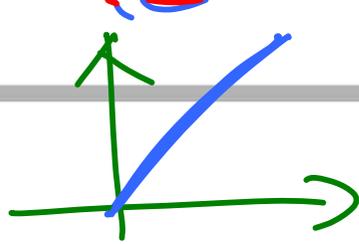
$= \frac{1}{s}$



$\text{Re}\{s\} > 0$

$1 \rightarrow \frac{1}{s}$

7.1: Example 2:



$$f(t) = t$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \mathcal{L}\{t\} =$$

$$\int_0^{\infty} e^{-st} t dt$$

$$u' v = uv - \int u v'$$

$$= \frac{1}{-s} e^{-st} t \Big|_0^{\infty} - \int_0^{\infty} \frac{e^{-st}}{-s} 1 dt$$

$$= \left[\frac{1}{-s} t e^{-st} \right]_0^{\infty} + \left[\frac{1}{s} \left[-\frac{1}{s} e^{-st} \right]_0^{\infty} \right]$$

$$\Rightarrow \frac{1}{-s} \left[\frac{\infty}{\infty} - \frac{0}{1} \right] = \frac{1}{s} \left[\frac{1}{s} e^{-st} \Big|_{t=\infty} - \frac{1}{s} \Big|_{t=0} \right] = 0$$

$$\Rightarrow \frac{1}{-s^2} e^{-st} \Big|_0^{\infty} = -\frac{1}{s^2} [0 - 1] = \frac{1}{s^2}$$

$$f(t) = t \rightarrow \text{First } \frac{1}{s^2} \quad \text{Re}(s) > 0$$

7.1: Example 3:

$$f(t) = \underline{e^{-3t}}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \mathcal{L}\{e^{-3t}\} = \int_0^{\infty} \underbrace{e^{-st} e^{-3t}} dt$$

$$= \int_0^{\infty} e^{-(s+3)t} dt$$

$$= \frac{1}{-(s+3)} \left[e^{-(s+3)t} \right]_0^{\infty}$$

$$= \frac{1}{-(s+3)} \left[\cancel{e^{-(s+3)\infty}} - 1 \right]$$

$$\underline{\operatorname{Re}\{s+3\} > 0}$$

$$\Rightarrow \boxed{\operatorname{Re}\{s\} > -3}$$

$$\underline{s+3} \rightarrow s+3=0 \Rightarrow s = \underline{-3}$$

7.1: Example 4:

$$f(t) = \sin^2 t$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \mathcal{L}\{\sin^2 t\} = \int_0^{\infty} e^{-st} \sin^2 t \, dt$$

$$= \frac{1}{(-s)} e^{-st} \sin^2 t \Big|_0^{\infty} - \int_0^{\infty} \frac{1}{(-s)} e^{-st} 2 \cos 2t \, dt$$

$$= \frac{1}{-s} (e^{-s\infty} \sin^2 \infty - e^{-0} \sin^2 0) + \frac{2}{s} \int_0^{\infty} e^{-st} \cos 2t \, dt$$

$$= \frac{2}{s} \left[\frac{1}{(-s)} e^{-st} \cos 2t \Big|_0^{\infty} - \int_0^{\infty} \frac{1}{(-s)} e^{-st} (-2) \sin 2t \, dt \right]$$

$$= \frac{2}{s} \left[\frac{1}{-s} (e^{-s\infty} \cos 2\infty - e^{-s \cdot 0} \cos 2 \cdot 0) - \frac{2}{s} \int_0^{\infty} e^{-st} \sin 2t \, dt \right]$$

$$= \frac{2}{s} \left[-\frac{1}{s} - \frac{2}{s} \int_0^{\infty} e^{-st} \sin 2t \, dt \right]$$

$v' \quad u = vu - \int vu'$

$\cos 2t$

$\frac{1}{s^2} \sin 2t$

$v' \quad u = vu - \int vu'$

\int

$$\Rightarrow \left(1 + \frac{4}{s^2}\right) S = \frac{2}{s}$$

$$\frac{s^2+4}{s^2}$$

$$\Rightarrow S = \frac{2}{s^2+4}$$

$$\boxed{\frac{2}{s^2+2^2}}$$



$$s = \pm j2$$

$$\text{Re}\{s\} > 0$$

$$(a) \mathcal{L}\{1\} = \frac{1}{s}$$

$$(b) \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad n = 1, 2, 3, \dots$$

$$(c) \mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$(d) \mathcal{L}\{\sin kt\} = \frac{k}{s^2 + k^2}$$

$$(e) \mathcal{L}\{\cos kt\} = \frac{s}{s^2 + k^2}$$

$$(f) \mathcal{L}\{\sinh kt\} = \frac{k}{s^2 - k^2}$$

$$(g) \mathcal{L}\{\cosh kt\} = \frac{s}{s^2 - k^2}$$

$$s = +k, -k$$

For others, see the Table of Laplace Transforms

- Laplace transform is a linear transform

$$\begin{aligned}
 & \mathcal{L}\{a f(t) + b g(t)\} \Rightarrow \textcircled{1} \\
 & = a \mathcal{L}\{f(t)\} + b \mathcal{L}\{g(t)\} \Rightarrow > \\
 & \textcircled{1} \Rightarrow \int_0^{\infty} \underline{e^{-st}} \cdot (\underline{a f(t)} + \underline{b g(t)}) dt \\
 & = a \int_0^{\infty} \underline{e^{-st}} f(t) dt + b \int_0^{\infty} \underline{e^{-st}} g(t) dt \\
 & = a \mathcal{L}\{f(t)\} + b \mathcal{L}\{g(t)\}
 \end{aligned}$$

$$\textcircled{1} \mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2 + \omega^2}$$

$$\mathcal{L}\{\cos \omega t\} = \frac{s}{s^2 + \omega^2}$$

$$a = \sigma + i\omega$$

$$e^{at} = e^{(\sigma + i\omega)t} = e^{\sigma t} e^{i\omega t}$$

$$\textcircled{1} \Rightarrow \mathcal{L}\{e^{at}\} = \frac{1}{s-a} = \frac{1}{s - (\sigma + i\omega)} = \frac{1}{(s-\sigma) - i\omega}$$

$$\sigma = 0 \quad \mathcal{L}\{e^{i\omega t}\} = \frac{1}{(s-i\omega)(s+i\omega)} = \frac{s+i\omega}{s^2 + \omega^2}$$

$$= \frac{s}{s^2 + \omega^2} + i \frac{\omega}{s^2 + \omega^2}$$

$$\begin{aligned} \mathcal{L}\{e^{i\omega t}\} &= \mathcal{L}\{\cos\omega t + i\sin\omega t\} \\ &= \mathcal{L}\{\cos\omega t\} + (i)\mathcal{L}\{\sin\omega t\} \end{aligned}$$

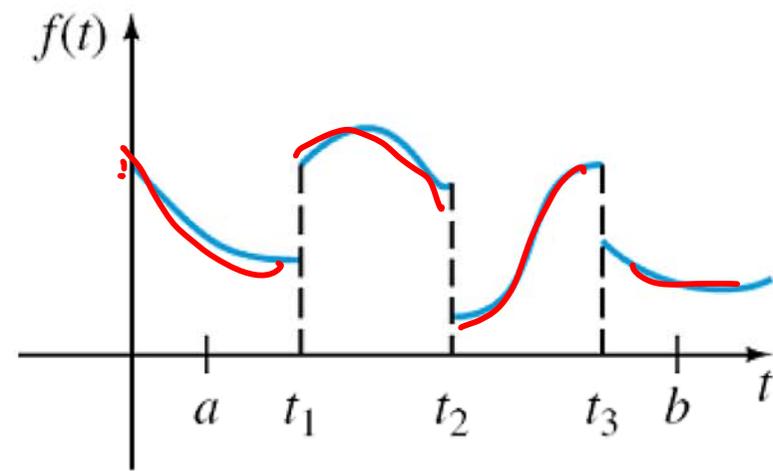
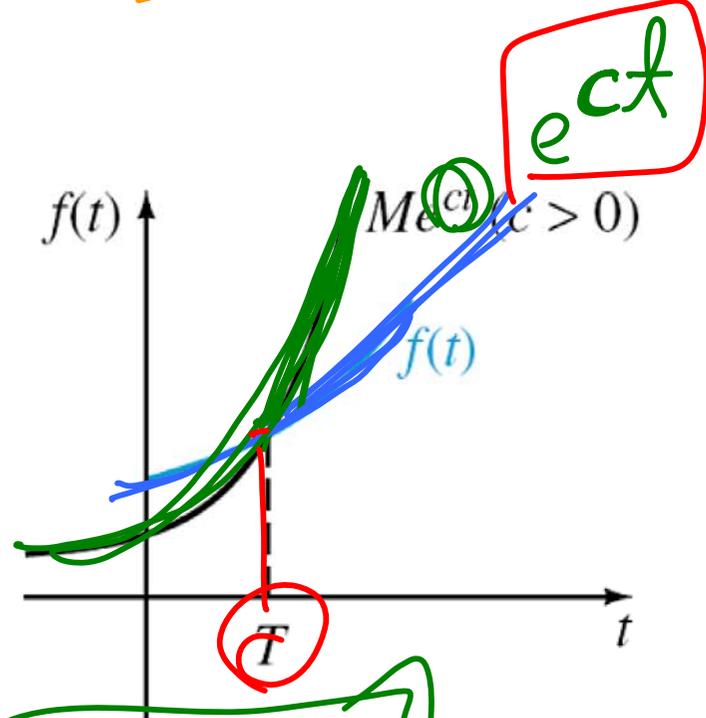
$$\mathcal{L}\{\cos\omega t\} = \frac{s}{s^2 + \omega^2}$$

$$\mathcal{L}\{\sin\omega t\} = \frac{\omega}{s^2 + \omega^2}$$

(1) $f(t)$ is piecewise continuous on $[0, \infty)$

(2) $f(t)$ is of exponential order ϵ for $t > T$

$\Rightarrow \mathcal{L}\{f(t)\}$ integration exists for $\text{Re}\{s\} > \epsilon$



$\Rightarrow |f(t)| \leq M e^{ct}, \forall t > T$

$\Rightarrow f(t)$ is continuous on $[t_k, t_{k+1}]$

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$$

$$= \int_0^T e^{-st} f(t) dt + \int_T^{\infty} e^{-st} f(t) dt$$

$I_1 \rightarrow \text{exist}$ (I_2)

$$|I_2| = \left| \int_T^{\infty} e^{-st} f(t) dt \right|$$

$$\leq \int_T^{\infty} |e^{-st}| |f(t)| dt$$

$$|f(t)| \leq M e^{ct}$$

$$\leq \int_T^{\infty} |e^{-st}| M e^{ct} dt$$

$$\operatorname{Re}\{s-c\} > 0$$

$$\leq \int_T^{\infty} |e^{-(s-c)t}| M dt$$

$$\frac{1}{-(s-c)} e^{-(s-c)t} \Big|_T^{\infty}$$

(1) $f(t)$ is piecewise continuous on $(0, \infty)$

(2) $f(t)$ is of exponential order $C \quad \forall t \geq T$

$$G(s) = g(t) \quad Y(s) \leftrightarrow y(t)$$

$$\underline{F(s)} = \mathcal{L}\{\underline{f(t)}\}$$

$$\Rightarrow \underline{\lim_{s \rightarrow \infty} F(s)} = 0$$

• **Proof:** ① $|f(t)| \leq M_1 = M_1 e^{0t} \quad \forall t$

② $|f(t)| \leq M_2 e^{ct} \quad \forall t \geq T$

$$\Rightarrow \underline{|f(t)| \leq M e^{at}}$$

$$M = \max(M_1, M_2)$$

$$a = \max\{0, c\}$$

$$|F(s)| = |\mathcal{L}\{f(t)\}| = \left| \int_0^{\infty} e^{-st} \underline{f(t)} dt \right|$$

$$\leq \int_0^{\infty} e^{-st} \underline{|f(t)|} dt$$

$$\leq \int_0^{\infty} e^{-st} M e^{at} dt$$

$$= M \int_0^{\infty} e^{-(s-a)t} dt$$

$$= M \frac{1}{-(s-a)} \left[e^{-(s-a)t} \right]_0^{\infty} \quad \text{Re}(s-a) > 0$$

$$= M \frac{1}{-(s-a)} \left[\cancel{e^{-(s-a)\infty}} - \cancel{e^0} \right]$$

$$\stackrel{s \rightarrow \infty}{=} \frac{M}{s-a} \rightarrow \frac{M}{\infty} = 0$$

$$\frac{(s+2)(s-3)}{(s+a)(s-b)(s+4)}$$

$$\mathcal{L}\{f(t)\} \triangleq \int_0^{\infty} e^{-st} f(t) dt = F(s)$$

$$(a) \mathcal{L}\{1\} = \frac{1}{s}$$

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