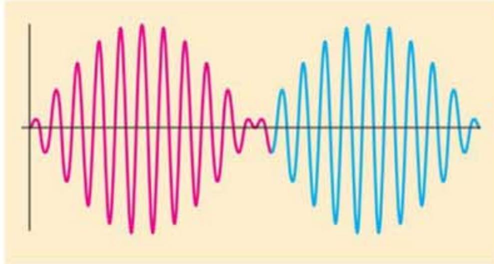
Fall 2019



微分方程
Differential Equations

Unit 05.1
Linear Models: Initial-Value Problems

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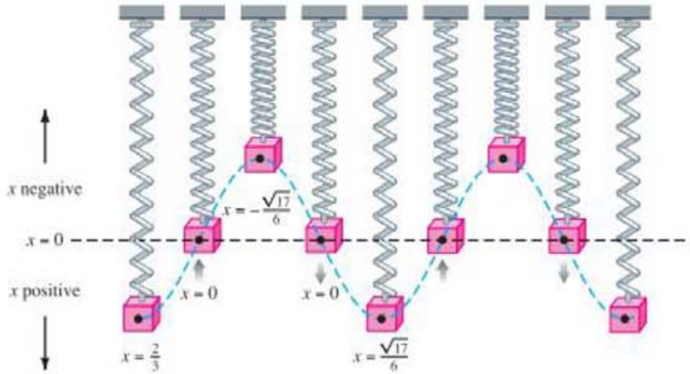
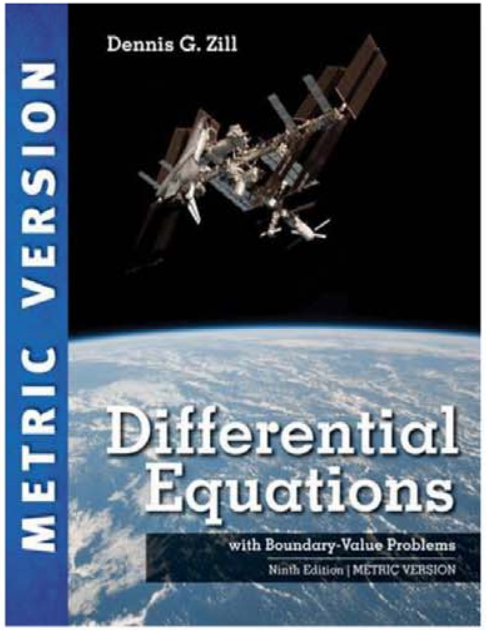
NTU-EE

Sep19 – Jan20

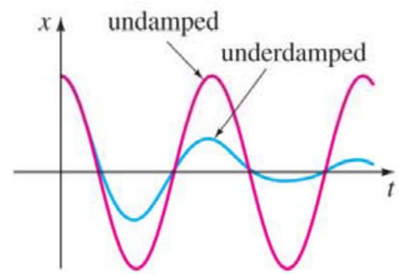


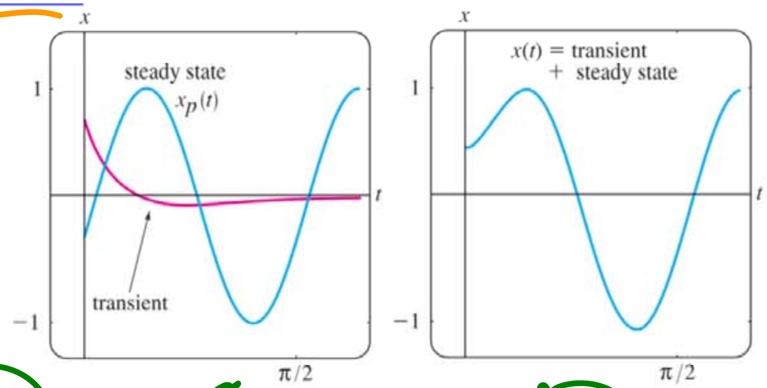


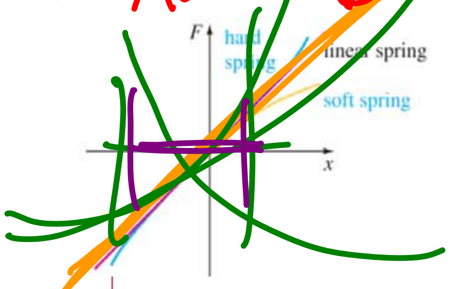
Figures and images used in these lecture notes are adopted from
Differential Equations with Boundary-Value Problems, 9th Ed., D.G. Zill, 2018 (Metric Version)

Outline

- **5.1: Linear Models: Initial-Value Problems**
  - **5.1.1: Spring/Mass Systems: Free Undamped Motion**
  - **5.1.2: Spring/Mass Systems: Free Damped Motion**
  - **5.1.3: Spring/Mass Systems: Driven Motion**
  - **5.1.4: Series Circuit Analogue**
- 5.2: Linear Models: ~~Boundary-Value Problems~~
- 5.3: Nonlinear Models







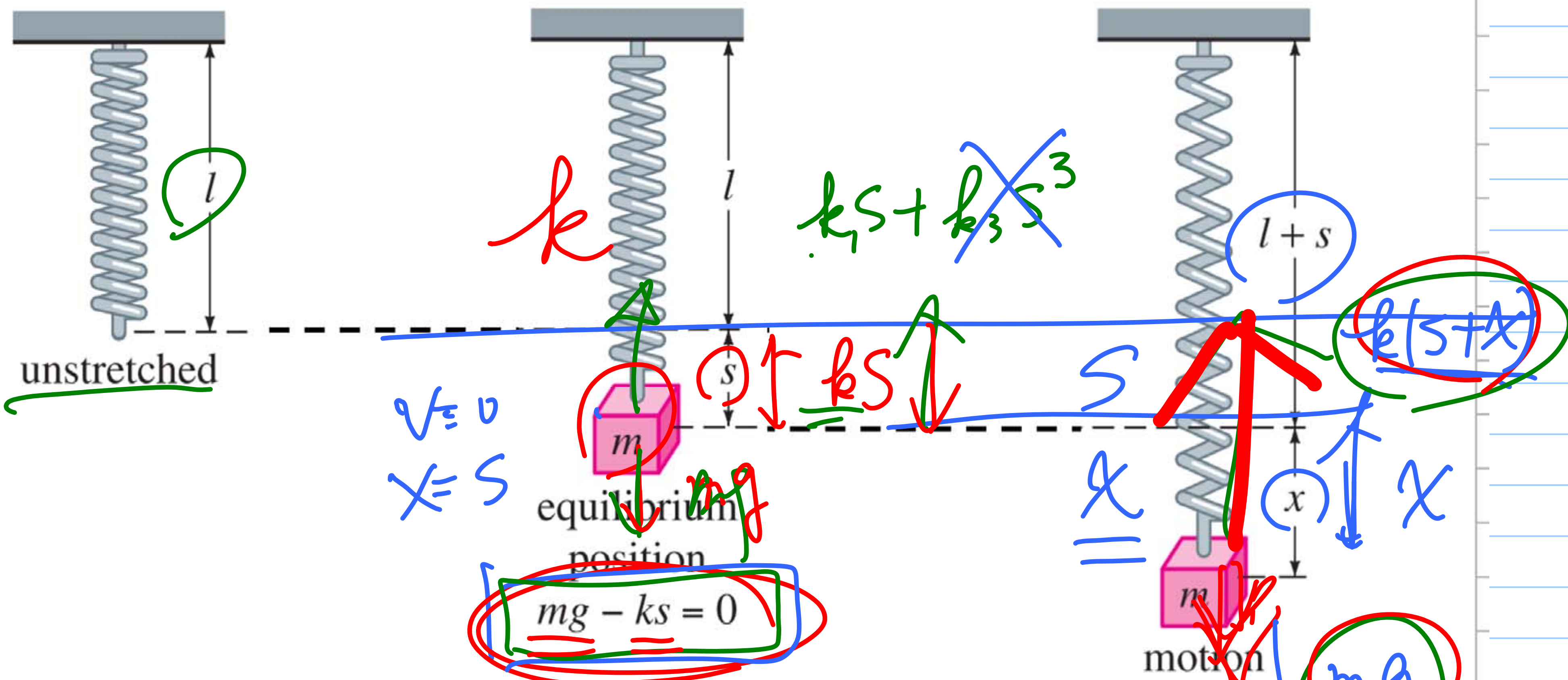
$$m\frac{dx^2(t)}{dt^2} + k\,x(t) = 0$$

$$m\frac{dx^2(t)}{dt^2} + \beta\frac{dx(t)}{dt} + k\,x(t) = 0$$

$$m\frac{dx^2(t)}{dt^2} + \beta\frac{dx(t)}{dt} + k\,x(t) = f(t)$$

$$\Rightarrow\ \boxed{L\frac{dq^2(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}q(t) = E(t)}$$

- free undamped motion:
  - → free: no external force
  - → undamped: no friction



- Hooke's law:
- Newton's 2nd law:

$ma = \sum F$

5.1.1:

$$\Rightarrow m\frac{d^2x}{dt^2} = \Sigma F = mg - k(s+x) = -kx + \boxed{mg - ks} = 0$$

$$= -kx$$

$$\Rightarrow m\frac{d^2x}{dt^2} + kx = 0$$

$$\omega = \sqrt{\frac{k}{m}} \qquad x \stackrel{\text{let}}{=} t^m$$

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{k}{m} x = 0 \iff x'' + \omega^2 x = 0, \qquad \frac{k}{m} = \omega^2$$

$$\boxed{x \stackrel{\text{let}}{=} e^{mt}}$$

aux. eq. $m^2 + \omega^2 = 0$

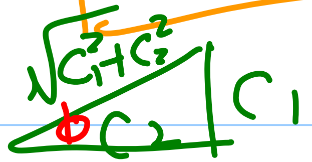
$$m_{1,2} = \pm i\omega$$

$$\Rightarrow x(t) =$$

x_c $= c_1 \cos \omega t + c_2 \sin \omega t$

$$= \boxed{A \sin(\omega t + \phi)}$$

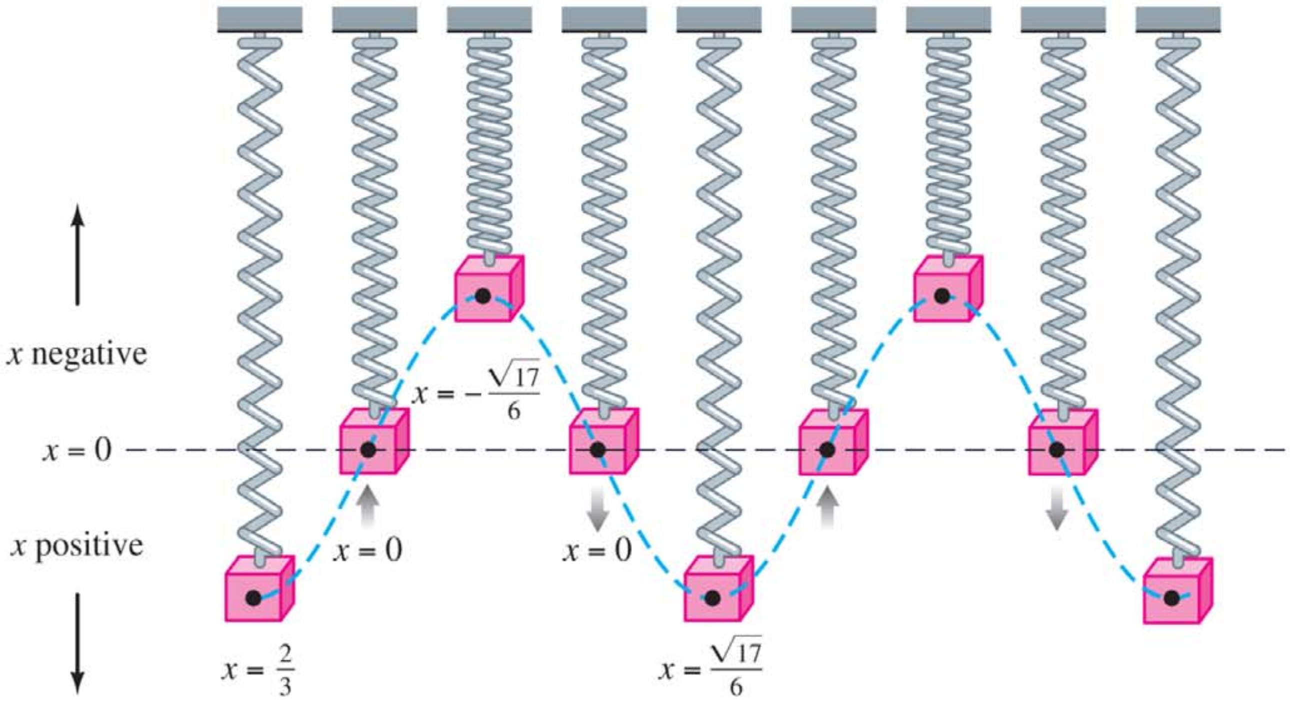
$$\omega = \sqrt{\frac{k}{m}}$$

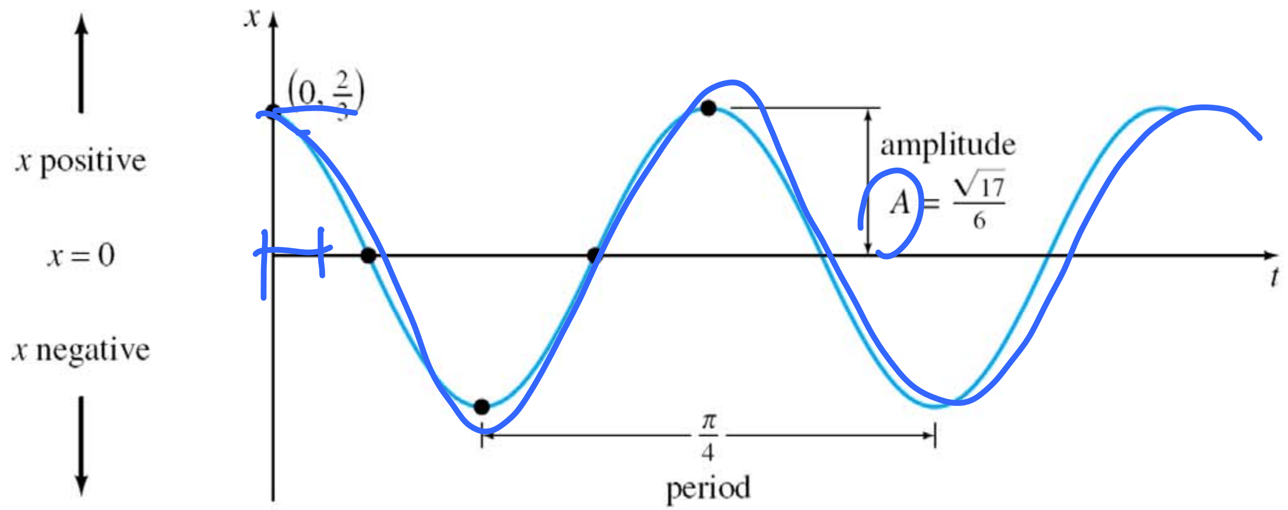


$$A = \sqrt{c_1^2 + c_2^2}$$

$$\phi = \tan^{-1} \frac{c_1}{c_2}$$

5.1.1:


x negative
x = 0
x positive
$x = -\frac{\sqrt{17}}{6}$
x = 0
x = 0
$x = \frac{2}{3}$
$x = \frac{\sqrt{17}}{6}$

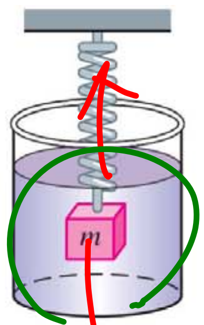

x
x positive
x = 0
x negative
$(0, \frac{2}{3})$
amplitude
$A = \frac{\sqrt{17}}{6}$
t
$\frac{\pi}{4}$
period

(b)

5.1.2: Spring/Mass Systems: Free Damped Motion

- free damped motion: → damped: friction

$\frac{dx}{dt}$



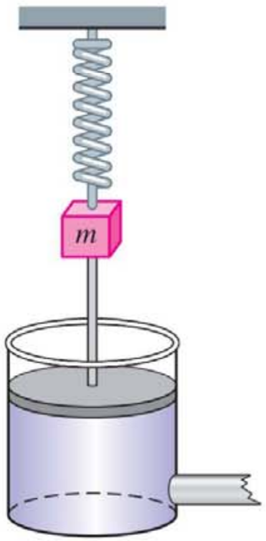
$$\Rightarrow m\frac{d^2x}{dt^2} = \cancel{mg} - k(\cancel{s}+x) - \beta\dot{x}$$

$$= -kx - \beta\dot{x}$$

$$\Rightarrow \frac{d^2x}{dt^2} + \boxed{\frac{\beta}{m}\dot{x}} + \frac{k}{m}x = 0$$

$2\lambda > 0$ $\qquad \omega^2 > 0$

$x \triangleq e^{mt}$



$$\Rightarrow m^2 + \boxed{\frac{\beta}{m}}\, m + \boxed{\frac{k}{m}} = 0$$

$$\Rightarrow m_{1,2} = \frac{-2\lambda \pm \sqrt{4\lambda^2 - 4\omega^2}}{2}$$

$$= -\lambda \pm \sqrt{\boxed{\lambda^2-\omega^2}}$$

(1) $\lambda^2 - w^2 > 0$ (over damped)

$\Rightarrow x(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t}$

$= e^{-\lambda t}\left[c_1 e^{\sqrt{\lambda^2 - w^2}\,t} + c_2 e^{-\sqrt{\lambda^2 - w^2}\,t}\right]$

(2) $\lambda^2 - w^2 = 0$ (critically damped)

$\Rightarrow x(t) = c_1 e^{m_1 t} + c_2 t e^{m_1 t}$

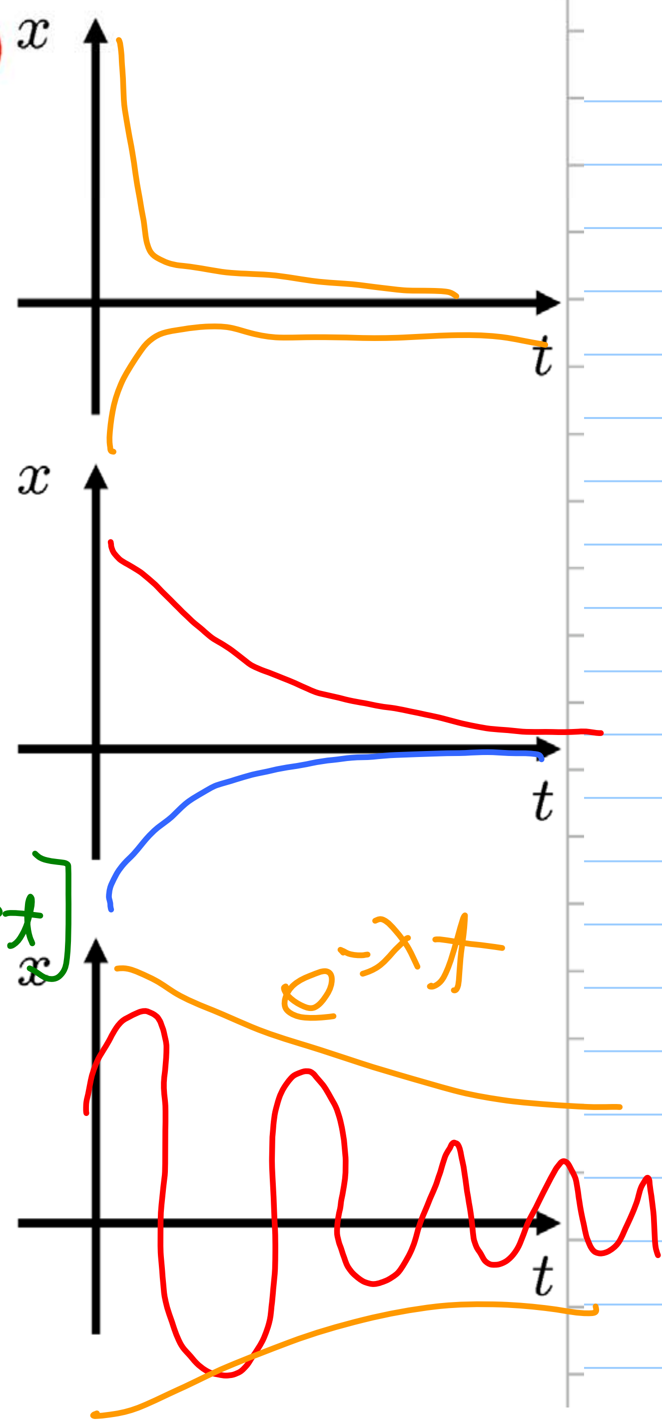
∞

$m_1 < 0$

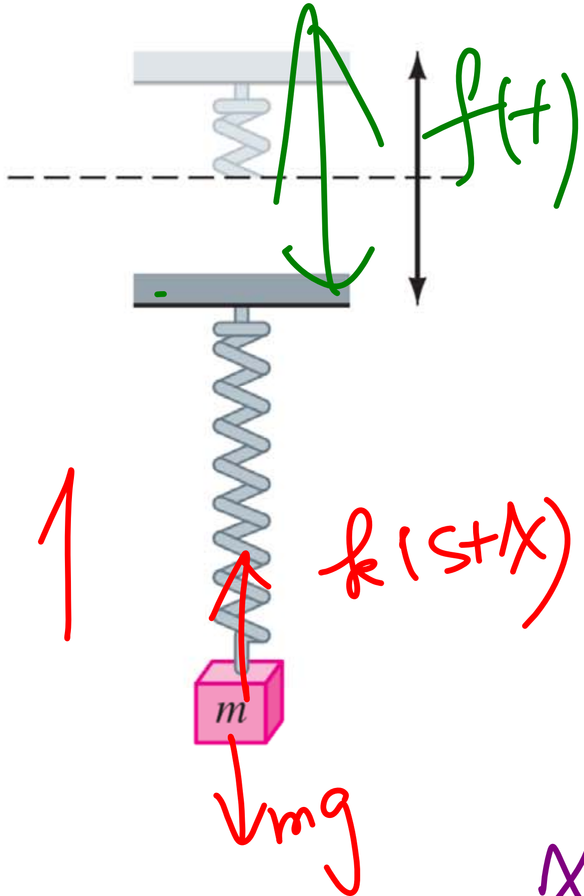
(3) $\lambda^2 - w^2 < 0$ (under damped)

$\Rightarrow x(t) = e^{-\lambda t}\left[c_1 \cos\sqrt{w^2 - \lambda^2}\,t + c_2 \sin\sqrt{w^2 - \lambda^2}\,t\right]$

or $x(t) = A e^{-\lambda t} \sin\left(\sqrt{w^2 - \lambda^2}\,t + \phi\right)$



- driven motion: $\rightarrow$ with external force



$$\Rightarrow m\frac{d^2x}{dt^2} = \cancel{mg} - k(x+\cancel{s}) - \beta\dot{x} + f(t)$$

$$= -kx - \beta\dot{x} + f(t)$$

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} + \overset{2\lambda}{\frac{\beta}{m}}\dot{x} + \overset{\omega^2}{\frac{k}{m}}x} = \frac{f(t)}{m} = F(t)$$

$x = e^{mt}$ $\qquad m^2 + 2\lambda m + \omega^2 = 0$

$$m = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$$

- consider underdamped case:
- IF $F(t) = F_0 \sin rt$

$r \neq \sqrt{\omega^2 - \lambda^2}$

$\Rightarrow x_c(t) = e^{-\lambda t}\left[c_1 \cos \sqrt{\omega^2-\lambda^2}\, t + c_2 \sin \sqrt{\omega^2-\lambda^2}\, t\right]$

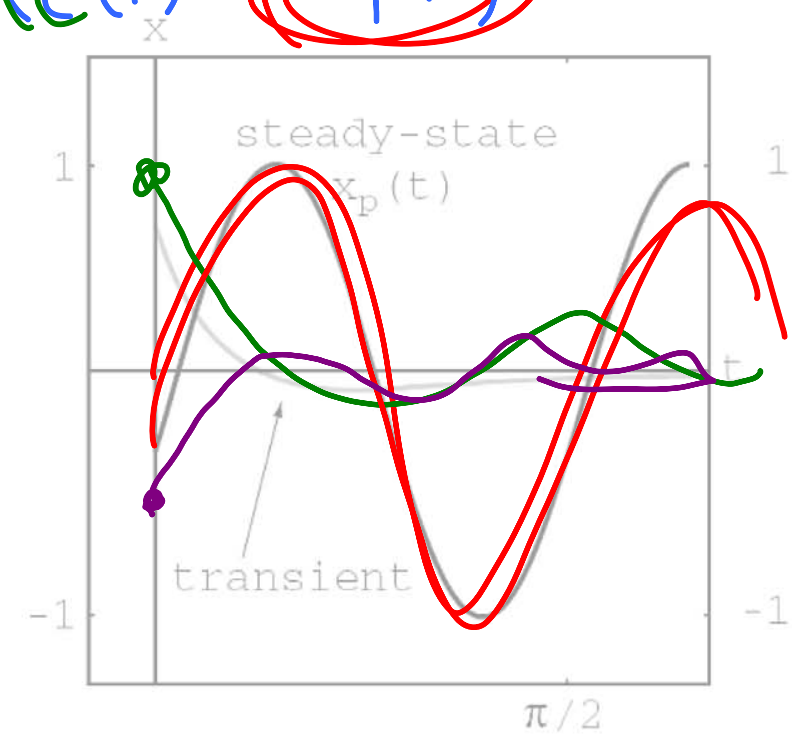
transient respn

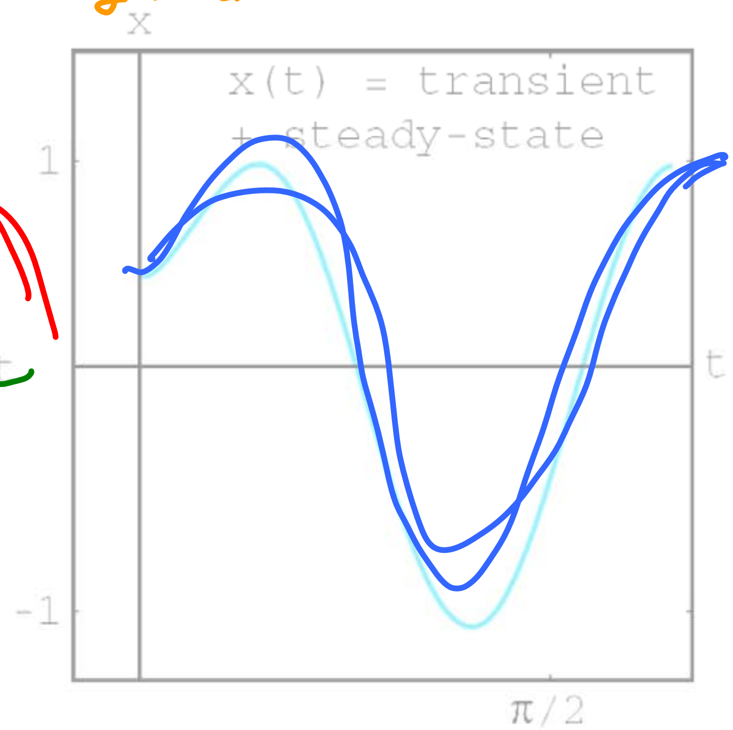
$\Rightarrow x_p(t) = A \cos rt + B \sin rt$

steady-state respn

I.C.

$\Rightarrow x(t) = x_c(t) + x_p(t)$





- Resonance

$\Rightarrow \lambda = 0$ $\qquad \omega = \sqrt{\frac{k}{m}}$

$\Rightarrow \dfrac{d^2x}{dt^2} + \omega^2 x = F_0 \sin \omega t$

$\Rightarrow x_c(t) = C_1 \cos \omega t + C_2 \sin \omega t$

$\Rightarrow x_p(t) = A t \cos \omega t + B t \sin \omega t$

$\Rightarrow x_p'(t) = A \cos \omega t + \cdots - A t \omega \cos \omega t$

$\Rightarrow x_p''(t) = A$

$$\Rightarrow x_p'' + w^2 x_p = -2wA\sin wt + 2Bw\cos wt = F_0 \sin wt$$

$$A = -\frac{F_0}{2w} \qquad B = 0$$

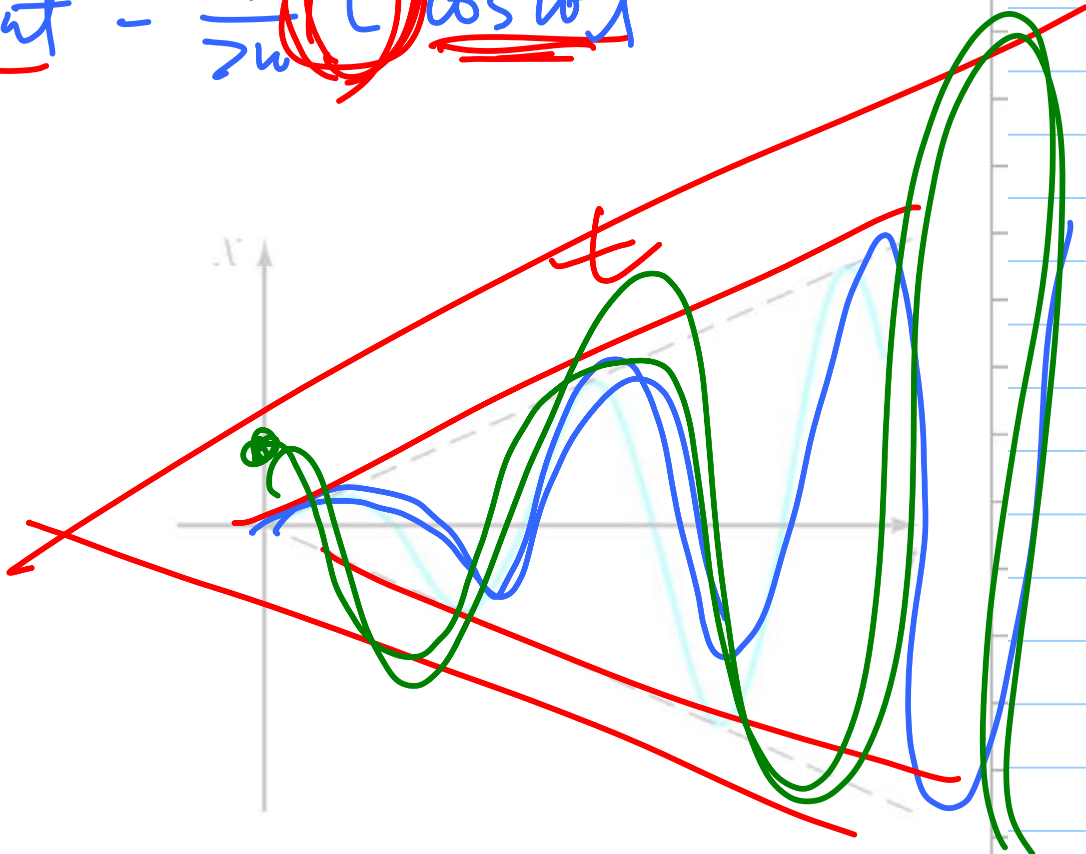
$$\Rightarrow$$

$$\Rightarrow x_p(t) = -\frac{F_0}{2w}\, t \cos wt$$

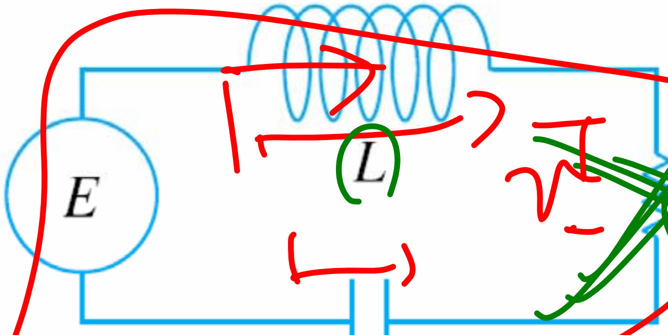
$$\Rightarrow x(t) = c_1 \cos wt + c_2 \sin wt - \frac{F_0}{2w}\, t \cos wt$$

$x(0)$ $\quad x'(0)$

$$\Rightarrow x(t) =$$



5.1.4: Series Circuit Analogue



$\frac{d}{dt}$

$$\Rightarrow \quad L\frac{di(t)}{dt} + R\,i(t) + \frac{1}{C}\int i(t)dt = E(t)$$

$$\Rightarrow \quad i(t) = \frac{dq(t)}{dt} \qquad \Rightarrow \quad L\frac{dq^2(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}q(t) = E(t)$$

OR $$L\frac{di^2(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{C}i(t) = \frac{dE(t)}{dt}$$

$$\boxed{m\frac{dx^2(t)}{dt^2} + \beta\frac{dx(t)}{dt} + k\,x(t) = f(t)}$$

- IF $E(t) = 0$ $\quad q(t) = e^{mt}$

  $\Rightarrow$ free electrical vibrations of the circuit

- auxiliary eqn: $Lm^2 + Rm + \frac{1}{C} = 0$

  $m = \dfrac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L}$

- IF $R \neq 0$, the circuit is

  over damped if $R^2 - \frac{4L}{C} > 0$

  critically damped if $R^2 - \frac{4L}{C} = 0$

  under damped if $R^2 - \frac{4L}{C} < 0$

- IF $E(t) = E_0 \sin rt$

  $\Rightarrow$ forced electrical vibrations of the circuit

  $\Rightarrow \boxed{L \dfrac{dq^2(t)}{dt^2} + R \dfrac{dq(t)}{dt} + \dfrac{1}{C} q(t)} \neq \boxed{E_0 \sin rt}$

  $\Rightarrow q_c(t) = c_1 q_1(t) + c_2 q_2(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t}$

  $\Rightarrow q_p(t) = A \cos rt + B \sin rt$

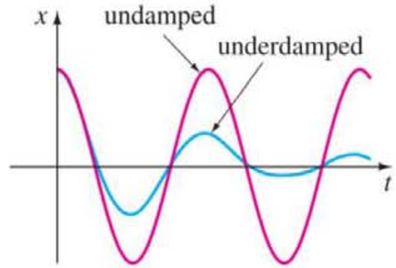
  $A\cancel{t} \cos rt + B \cancel{t} \sin rt$

  $\Rightarrow i_p(t) = q_p'(t)$

Summary - 5.1: Linear Models: Initial-Value Problems

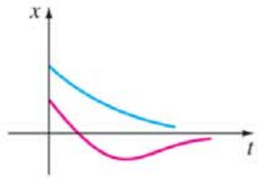
$$\frac{dx^2(t)}{dt^2} + 2\lambda \frac{dx(t)}{dt} + w^2\, x(t) = f(t)$$

- free undamped motion:



$$\Rightarrow\ x(t) = c_1 \cos wt + c_2 \sin wt$$

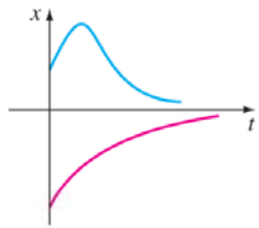
- free damped motion:



(1) $\lambda^2 - w^2 > 0$ (overdamped)

$$\Rightarrow\ x(t) = e^{-\lambda t}\left(c_1 e^{\sqrt{\lambda^2 - w^2}\,t} + c_2 e^{-\sqrt{\lambda^2 - w^2}\,t}\right)$$

(2) $\lambda^2 - w^2 = 0$ (critical damped)



$$\Rightarrow\ x(t) = e^{-\lambda t}\,(c_1 + c_2 t)$$

(3) $\lambda^2 - w^2 < 0$ (underdamped)

$$\Rightarrow\ x(t) = e^{-\lambda t}\left(c_1 \cos\sqrt{w^2 - \lambda^2}\,t + c_2 \sin\sqrt{w^2 - \lambda^2}\,t\right)$$

$$\frac{dx^2(t)}{dt^2} + 2\lambda \frac{dx(t)}{dt} + w^2\, x(t) = f(t)$$

- driven motion:

$$\Rightarrow x(t) = x_c(t) + x_p(t)$$

- Resonance

$$x_c(t) = c_1 \cos wt + c_2 \sin wt$$

$$f(t) = F_1 \cos wt + F_2 \sin wt$$

$$\Rightarrow x_p(t) = A\, t \cos wt + B\, t \sin wt$$