

Fall 2019

# 微分方程 Differential Equations

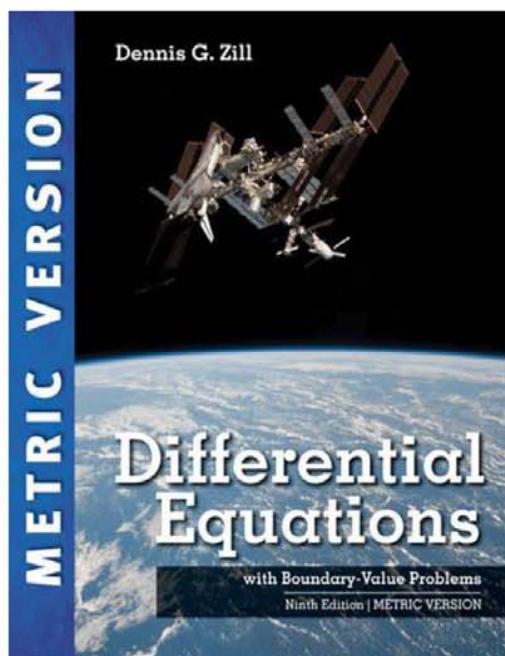
## Unit 04.6 Variation of Parameters

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$$y_c(x) = c_1 \underline{y_1} + c_2 \underline{y_2}$$
$$y_p(x) = \underline{u_1} \underline{y_1} + \underline{u_2} \underline{y_2}$$



- 4.1: Linear Differential Equations: Basic Theory
  - 4.1.1: Initial-Value and Boundary-Value Problems
  - 4.1.2: Homogeneous Equations
  - 4.1.3: Nonhomogeneous Equations
- 4.2: Reduction of Order
- 4.3: Homogeneous Linear Eqns w/ Constant Coefficients
- 4.4: Undetermined Coefficients – Superposition Approach ✓
- 4.5: Undetermined Coefficients – Annihilator Approach ✓
- **4.6: Variation of Parameters**
- 4.7: Cauchy-Euler Equations
- 4.8: Green's Functions
- 4.9: Solving Systems of Linear Equations by Elimination
- 4.10: Nonlinear Differential Equations

- Underdetermined coefficients: (4.4, 4.5)

need  $\left\{ \begin{array}{l} (1) \text{ constant coefficients} \\ (2) g(x): \text{polynomial, exponent, } \sin, \cos \end{array} \right.$

- If  $g(x)$  does NOT belongs to these types of functions,  


we CANNOT use undetermined coefficients



⇒ Use Variation of Parameters

$$y_p(x) = \underline{u_1} y_1 + \underline{u_2} y_2$$

Joseph Louis Lagrange (1736-1813)

- Consider a 2nd-order DE:

$$\underline{a_2(x) \underline{y''(x)}} + \underline{a_1(x) \underline{y'(x)}} + \underline{a_0(x) \underline{y(x)}} = \underline{g(x)}$$

$\Rightarrow$  standard form

$$a_2(x) \neq 0 \quad \forall x \in I$$

$$y''(x) + \frac{a_1}{a_2} y'(x) + \frac{a_0}{a_2} y(x) = \frac{g(x)}{a_2}$$

$$y'' + P(x) y' + Q(x) y = f(x)$$

AIIE =  $\{y_1, y_2\}$  : linearly independent solutions

$$\underline{y_c} = c_1 y_1 + c_2 y_2 \quad c_1, c_2 \text{ constants}$$

$$+ Q(x) y_p = (u_1^{(x)} y_1 + u_2^{(x)} y_2) Q$$

$$+ P(x) \frac{y_p'}{y_p} = u_1' y_1 + u_1'' y_1' + u_2' y_2 + u_2'' y_2' \\ (y_p') = u_1' y_1 + u_1'' y_1' + u_2' y_2 + u_2'' y_2' \\ + u_1'' y_1 + u_2' y_2 + u_2'' y_2' + u_1' y_1' + u_2' y_2' + u_2'' y_2' \\ + u_1' y_1 + u_2' y_2 + u_2'' y_2' + u_1'' y_1' + u_2' y_2' + u_2'' y_2'$$

$$\boxed{u_1 [Qy_1 + Py_1' + y_1''] + u_2 [Qy_2 + Py_2' + y_2''] \\ \rightarrow u_1' [Py_1 + 2Py_1'] \\ \rightarrow u_1'' [y_1] \\ P [u_1' y_1 + u_2' y_2] + [2u_1' y_1' + 2u_2' y_2' + u_1'' y_1 + u_2'' y_2]} = f(x)$$

$\frac{u_1' y_1}{u_1' y_1} \quad \frac{u_2' y_2'}{u_2' y_2}$

$$\Rightarrow \boxed{P \left[ u_1' y_1 + u_2' y_2 \right] + \left[ u_1' y_1' + u_2' y_2' + u_1'' y_1 + u_2'' y_2 \right]} \\ h(x) + \underline{u_1' y_1 + u_2' y_2} = f(x)$$

$$\frac{d}{dx} \left[ u_1' y_1 + u_2' y_2 \right]$$

$$P \left[ h(x) \right] + \frac{d}{dx} \left( h(x) \right) + u_1' y_1' + u_2' y_2' = f(x)$$

$$\underline{h(x) = 0} \Rightarrow$$

$$\left[ \begin{array}{l} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = f(x) \end{array} \right]$$

Cramer's rule

$$u_1' = - \frac{|f \quad y_2'|}{|y_1 \quad y_2'|}$$

$$u_2' = \frac{|y_1 \quad 0|}{|y_1 \quad y_2'|}$$

$$u_1 = \int u_1' dx \quad u_2 = \int u_2' dx \Rightarrow y_p = u_1 y_1 + u_2 y_2$$

## 4.6: Example 2

$$4y'' + 36y = \csc 3x$$

⇒ standard form  $y'' + 9y = \frac{1}{4} \csc 3x$

$$AHE = \underline{\underline{y} = e^{mx}} \Rightarrow m^2 + 9 = 0 \Rightarrow m = \pm 3i$$

$$\Rightarrow y_c = \underline{\underline{C_1 \cos 3x + C_2 \sin 3x}}$$

$$y_p = \underline{\underline{u_1 y_1 + u_2 y_2}}, \quad y_p' = \underline{\underline{y_1' u_1 + y_2' u_2}}, \quad y_p'' = \underline{\underline{y_1'' u_1 + y_2'' u_2}}$$

$$\Rightarrow \begin{cases} y_1 u_1' + y_2 u_2' = 0 \\ y_1' u_1 + y_2' u_2 = f(x) = \frac{1}{4} \csc 3x \end{cases}$$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{\begin{vmatrix} 0 & \sin 3x \\ \frac{1}{4} \csc 3x & 3 \cos 3x \end{vmatrix}}{\begin{vmatrix} \cos 3x & \sin 3x \\ -3 \sin 3x & 3 \cos 3x \end{vmatrix}} = \frac{-\frac{1}{4} \csc 3x}{3} = -\frac{1}{12}$$

$$\begin{pmatrix} y_1 & 0 \\ y'_1 & f \end{pmatrix}$$

$$\frac{1}{3} = -\frac{1}{3} \cdot \begin{vmatrix} \cos 3x & 0 \\ -3 \sin 3x & \frac{1}{4} \csc 3x \end{vmatrix} = \frac{1}{3} \frac{\cos 3x}{\sin 3x}$$

$$\boxed{\frac{1}{12} \frac{\cos 3x}{\sin 3x}}$$

$$\underline{u_1} = -\frac{1}{12} x$$

$$\underline{u_2} = \frac{1}{12} \frac{1}{3} \ln |\sin 3x| = \frac{1}{36} \ln |\sin 3x|$$

$$y_p = u_1 \underline{y_1} + u_2 \underline{y_2} = \left( -\frac{1}{12} x \right) \cos 3x + \frac{1}{36} \ln |\sin 3x| (\sin 3x)$$

$$y = \underline{y_c} + \underline{y_p} = C_1 \cos 3x + C_2 \sin 3x$$

$$+ \left( -\frac{1}{12} x \right) \cos 3x + \frac{1}{36} \ln |\sin 3x| (\sin 3x)$$

$$y'' + P(x) y' + Q(x) y = f(x)$$

- Given  $\underline{y_1}, \underline{y_2}$ , satisfy AHE
- And  $\{y_1, y_2\}$ : linearly independent  $\Rightarrow \underline{y_c} = c_1 \underline{y_1} + c_2 \underline{y_2}$
- How to find  $y_p$ ?

$$\begin{aligned} Q(y_p) &= u_1 \underline{y_1} + u_2 \underline{y_2} \\ P(\underline{y_p}) &= u_1 \underline{y'_1} + u_2 \underline{y'_2} \\ y_p^H &= u_1 \underline{y''_1} + u_2 \underline{y''_2} \end{aligned}$$

$$\begin{aligned} &+ \boxed{u_1' y_1 + u_2' y_2} \\ &+ u_1' y'_1 + u_2' y'_2 \end{aligned}$$

0

$$u_1 [y_1'' + P y_1' + Q y_1] \\ + u_2 [y_2'' + P y_2' + Q y_2]$$

$$+ [u_1' y_1' + u_2' y_2'] = f(x)$$

$$\boxed{y''' + P(x)y'' + Q(x)y' + R(x)y} = f(x)$$

- Given  $y_1, y_2, y_3$ : AHE Linearly Independent Solutions

$$y_c = C_1 y_1 + C_2 y_2 + C_3 y_3$$

- Assume  $y_p = \underline{\underline{u_1 y_1 + u_2 y_2 + u_3 y_3}}$

$+ R(y_p) = \underline{\underline{u_1 y_1' + u_2 y_2' + u_3 y_3'}} = 0$

$+ Q(y_p) = \underline{\underline{u_1 y_1' + u_2 y_2' + u_3 y_3'} + \underline{\underline{u_1' y_1 + u_2' y_2 + u_3' y_3}}} = 0$

$+ P(y_p) = \underline{\underline{u_1 y_1'' + u_2 y_2'' + u_3 y_3''}} + \underline{\underline{u_1' y_1' + u_2' y_2' + u_3' y_3'}} = 0$

$(y_p) = \underline{\underline{u_1 y_1''' + u_2 y_2''' + u_3 y_3'''}} + \underline{\underline{u_1' y_1'' + u_2' y_2'' + u_3' y_3''}} = 0$

$\underline{\underline{u_1 [y_1''' + P y_1'' + Q y_1' + R y_1]}} + \dots = f(x)$

$$\begin{aligned}
 & u_1' y_1 + u_2' y_2 + u_3' y_3 = 0 \\
 & u_1' y_1 + u_2' y_2 + u_3' y_3 = 0 \\
 & u_1' y_1 + u_2' y_2 + u_3' y_3 = f(x) \\
 u_1' &= \frac{f}{\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}} \quad u_2' = \dots \quad u_3' = \dots
 \end{aligned}$$

$$u_1 = \int u_1' dx$$

$$y_p = u_1 y_1 + u_2 y_2 + u_3 y_3$$

$$\Rightarrow y = y_c + y_p$$

- Consider a 2nd-order DE:

$$\underline{y''(x) + P(x)y'(x)} + \underline{Q(x)y(x)} = f(x)$$

- $\left\{ \underline{y_1(x), y_2(x)} \right\}$ : are linearly independent solutions of AHE

$$\Rightarrow y_p(x) = u_1(x) \underline{y_1(x)} + u_2(x) \underline{y_2(x)}$$

$$\rightarrow \begin{cases} w_1(x) = u'_1(x) \\ w_2(x) = u'_2(x) \end{cases}$$

$$\Rightarrow \begin{cases} w_1 y_1 + w_2 y_2 = 0 \\ w_1 y'_1 + w_2 y'_2 = f \end{cases} \Rightarrow w_1 = \frac{\begin{vmatrix} 0 & y_2 \\ f & y'_2 \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}}, w_2 = \frac{\begin{vmatrix} y_1 & 0 \\ y'_1 & f \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}}$$

$$\Rightarrow y_p(x) = \left( \int w_1 dx \right) y_1(x) + \left( \int w_2 dx \right) y_2(x)$$