

Fall 2019

# 微分方程 Differential Equations

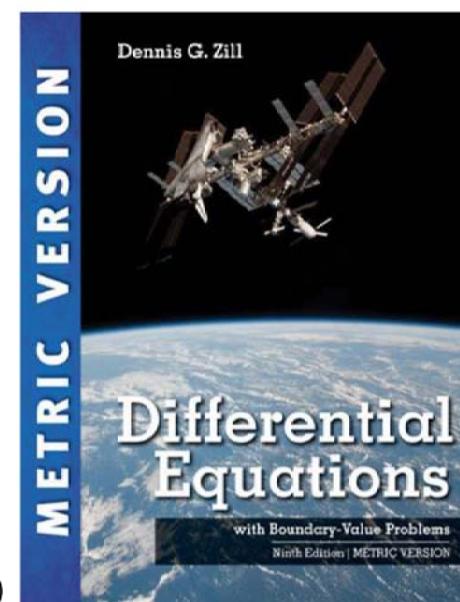
Unit 04.4

Undetermined Coefficients – Superposition

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Figures and images used in these lecture notes are adopted from  
[Differential Equations with Boundary-Value Problems](#), 9th Ed., D.G. Zill, 2018 (Metric Version)

- 4.1: Linear Differential Equations: Basic Theory
  - 4.1.1: Initial-Value and Boundary-Value Problems
  - 4.1.2: Homogeneous Equations
  - 4.1.3: Nonhomogeneous Equations
- 4.2: Reduction of Order
- 4.3: Homogeneous Linear Eqns w/ Constant Coefficients
- 4.4: Undetermined Coefficients– Superposition Approach
- 4.5: Undetermined Coefficients – Annihilator Approach
- 4.6: Variation of Parameters
- 4.7: Cauchy-Euler Equations
- 4.8: Solving Systems of Linear Equations by Elimination
- 4.9: Nonlinear Differential Equations

## 4.4: Undetermined Coefficients – Superposition Approach

$$y'' + 4y' - 2y = 2x^2 - 6 + e^{-3x} - 4\cos(5x)$$

$$\Rightarrow y = y_c + y_p$$

$$\Rightarrow \text{AHE} \rightarrow y_c = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

$\Rightarrow \text{RHS} \rightarrow y_p =$

$$\Rightarrow y_p = A + BX + CX^2$$

$$\Rightarrow y_p = A + BX + CX^2 + DX^3 + EX^4$$

$$\Rightarrow y_p = Ae^{-3x} y_p^1 y_p^2 y_p^3$$

$$\Rightarrow y_p = A e^{(-3+5j)x} e^{(-3-5j)x} \\ A \cos 5x + B \sin 5x$$

$$y'' + 4y' - 2y = 6$$

$$y'' + 4y' - 2y = 2x^2$$

$$y'' + 4y' - 2y = e^{-3x}$$

$$y'' + 4y' - 2y = \cos(5x)$$

$$y'' + 4y' - 2y = \sin(5x)$$

$$y'' + 4y' - 2y = \ln x, \frac{1}{x}, \tan(x), \sin^{-1}(x), \dots$$

$$\boxed{y'' + 4y' - 2y} = \boxed{2x^2 - 3x + 6}$$

$$\Rightarrow y = y_c + y_p \quad \Rightarrow m_{1,2} = -2 \pm \sqrt{6}$$

$$\Rightarrow \text{AHE} \rightarrow y_c = C_1 e^{(-2+\sqrt{6})x} + C_2 e^{(-2-\sqrt{6})x}$$

$$\Rightarrow \text{RHS} \rightarrow y_p = Ax^2 + Bx + C$$

$$y'_p = 2Ax + B$$

$$y''_p = 2A$$

$$\Rightarrow y_p = x^2 - \frac{5}{2}x - 9$$

$$2x^2 - 3x + 6$$

$$(2A) + 4(2A) - 2(Ax^2 + Bx + C) =$$

$$\left\{ \begin{array}{l} 2 = -2A \\ -3 = 8A - 2B \\ 6 = 2A + 4B \end{array} \right. \Rightarrow \begin{array}{l} A = -1 \\ B = -\frac{5}{2} \\ C = -9 \end{array}$$

$$y = y_c + y_p$$

$$= C_1 e^{(-\sigma + \sqrt{\beta})x} + C_2 e^{(-\sigma - \sqrt{\beta})x}$$

$$= C_1 e^{\sigma x} + C_2 e^{-\sigma x} + (-\frac{x^2}{2} \leq 0 - q)$$

I.C.  
C<sub>1</sub>, C<sub>2</sub>

\*

AHE

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = g(x)$$

$$\Rightarrow y =$$

$$y_c + y_p$$



## 4.4: Example 2

$$\begin{aligned}
 & y''(-)y' + y = 2\sin(3x) + 0\cos(3x) \\
 \Rightarrow y &= y_c + y_p \\
 \Rightarrow \text{AHE} &\rightarrow y_c \\
 \Rightarrow \text{RHS} \rightarrow y_p &= (A \cos(3x) + B \sin(3x)) \\
 y'_p &= (-3A \sin(3x) + 3B \cos(3x)) \\
 y''_p &= (-9A \cos(3x) - 9B \sin(3x)) \\
 y''_p - y'_p + y_p &= \cos(3x)(-8A - 3B) = 0 \\
 &+ \sin(3x)(-8B + 3A) = 2 \\
 \left. \begin{array}{l} -8A - 3B = 0 \\ -8B + 3A = 2 \end{array} \right\} \Rightarrow A &= \frac{6}{13} \\
 &B = -\frac{16}{13} \quad y_p = \frac{6}{13} \cos(3x) \\
 &\quad - \frac{16}{13} \sin(3x)
 \end{aligned}$$

## 4.4: Example 3

$$y'' - 2y' - 3y = 6e^{2x}$$

$$\Rightarrow y = y_c + y_p$$

$$\Rightarrow \text{AHE} \rightarrow y_c \approx$$

$$\Rightarrow \text{RHS} \rightarrow y_p =$$

$$y'_p =$$

$$y''_p =$$

$$y''_p - 2y'_p + 3y_p = e^{2x} \left( A e^{2x} \right)$$

$$= e^{2x} \left( A e^{2x} \times (-3) \right) + e^{2x} \left( 2A e^{2x} \times (-2) \right) + e^{2x} \left( 2^2 A e^{2x} \times (1) \right)$$

$$= e^{2x} (4A - 4A - 3A) = -3A e^{2x}$$

$$\Rightarrow -3A = 6 \Rightarrow A = -2$$

$$y_p = -2 e^{2x}$$

$$y'' = \underline{2y'} - \underline{3y} = 6xe^{2x}$$

$$\Rightarrow y = y_c + y_p$$

$$\Rightarrow \text{AHE} \rightarrow y_c$$

$$\Rightarrow \text{RHS} \rightarrow y_p =$$

$$y'_p = \left( A \cancel{x e^{2x}} + B e^{2x} \right) \times (-3)$$

$$y''_p = \left( A \cancel{e^{2x}} + 2A \cancel{x e^{2x}} + 2B e^{2x} \right) (-2)$$

$$= \left( 2A \cancel{e^{2x}} + 2A \cancel{e^{2x}} + 4A \cancel{x e^{2x}} + 4B e^{2x} \right) \times 1$$

$$y''_p - 2y'_p + 3y_p = e^{2x} ( \cancel{4A} + \cancel{4B} - \cancel{2A} - \cancel{4B} - 3B ) = 0$$

$$+ xe^{2x} \left( \frac{-3A}{-} = 6 \right) = 6xe^{2x}$$

$$\left\{ \begin{array}{l} A = -2 \\ B = -\frac{4}{3} \end{array} \right.$$

$$y_p = -2 \cancel{xe^{2x}} - \frac{4}{3} \cancel{e^{2x}}$$

$$\begin{aligned}
 & y'' - 5y' + 4y = e^x \\
 \Rightarrow y &= y_c + y_p \\
 \Rightarrow \text{AHE} \rightarrow y_c &= C_1 e^x + C_2 e^{4x} \\
 \Rightarrow \text{RHS} \rightarrow y_p &= (A e^x) \times 4 \Rightarrow A e^x \\
 y'_p &= (A e^x) \times (-5) \Rightarrow A e^x + A x e^x \\
 y''_p &= (A e^x) \times 1, \Rightarrow A e^x + A e^x + A x e^x \\
 y''_p - y'_p + y_p &= e^x (A - 5A + 4A) = e^x (D) = 0 e^x \\
 &= 0 e^x
 \end{aligned}$$

(X. e<sup>2x</sup>)

$$y'' - 5y' + 4y = 8e^x$$

$$\Rightarrow y = y_c + y_p$$

$$\Rightarrow \text{AHE} \rightarrow y_c = c_1 e^x + c_2 e^{4x}$$

$$\Rightarrow \text{RHS} \rightarrow y_p = (Ax e^x) \times (4)$$

$$y'_p = (A e^x + Ax e^x) \times (-5)$$

$$y''_p = (A e^x + A e^x + Ax e^{2x}) \times (1)$$

$$y''_p - 5y'_p + 4y_p = x e^x (4A - 5A + A) = 0$$

$$e^x (2A - 5A + 8) = 8e^x$$

$$2A - 5A = 8 \quad A = -\frac{8}{3}$$

$$y = y_c + y_p = c_1 e^x + c_2 e^{4x} - \frac{8}{3} x e^x$$

- Case I: Candidate particular solution

is NOT a complementary function

**TABLE 4.1** Trial Particular Solutions

$g(x)$	Form of $y_p$
1. <u>1 (any constant)</u>	<u><math>A</math></u>
2. <u><math>5x + 7</math></u>	<u><math>Ax + B</math></u>
3. <u><math>3x^2 - 2</math></u>	<u><math>Ax^2 + Bx + C</math></u>
4. <u><math>x^3 - x + 1</math></u>	<u><math>Ax^3 + Bx^2 + Cx + E</math></u>
5. <u><math>\sin 4x</math></u>	<u><math>A \cos 4x + B \sin 4x</math></u>
6. <u><math>\cos 4x</math></u>	<u><math>A \cos 4x + B \sin 4x</math></u>
7. <u><math>e^{5x}</math></u>	<u><math>Ae^{5x}</math></u>
8. <u><math>(9x - 2)e^{5x}</math></u>	<u><math>(Ax + B)e^{5x}</math></u>
9. <u><math>x^2 e^{5x}</math></u>	<u><math>(Ax^2 + Bx + C)e^{5x}</math></u>
10. <u><math>e^{3x} \sin 4x</math></u>	<u><math>Ae^{3x} \cos 4x + Be^{3x} \sin 4x</math></u>
11. <u><math>5x^2 \sin 4x</math></u>	<u><math>(Ax^2 + Bx + C) \cos 4x + (Ex^2 + Fx + G) \sin 4x</math></u>
12. <u><math>x e^{3x} \cos 4x</math></u>	<u><math>(Ax + B)e^{3x} \cos 4x + (Cx + E)e^{3x} \sin 4x</math></u>

- Case II: Candidate particular solution

is ALSO a complementary function

$$y'' - 2y' + y = \underline{e^x} \rightarrow m^2 - 2m + 1$$

$$y_c = c_1 \underline{e^x} + c_2 x \underline{e^x}$$

$$(m-1)^2 = 0$$

$$y_p = \underline{\underline{A}} x^2 e^x$$

$$y = y_c + y_p = c_1 \underline{e^x} + c_2 \underline{x e^x} + \frac{1}{2} \underline{\underline{x^2 e^x}}$$



$$\cancel{y'' + y} = \cancel{4x + 10\sin x}$$

$m^2 + 1$

$$y(\pi) = 0$$

$$y'(\pi) = 2$$

- $y_c = C_1 \cos x + C_2 \sin x$

- $y_p = \cancel{Ax+B} + Cx\cos x + D\cancel{x\sin x}$

$$y_p' =$$

$$y_p'' =$$

$$y = y_c + y_p = C_1 \cos x + C_2 \sin x + 4x - 5x \cos x$$

$$y(\pi) = 0 \quad y'(\pi) = -C_1 + C_2 \cdot 0 + 4\pi - 5\pi(-1) = 0$$

$$C_1 = \underline{\underline{9\pi}}$$

$$y'(\pi) = 2 \quad \dots \quad C_2 = \underline{\underline{+7}}$$

$$y(x) = 9\pi \cos x + 7 \sin x + 4x - 5x \cos x$$



$$0(y'' + 4y' - 2y) = 0(2x^2 + 6 + 10\sin 3x - 3e^{-2x})$$

$$\Rightarrow y = y_c + \underline{y_p}$$

$x$      $x^2$  . -

4.4:  $\Rightarrow y_p = (Ax^2 + Bx + C) + (E\cos 3x + F\sin 3x) + (Ge^{-2x})$

4.5:  $\Rightarrow P(D)(y'' + 4y' - 2y) = 0$

$$= P(D)(2x^2 + 6 + 10\sin 3x - 3e^{-2x}) = 0$$

$$e.g., \Rightarrow y^{(4)} - 9y^{(3)} + 7y'' - 6y' + 12y = 0$$

$$\Rightarrow y = y_c + y_p$$