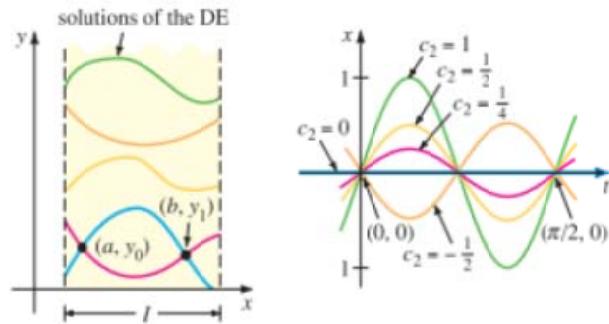


Fall 2019

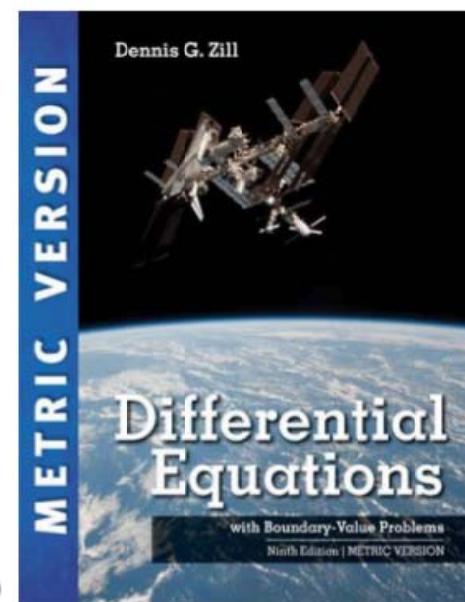
# 微分方程 Differential Equations

## Unit 04.1 Preliminary Theory - Linear Equations



Feng-Li Lian  
NTU-EE  
Sep19 – Jan20

Figures and images used in these lecture notes are adopted from  
[Differential Equations with Boundary-Value Problems](#), 9th Ed., D.G. Zill, 2018 (Metric Version)



- **4.1: Linear Differential Equations: Basic Theory**
  - 4.1.1: Initial-Value and Boundary-Value Problems
  - 4.1.2: Homogeneous Equations
  - 4.1.3: Nonhomogeneous Equations
- 4.2: Reduction of Order
- 4.3: Homogeneous Linear Eqns w/ Constant Coefficients
- 4.4: Undetermined Coefficients – Superposition Approach
- 4.5: Undetermined Coefficients – Annihilator Approach
- 4.6: Variation of Parameters
- 4.7: Cauchy-Euler Equations
- 4.8: Solving Systems of Linear Equations by Elimination
- 4.9: Nonlinear Differential Equations

- $n$ th-Order Initial-Value Problems (IVP):

Solve:

$$a_n(x) \frac{d^n y(x)}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y(x)}{dx^{n-1}} + \cdots + a_1(x) \frac{dy(x)}{dx} + a_0(x) y(x) = g(x)$$

Subject to:

$$\begin{aligned} y(x_0) &= y_0 \\ y'(x_0) &= y_1 \\ y''(x_0) &= y_2 \\ &\vdots \\ y^{(n-1)}(x_0) &= y_{n-1} \end{aligned}$$

initial conditions

$$y(x) = C e^{kx}$$

Solution:

$$y(x) = C_1 e^{k_1 x} + C_2 e^{k_2 x} + C_3 e^{k_3 x} + \cdots + C_n e^{k_n x}$$

## 4.1.1: Theorem 4.1.1: Existence of a Unique Solution

$$a_n(x) \frac{d^n y(x)}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y(x)}{dx^{n-1}} + \cdots + a_1(x) \frac{dy(x)}{dx} + a_0(x) y(x) = g(x)$$

subject to:  $y(x_0) = y_0, y'(x_0) = y_1, y^{(2)}(x_0) = y_2, \dots, y^{(n-1)}(x_0) = y_{n-1}$

- Let  $a_n(x), a_{n-1}(x), \dots, a_1(x), a_0(x)$  &  $g(x)$  be continuous on  $I$
  - And  $a_n(x) \neq 0, \forall x \in I$
  - Then IVP exists a unique solution  $\underline{y(x)}$  on  $I$
-

$$3y''' + 5y'' - y' + 7y = 0$$

$$\Rightarrow y''' = 0$$

$$y(1) = 0$$

$$y'(1) = 0,$$

$$y''(1) = 0$$

$$y'' - 4y = 12x$$

$$y(x) = 3e^{2x} + e^{-2x} - 3x$$

$$y(0) = 4,$$

$$y'(0) = 1$$

$$\frac{y''}{x^2} - \frac{2xy'}{x^2} + \frac{2y}{x^2} = 6$$

$$y(0) = 3,$$

$$y'(0) = 1$$

$$y_1(x) = cx^2 + x + 3$$

~~$$y_2(x) = 2cx + 1$$~~

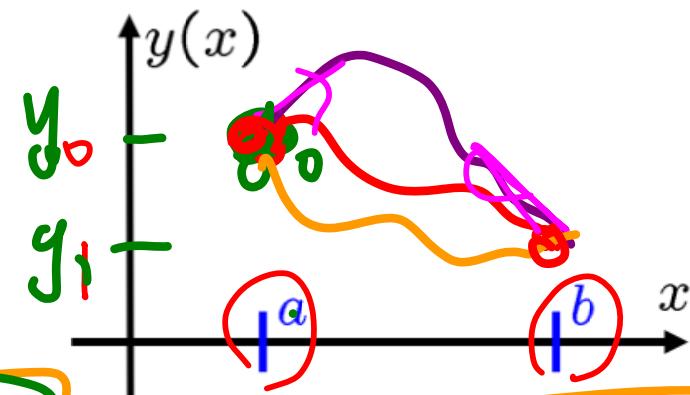
~~$$y''(x) = 2c$$~~

- 2nd-Order Boundary-Value Problems (BVP):

Solve 
$$a_2(x) \frac{d^2y(x)}{dx^2} + a_1(x) \frac{dy(x)}{dx} + a_0(x)y(x) = g(x)$$

Subject to:

$$\begin{cases} y(a) = y_0 \\ y(b) = y_1 \end{cases}$$



Other possible BCs:

$$\begin{cases} y'(a) = y_0 \\ y(b) = y_1 \end{cases}$$

$$\begin{cases} y(a) = y_0 \\ y'(b) = y_1 \end{cases}$$

$$\begin{cases} y(a) = y_0 \\ y'(b) = y_1 \end{cases}$$

General BCs:

$$\begin{cases} y(a) + r_1 \\ y(b) + r_2 \end{cases}$$

$$\begin{cases} y'(a) = r_1 \\ y'(b) = r_2 \end{cases}$$

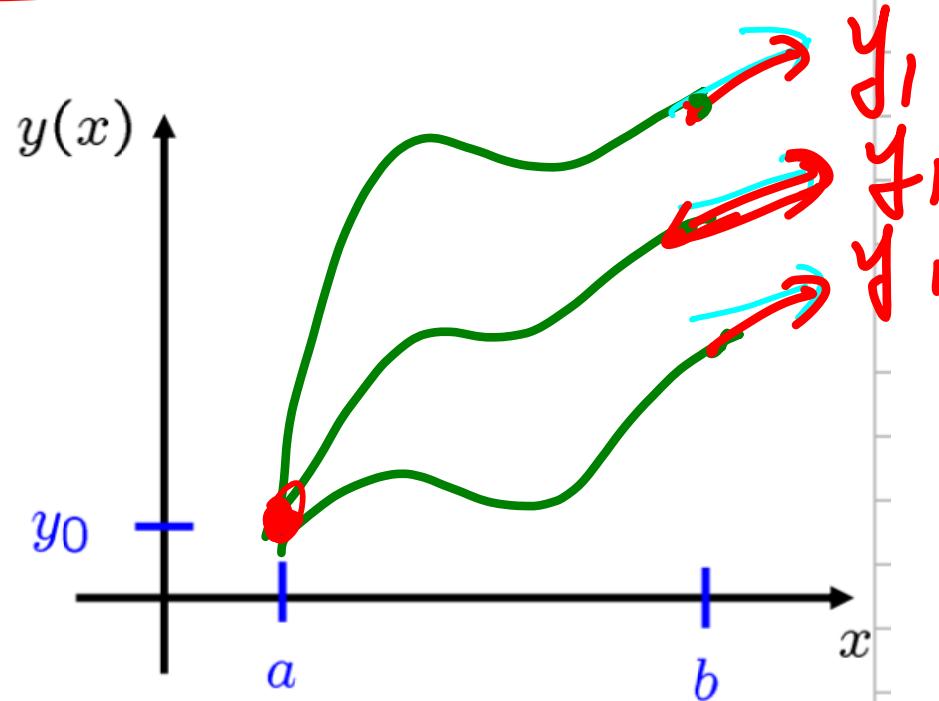
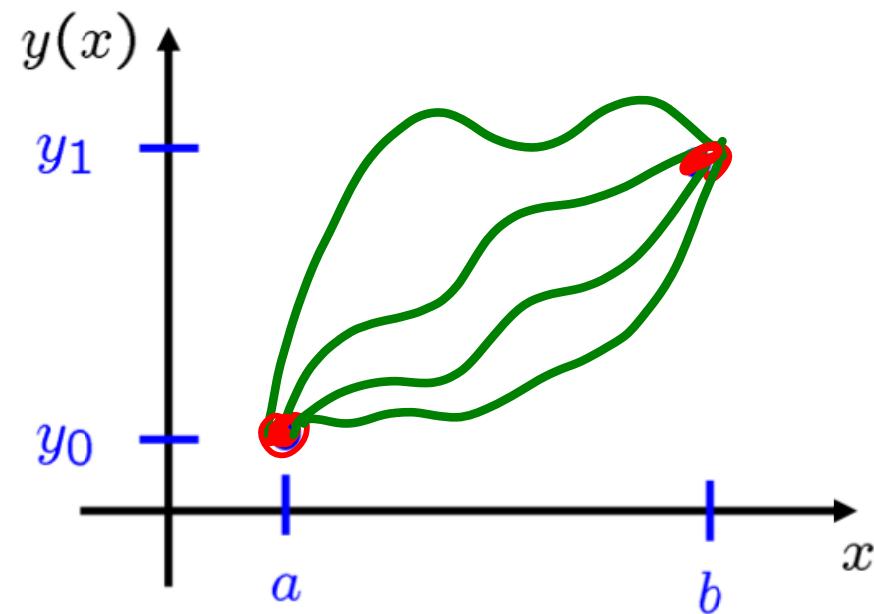
$a$   $y(a)$   
 $b$   $y(b)$   
 $y_i$   
 $y'_i$

- A BVP may have:

(1) several solutions

or (2) a unique solution

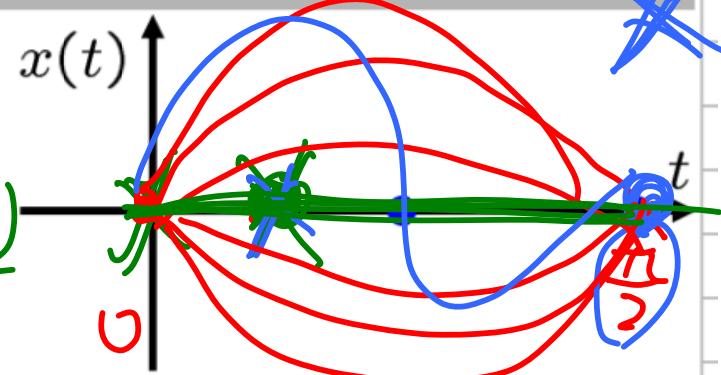
or (3) no solution



### 4.1.1: Example 3

$$x'' + 16x = 0$$

$$\Rightarrow \text{solution: } x(t) = C_1 \cos(4t) + C_2 \sin(4t)$$



$$(a) \begin{cases} x(0) = 0 \\ x(\frac{\pi}{2}) = 0 \end{cases} \quad C_1 + 0 = 0 \Rightarrow C_1 = 0 \quad x(t) = C_2 \sin(4t)$$

$$0 + C_2 x_0 = 0 \Rightarrow 0 = 0$$

$$(b) \begin{cases} x(0) = 0 \\ x(\frac{\pi}{8}) = 0 \end{cases} \Rightarrow C_1 + 0 = 0 \Rightarrow C_1 = 0 \quad x(t) = 0$$

$$0 + C_2 = 0 \Rightarrow C_2 = 0$$

$$(c) \begin{cases} x(0) = 0 \\ x(\frac{\pi}{2}) = 1 \end{cases} \Rightarrow C_1 = 0 \quad \text{nb solution}$$

$$0 + C_2 x_0 = 1 \Rightarrow \times$$

## 4.1.2: Homogeneous Equations

$$a_n(x) \frac{d^n y(x)}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y(x)}{dx^{n-1}} + \cdots + a_1(x) \frac{dy(x)}{dx} + a_0(x) y(x) = 0$$

homogeneous eqn

$$a_n(x) \frac{d^n y(x)}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y(x)}{dx^{n-1}} + \cdots + a_1(x) \frac{dy(x)}{dx} + a_0(x) y(x) = g(x)$$

$\cancel{g(x) \neq 0}$

eqn

- associated homogeneous equation:

$$F(x, y, y', \dots, y(n)) = g(x)$$

$$F(x, y, y', \dots, y(n)) = 0$$

AHE

- In the following, we assume on some common interval  $I$

(1)  $\underline{a_i(x)}$ ,  $i = 0, 1, \dots, n$ , are Continuous for  $x \in I$

(2)  $g(x)$  is Continuous for  $x \in I$

(3)  $\boxed{a_n(x)} \neq 0$  for  $x \in \textcircled{I}$

$$\begin{aligned} I &= (3, \infty) \\ I &= (-\infty, -2) \end{aligned}$$

$$\boxed{Dy \triangleq \frac{dy}{dx}}$$

*define*

$$\underline{\underline{D^2y}} \triangleq \frac{d^2y}{dx^2}$$

$$\underline{\underline{D^n y}} \triangleq \frac{d^ny}{dx^n}$$

$$D : y \rightarrow y'$$

$$D^2 : y \rightarrow y''$$

$$D^n : y \rightarrow y^{(n)}$$

$$y = 2x^2 + 3$$

$$\begin{aligned} \underline{\underline{Dy}} &= D(\underline{\underline{2x^2+3}}) \\ &= \underline{\underline{4x}} \end{aligned}$$

$$\underline{\underline{(D+3)y}} \triangleq \frac{dy}{dx} + 3y$$

$$\underline{\underline{D+3}} : y \rightarrow y' + 3y$$

$$L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x)$$

$$\Rightarrow L(y) = \underbrace{L\{y\}}_{=} = a_n(x) \frac{d^ny}{dx^n} + a_{n-1}(x) \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)y + a_0(x)$$

for example,

$$\boxed{L = (x+5)D + x}$$

$$L = (x+2)D^2 + x^3 + \sin x$$

$$\Rightarrow L(y) = (x+5) \frac{dy}{dx} + x^3 y$$

- homogeneous equation:

$$L(y) = 0$$

- nonhomogeneous equation:

$$L(y) = g(x)$$

- $D$  is a linear operator

$$\begin{cases} \underline{\underline{D}} \left\{ \underline{\underline{f(x)}} + \underline{\underline{g(x)}} \right\} = D\{f(x)\} + D\{g(x)\} \\ D \left\{ \underline{\underline{c f(x)}} \right\} = c D\{f(x)\} \end{cases}$$

or  $D \left\{ \underline{\underline{a f(x) + b g(x)}} \right\} = a D\{f(x)\} + b D\{g(x)\}$

$\Rightarrow L$  is a linear operator

$$\Rightarrow L \left\{ \underline{\underline{a f(x) + b g(x)}} \right\} = a L\{f(x)\} + b L\{g(x)\}$$

- Let  $y_1(x), y_2(x), \dots, y_k(x)$  be solutions of  $L(y)=0$  on  $I$



$$\boxed{L(y)=0}$$

- Then  $y(x) = C_1 \underline{y_1(x)} + C_2 \underline{y_2(x)} + \dots + C_k \underline{y_k(x)}$

is also a solution of  $L(y)=0$ , on  $I$

where  $C_1, C_2, \dots, C_k$  are arbitrary constants

- Proof:  $L(y_i) = 0 \quad i=1, \dots, k$

$$L(\underline{\underline{c_i y_i}}) = c_i \underline{\underline{L(y_i)}} = c_i \cdot 0 = \underline{\underline{0}}$$

$$L(\underline{\underline{\sum c_i y_i}}) = \sum c_i \underline{\underline{L(y_i)}} = \sum c_i \cdot 0 = 0$$

### 4.1.2: Example 4

$$x^3 y''' - 2x y' + 4y = 0$$

$$y_1(x) = x^2$$

$$\underline{y_2(x)} = \underline{x^2 \ln x}$$

$$\boxed{\begin{aligned} y_1' &= 2x \\ y_1'' &= 2 \\ y_1''' &= 0 \end{aligned}}$$

$$\Rightarrow y(x) = C_1 y_1(x) + C_2 y_2(x)$$

$$= \boxed{C_1 x^2 + C_2 x^2 \ln x}$$

$$\begin{aligned} y' &= \dots \\ y'' &= \dots \end{aligned}$$

$$\begin{aligned} y''' &= \dots \end{aligned}$$

$$\begin{aligned} y_2' &= 2x \ln x + x^2 \frac{1}{x} \\ y_2'' &= 2 \ln x + 2x \frac{1}{x} + 1 \\ y_2''' &= 2 \ln x + 2 + 1 = 2 \ln x + 3 \end{aligned}$$

on  $(0, \infty)$

$$y_2''' = \underline{2 \frac{1}{x}}$$

$$\cancel{x^3 \frac{2}{x}} - \cancel{2x(2 \ln x + 3)} + \cancel{4x^2} = 0$$

$$\Rightarrow 0$$

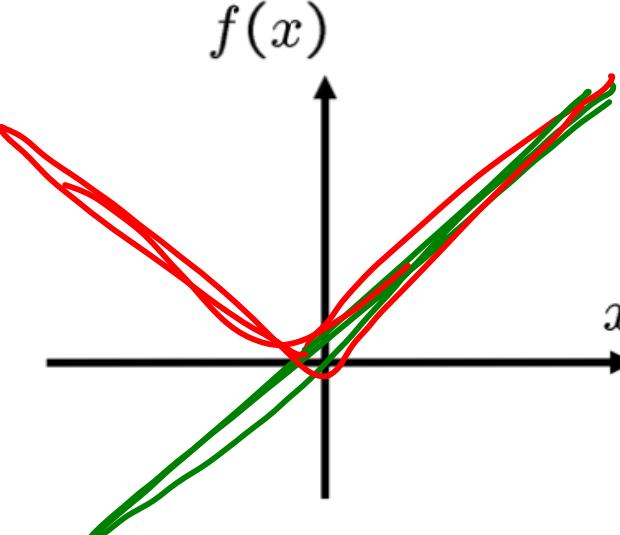
$\cos, \sin e$

- $\{f_1(x), f_2(x), \dots, f_n(x)\}$  is said to be linearly dependent on  $I$
- If there exist constants  $c_1, c_2, \dots, c_n$ , not all equal to  $0$   $\forall x \in I$
- such that  $c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x) \equiv 0 \quad \forall x \in I$
- If  $\{f_1(x), f_2(x), \dots, f_n(x)\}$  is NOT linearly dependent
- Then, the set of functions is linearly independent
- That is,  $c_1 = c_2 = c_3 = \dots = c_n = 0 \quad \forall x \in I$

- $\left\{ \underline{\sin(2x)}, \underline{\sin(x)\cos(x)} \right\}$  linearly dependent  $I = (-\infty, \infty)$   
 $C_1 \sin(2x) + C_2 \sin(x)\cos(x) = 0$   $C_1 = \frac{1}{2} C_2$

$$\textcircled{C}_1 \sin(2x) + \textcircled{C}_2 \sin(x)\cos(x) = 0$$

- $\left\{ \underline{x}, \underline{|x|} \right\}$   $x \in (-\infty, \infty)$  linearly independent  
 $C_1 x + C_2 |x| = 0 \quad \forall$



- $\left\{ \underline{x}, \underline{|x|} \right\}$   $x \in [0, \infty)$  linearly dependent

$$x = |x| \quad \forall x \in [0, \infty)$$

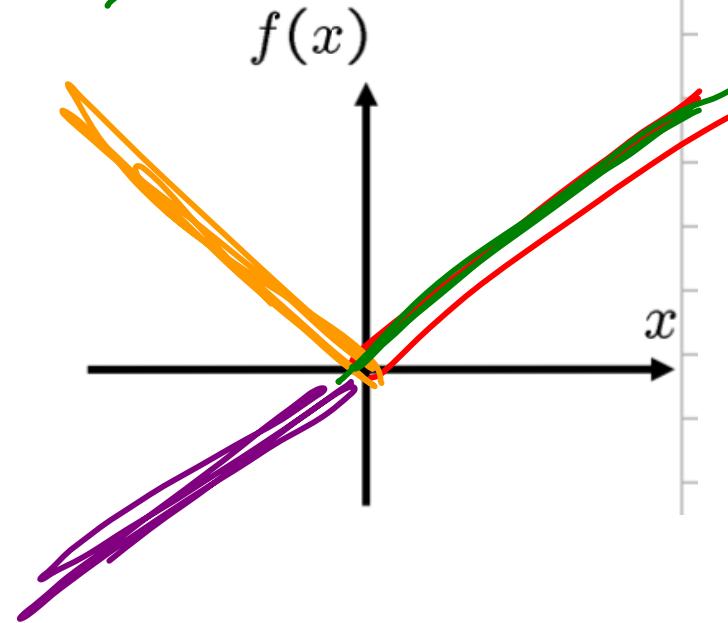
- $\left\{ \underline{x}, \underline{|x|} \right\}$   $x \in (-\infty, 0]$

$$x - |x| = 0$$

$$-\underline{x} = |x|$$

$$x + |x| = 0$$

linearly dependent



- Suppose each of  $f_1(x), f_2(x), \dots, f_n(x)$  has at least

$(n-1)$  derivatives

$f, f', f'' \dots f^{(n-1)}$

- Define:

$$\underline{W(f_1, f_2, \dots, f_n)} \triangleq \det$$

scalar function

$$\begin{bmatrix} f_1 & f_2 & \dots & f_n \\ f'_1 & f'_2 & \dots & f'_n \\ f''_1 & f''_2 & \dots & f''_n \\ \vdots & \vdots & \ddots & \vdots \\ f^{(n-1)}_1 & f^{(n-1)}_2 & \dots & f^{(n-1)}_n \end{bmatrix}$$

$n \times n$

as the Wronskian  
Wronski

of  $f_1(x), f_2(x), \dots, f_n(x)$

- Let  $y_1(x), y_2(x), \dots, y_n(x)$  be n solutions of  $L[y] = 0$  on  $I$
- Then  $\{y_1(x), y_2(x), \dots, y_n(x)\}$  is linearly independent solutions on  $I$
- IFF (if and only if)

$$W(y_1, y_2, \dots, y_n) \neq 0 \quad \forall x \in I$$

$$c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) = 0$$

$$c_1 y'_1(x) + c_2 y'_2(x) + \cdots + c_n y'_n(x) = 0$$

⋮

$$c_1 y''_1(x) + c_2 y''_2(x) + \cdots + c_n y''_n(x) = 0$$

$$c_1 y_1^{(n-1)}(x) + c_2 y_2^{(n-1)}(x) + \cdots + c_n y_n^{(n-1)}(x) = 0$$

$$\begin{bmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y'_1(x) & y'_2(x) & \cdots & y'_n(x) \\ y''_1(x) & y''_2(x) & \cdots & y''_n(x) \\ & & \vdots & \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = 0$$

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = 0$$

$$a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = -a_n \frac{d^n y}{dx^n}$$

$$a_{n-1} \frac{d^{n-1} y_1}{dx^{n-1}} + \cdots + a_1 \frac{dy_1}{dx} + a_0 y_1 = -a_n \frac{d^n y_1}{dx^n}$$

$$a_{n-1} \frac{d^{n-1} y_2}{dx^{n-1}} + \cdots + a_1 \frac{dy_2}{dx} + a_0 y_2 = -a_n \frac{d^n y_2}{dx^n}$$

⋮

$$a_{n-1} \frac{d^{n-1} y_n}{dx^{n-1}} + \cdots + a_1 \frac{dy_n}{dx} + a_0 y_n = -a_n \frac{d^n y_n}{dx^n}$$

$$\begin{bmatrix} y_1^{(n-1)} & y_1^{(n-2)} & \cdots & y'_1 & y_1 \\ y_2^{(n-1)} & y_2^{(n-2)} & \cdots & y'_2 & y_2 \\ & & \vdots & & \\ y_n^{(n-1)} & y_n^{(n-2)} & \cdots & y'_n & y_n \end{bmatrix} \begin{bmatrix} a_{n-1} \\ a_{n-2} \\ \vdots \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} -a_n y_1^{(n)} \\ -a_n y_2^{(n)} \\ \vdots \\ -a_n y_n^{(n)} \end{bmatrix}$$

- Remark:

- Given  $y_1(x), y_2(x), \dots, y_n(x)$  be n solutions of  $L(y) = 0$  on  $I$

$W(f_1, f_2, \dots, f_n)$  is:

either      nonzero     $\forall x \in I$

IFF  $\{y_1(x), y_2(x), \dots, y_n(x)\}$  is      linearly independent

or      identically zero     $\forall x \in I$

IFF  $\{y_1(x), y_2(x), \dots, y_n(x)\}$  is      linearly dependent

- $\{\underline{y_1(x)}, \underline{y_2(x)}, \dots, \underline{y_n(x)}\}$  is a fundamental set of solutions on  $\mathbb{I}$
- IF (1)  $y_i(x)$ ,  $i = 1, 2, \dots, n$  are solutions of  $L(y) = 0$
- (2)  $\{\underline{y_1(x)}, \underline{y_2(x)}, \dots, \underline{y_n(x)}\}$  is linearly independent

- There exists a fundamental set of solutions for the homogeneous linear  $n$ th-order DE on  $I$

$$L(y) = 0$$

$$a_n(x) \frac{d^n y(x)}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y(x)}{dx^{n-1}} + \cdots + a_1(x) \frac{dy(x)}{dx} + a_0(x) y(x) = 0$$

(1)  $a_i(x)$ ,  $i = 0, 1, \dots, n$ , are continuous on  $I$

(2)  $\underline{a_n(x) \neq 0}$ ,  $\forall x \in I$

P

- $\underbrace{\{y_1(x), y_2(x), \dots, y_n(x)\}}_{\text{a fundamental set of solutions for the DE on } I} :$
- The general solution is  $L(y) = 0$   
 $y(x) = C_1 y_1(x) + C_2 y_2(x) + \dots + C_n y_n(x)$   
 $C_1, C_2, \dots, C_n \text{ constants}$

$$y'' - 9y = 0 \quad I = (-\infty, \infty)$$

•  $\{e^{3x}, e^{-3x}\}$ :

$$\begin{aligned} y &= e^{3x} \\ y' &= 3e^{3x} \\ y'' &= 9e^{3x} \end{aligned}$$

$$9e^{3x} - 9e^{3x} = 0$$

① Solution set

② Linearly independent

$$W(e^{3x}, e^{-3x})$$

$$\det \begin{bmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{bmatrix} = (-3)e^{3x}e^{-3x} - 3e^{3x}e^{-3x} = -6 \stackrel{c.}{=} 0$$

$$\Rightarrow y(x) = c_1 e^{3x} + c_2 e^{-3x}$$

### 4.1.3: Nonhomogeneous Equations: Thm 4.1.6: General Solutions

$$a_n(x) \frac{d^n y(x)}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y(x)}{dx^{n-1}} + \cdots + a_1(x) \frac{dy(x)}{dx} + a_0(x) y(x) = g(x)$$

$\text{AHE}$

$g(x) \neq 0$

the general solution

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x)$$

complementary function

particular solution

where

(1)  $\{y_1(x), y_2(x), \dots, y_n(x)\}$  is fundamental set of solutions on  $I$

(2)  $\underline{y_p(x)}$   $\underline{\underline{L(y_p)}} = \underline{\underline{g(x)}}$

(3)  $c_i, i = 1, 2, \dots, n,$

## 4.1.3: Thm 4.1.7: Superposition Principle – Nonhomogeneous Eqns

• IF

$$a_n(x) y^{(n)} + a_{n-1}(x) y^{(n-1)} + \cdots + a_1(x) y' + a_0(x) y =$$

has a

$$y_{p1}(x) = 2x$$

$$y_{p2}(x) = 3x^2 + 5x$$

$$y_{p3}(x) = 5e^{2x}$$

• THEN

$$a_n(x) y^{(n)} + a_{n-1}(x) y^{(n-1)} + \cdots + a_1(x) y' + a_0(x) y =$$

$$= \sum_{i=1}^k g_i(x)$$

has a

$$\begin{aligned} y_p &= \sum y_{pi}(x) \\ &= y_{p1} + y_{p2} + \cdots + y_{pk} \end{aligned}$$

$$\begin{aligned} g_1(x) &= x \\ g_2(x) &= x^2 \\ g_3(x) &= e^{2x} \end{aligned}$$

## 4.1.3: Example 11

$$\underline{y'' - 3y' + 4y} = \underline{-16x^2 + 24x - 8} = g_1(x)$$

$$y_{p1} = \underline{4x^2}$$

$$\underline{y'' - 3y' + 4y} = \underline{2e^{2x}} = g_2(x) \Rightarrow y_{p2} = \underline{e^{2x}}$$

$L(y_p, g_2)$

$$\underline{y'' - 3y' + 4y} = \underline{2xe^x - e^x} = g_3(x) \Rightarrow y_{p3} = \underline{xe^x}$$

$$\underline{y'' - 3y' + 4y} = \underline{(-16x^2 + 24x - 8)} + \underline{(2e^{2x})} + \underline{(2xe^{2x} - e^x)}$$

$$y_p = y_{p1} + y_{p2} + y_{p3}$$

- 2nd-order DE:

Solve:

$$a_2(x) \frac{d^2y(x)}{dx^2} + a_1(x) \frac{dy(x)}{dx} + a_0(x) y(x) = g(x)$$

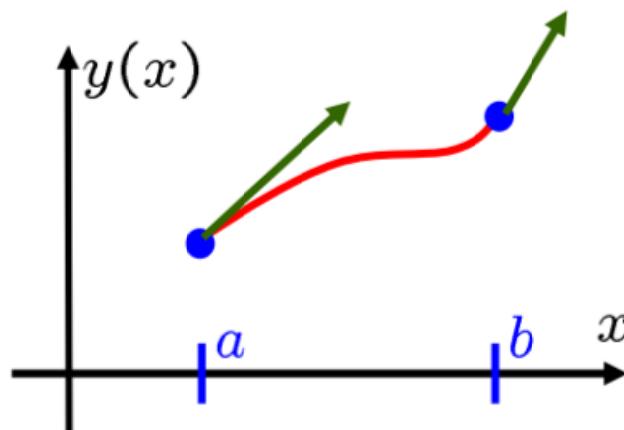
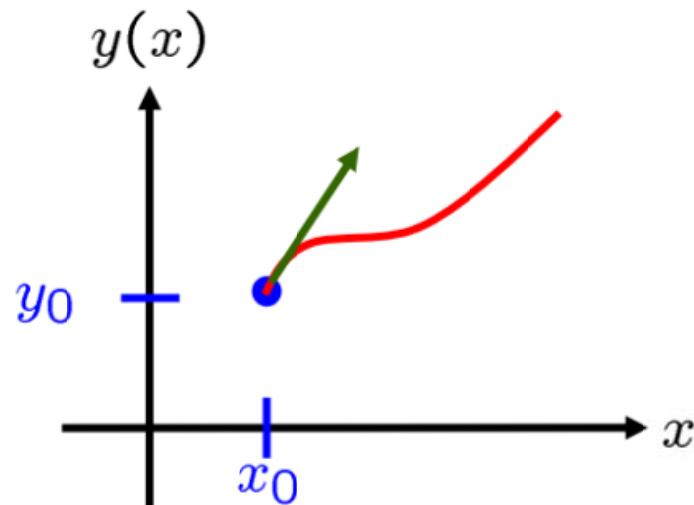
Subject to:

- Initial Condition

$$\begin{cases} y(x_0) = y_0 \\ y'(x_0) = y_1 \end{cases}$$

- Boundary Condition

$$\begin{cases} \alpha_1 y(a) + \beta_1 y'(a) = \gamma_1 \\ \alpha_2 y(b) + \beta_2 y'(b) = \gamma_2 \end{cases}$$



$$a_n(x) \frac{d^n y(x)}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y(x)}{dx^{n-1}} + \cdots + a_1(x) \frac{dy(x)}{dx} + a_0(x) y(x) = g(x)$$

subject to:  $y(x_0) = y_0, \quad y'(x_0) = y_1, \quad y^{(2)}(x_0) = y_2, \quad \dots, \quad y^{(n-1)}(x_0) = y_{n-1}$

- Let  $a_n(x), a_{n-1}(x), \dots, a_1(x), a_0(x)$  &  $g(x)$  be continuous on I
- And  $a_n(x) \neq 0, \quad \forall x \in I, \quad x_0 \in I$
- Then IVP exists a unique solution  $y(x)$  on I

$$a_n(x) \frac{d^n y(x)}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y(x)}{dx^{n-1}} + \cdots + a_1(x) \frac{dy(x)}{dx} + a_0(x) y(x) = 0$$

- $\{y_1(x), y_2(x), \dots, y_n(x)\}$  :  
a fundamental set of solutions for the DE on  $I$

(1)  $y_i(x)$ ,  $i = 1, 2, \dots, n$  are solutions

(2)  $\{y_1(x), y_2(x), \dots, y_n(x)\}$  is linearly independent

$$W(y_1, y_2, \dots, y_n) \triangleq \det \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y'_1 & y'_2 & \cdots & y'_n \\ \vdots & & \ddots & \\ y_1^{n-1} & y_2^{n-1} & \cdots & y_n^{n-1} \end{bmatrix} \neq 0, \quad \forall x \in I$$

- The general solution:

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$$

$$a_n(x) \frac{d^n y(x)}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y(x)}{dx^{n-1}} + \cdots + a_1(x) \frac{dy(x)}{dx} + a_0(x) y(x) = g(x)$$
$$g(x) \not\equiv 0$$

- The general solution:

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x)$$