

Fall 2019

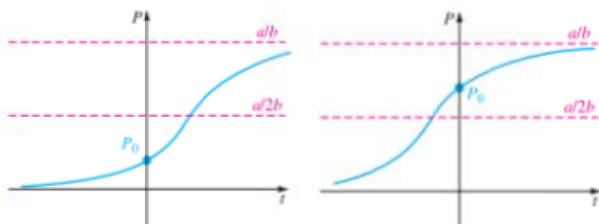
# 微分方程 Differential Equations

## Unit 03.2 Nonlinear Models

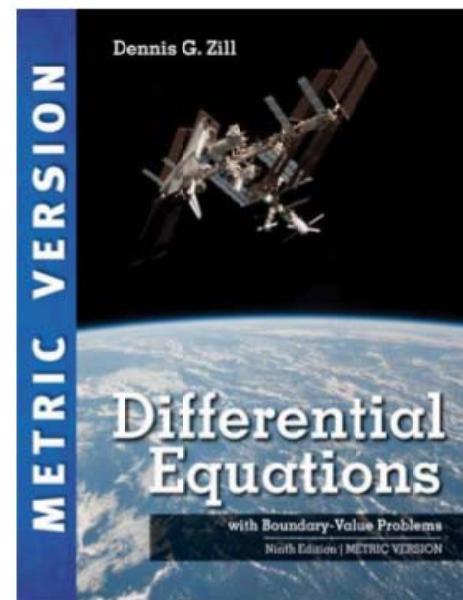
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Sep19 – Jan20



Figures and images used in these lecture notes are adopted from  
[Differential Equations with Boundary-Value Problems](#), 9th Ed., D.G. Zill, 2018 (Metric Version)



- 3.1: Linear Models
- **3.2: Nonlinear Models**
- 3.3: Modeling with Systems of DEs

$P(t)$ : size of population at time  $t$

$$\Rightarrow \frac{dP(t)}{dt} = k P(t), \quad k > 0, \quad P(0) = P_0$$

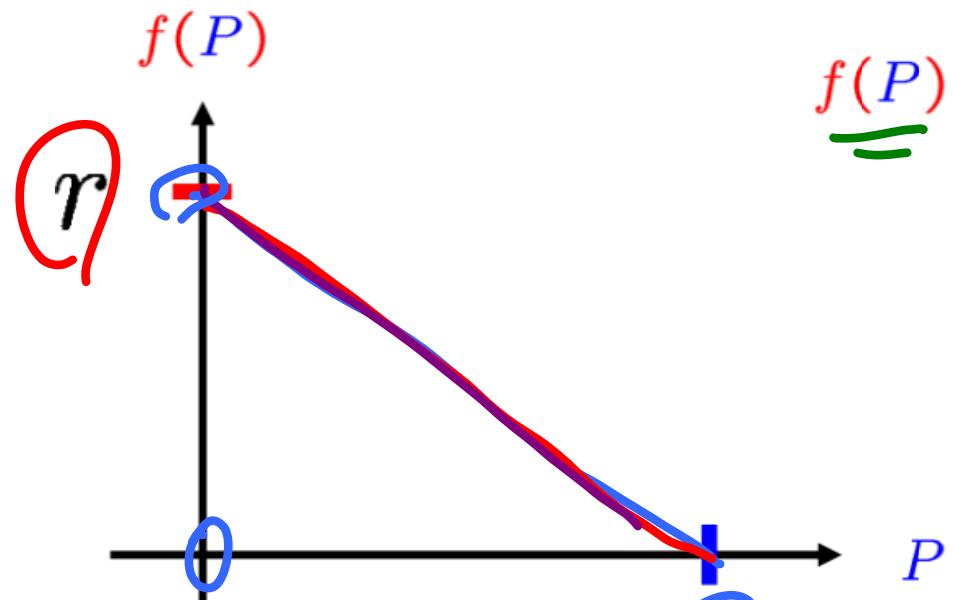
$$\Rightarrow P(t) = P_0 e^{kt}, \quad t \geq 0$$

- density-dependent hypothesis:

$$\frac{dP(t)}{dt} = f(P)$$

$$\frac{dP}{dt} = P f(P) \\ (Ch - P)$$

- for example



$$f(P) = r - \frac{r}{K}P = \underline{\underline{a}} - \underline{\underline{b}}P = 0$$

$$0 \leq P \leq K$$

$$P(0) = P_0$$

$$\begin{aligned} \Rightarrow \frac{dP(t)}{dt} &= P(a-bP) \\ \Rightarrow \frac{dP}{dt} &= P(a-bP) = \boxed{ap - bp^2} = 0 \\ \Rightarrow \text{equilibrium points: } &P=0 \text{ or } \frac{a}{b} \end{aligned}$$

$$\frac{dP}{P(a-bP)} = dt$$

$$\frac{dP}{P(a-bP)} = dt$$

$\frac{1}{(P_0)} + \frac{1}{(P-B)}$

$$\frac{b}{a} \left( \frac{1}{P} + \frac{1}{\frac{a}{b}-P} \right) dP = dt$$

$$\int \frac{1}{a} \left( \frac{1}{P} + \frac{1}{\frac{a}{b}-P} \right) dP = \int dt$$

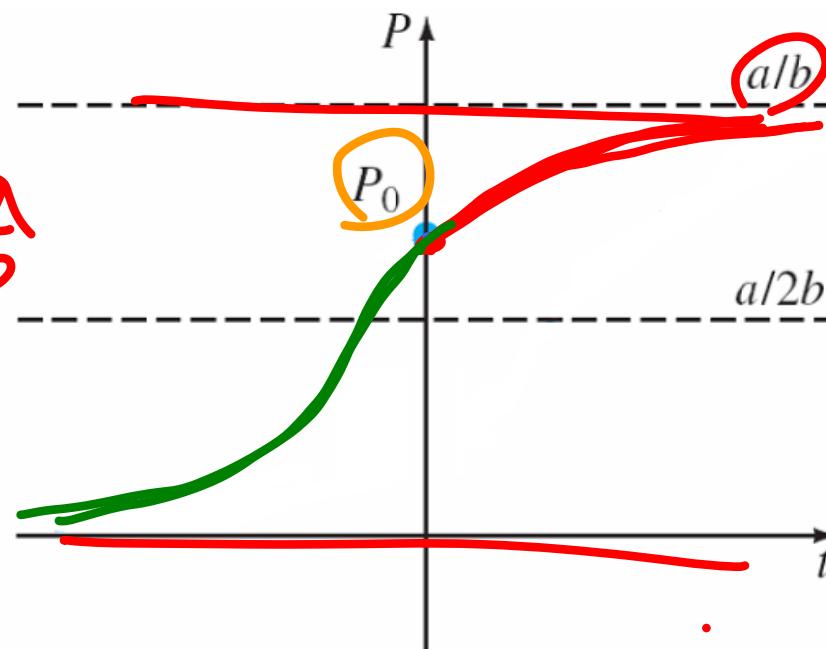
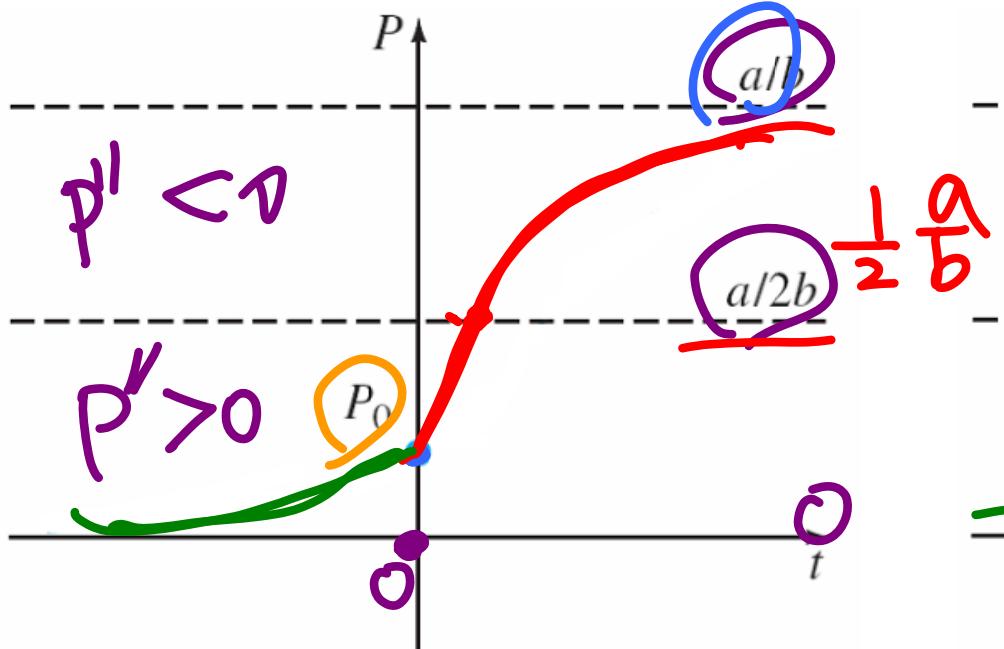
$$\int \frac{1}{P} dP + \int \frac{1}{\frac{a}{b}-P} dP = \int a dt$$

$$\ln|P| - \ln|\frac{a}{b}-P| = at + C,$$

$$\ln \left| \frac{P}{\frac{a}{b} - P} \right| = at + C_1 \quad \pm e^{C_1} = C_2$$

$$\left( \frac{P}{\frac{a}{b} - P} \right) \cdot e^{at + C_1} = e^{at} = e^{at} e^{C_1} \quad C_1 = \frac{C_2}{b}$$
~~$$\frac{P}{\frac{a}{b} - P} \cdot e^{at + C_2} \Rightarrow P = \frac{ac}{bc + e^{-at}}$$~~

$$P(t) = \frac{a P_0}{b P_0 + (a - b P_0) e^{-at}} \quad P(0) = P_0$$



$$\frac{dP(t)}{dt} = P(a - b\underline{\underline{P}})$$

~~-h~~ rate

$$\frac{dP(t)}{dt} = P(a - b\underline{\underline{P}})$$

+h

$$\frac{dP(t)}{dt} = P(a - b\underline{\underline{P}}) - cP$$

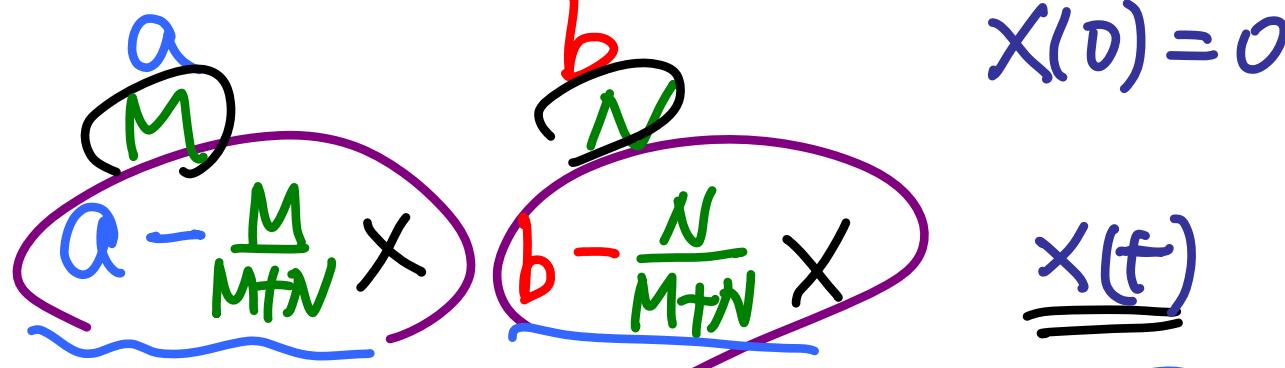
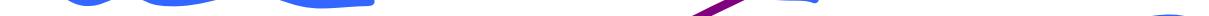
$$\frac{dP(t)}{dt} = P(a - b\underline{\underline{P}}) + ce^{-kp}$$

$$\frac{dP(t)}{dt} = P(a - b\underline{\underline{P}})$$

Gompertz DE

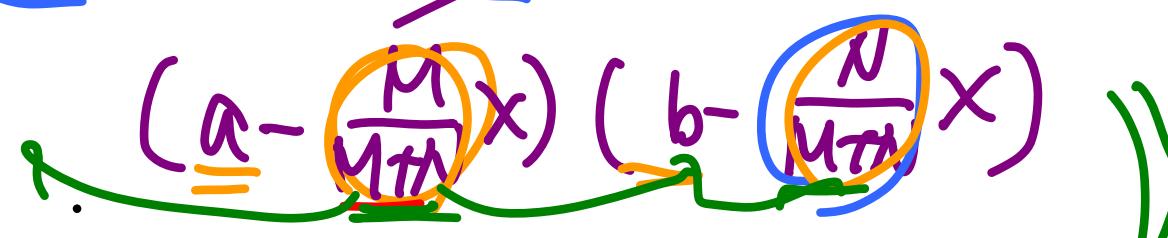


time = 0

time =  $t$ 

• Law:

$$\frac{dX(t)}{dt}$$

 $\propto$ 

$$\Rightarrow \frac{dX(t)}{dt} = k(\alpha - x)(\beta - x) = 0$$

$$\Rightarrow \frac{dx}{(\alpha - x)(\beta - x)} = k dt$$

$$\begin{aligned} x &= \alpha \\ x &= \beta \end{aligned}$$

$$\int \frac{1}{\beta-\alpha} \left( \frac{1}{\alpha-x} - \frac{1}{\beta-x} \right) dx = \int k dt$$

$$\frac{1}{\beta-\alpha} \left[ (-1)\ln|\alpha-x| - (-1)\ln|\beta-x| \right] = kt + C_1$$

$$\ln \left| \frac{\alpha-x}{\beta-x} \right| = -k(\beta-\alpha)t + C_1(\beta-\alpha)$$

$$\left| \frac{\alpha-x}{\beta-x} \right| = e^{-k(\beta-\alpha)t} e^{C_1(\beta-\alpha)}$$

$$\frac{\alpha-x}{\beta-x} = C_2 e^{-k(\beta-\alpha)t} \stackrel{+e^{-C_1(\beta-\alpha)}}{=} C_2$$

$$x(t) = \frac{\beta - \alpha C_2 e^{+k(\beta-\alpha)t}}{1 - C_2 e^{+k(\beta-\alpha)t}}$$

$$\frac{dP(t)}{dt} = P(a - b \frac{P}{P})$$

$P=0, \frac{\alpha}{b}$

Or  $k(\underline{P-\alpha})(\underline{\beta-P})$

$P=\alpha, \text{ or } \beta$

$$\frac{dP(t)}{dt} = P(a - b P) - h$$

$$\frac{dP(t)}{dt} = P(a - b P) + h$$

$$\frac{dP(t)}{dt} = P(a - b P) - c P$$

$$\frac{dP(t)}{dt} = P(a - b P) + c e^{-kP}$$

$$\frac{dP(t)}{dt} = P(a - b \ln P)$$

