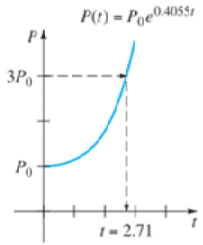
Fall 2019

微分方程
Differential Equations

Unit 03.1
Linear Models



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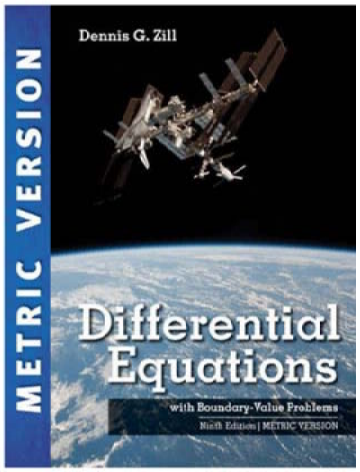
Sep19 – Jan20



Figures and images used in these lecture notes are adopted from
Differential Equations with Boundary-Value Problems, 9th Ed., D.G. Zill, 2018 (Metric Version)

Outline

- **3.1: Linear Models**
- 3.2: Nonlinear Models
- 3.3: Modeling with Systems of DEs

3.1: LINEAR MODELS: Growth and Decay

Given $\dfrac{dx(t)}{dt} = kx(t)$ with $\boxed{x(t_0) = x_0}$

$x=0$

Solution $\left(\dfrac{dx}{dt} \boxed{- kx}\right) = 0$

$P(x)$

$$I.F. = e^{\int P(x)dx} = e^{\int(-k)dx} = e^{-kx}$$

$$e^{-kt}\left[\frac{dx}{dt} - kx\right] = 0$$

$$e^{-kt}\frac{dx}{dt} - kxe^{-kt} = 0$$

$$\frac{d}{dt}\left[e^{-kt}x\right] = 0$$

$k > 0$

$k = 0$

$k < 0$

$$e^{-kt}x = c$$

$$\boxed{x = c\,e^{kt}}$$

$$x_0 = c\,e^{kt_0}$$

$$c = x_0 e^{-kt_0}$$

$$\boxed{x(t)} = x_0 e^{-kt_0} e^{kt} = x_0 e^{k(t-t_0)}$$

Δt

x_0

1

$e^{-k(t-t_0)}$

3.1: Growth and Decay

$$x(t) = x_0 \, e^{k(t-t_0)}$$

- growth: $k>0$

  or diverge

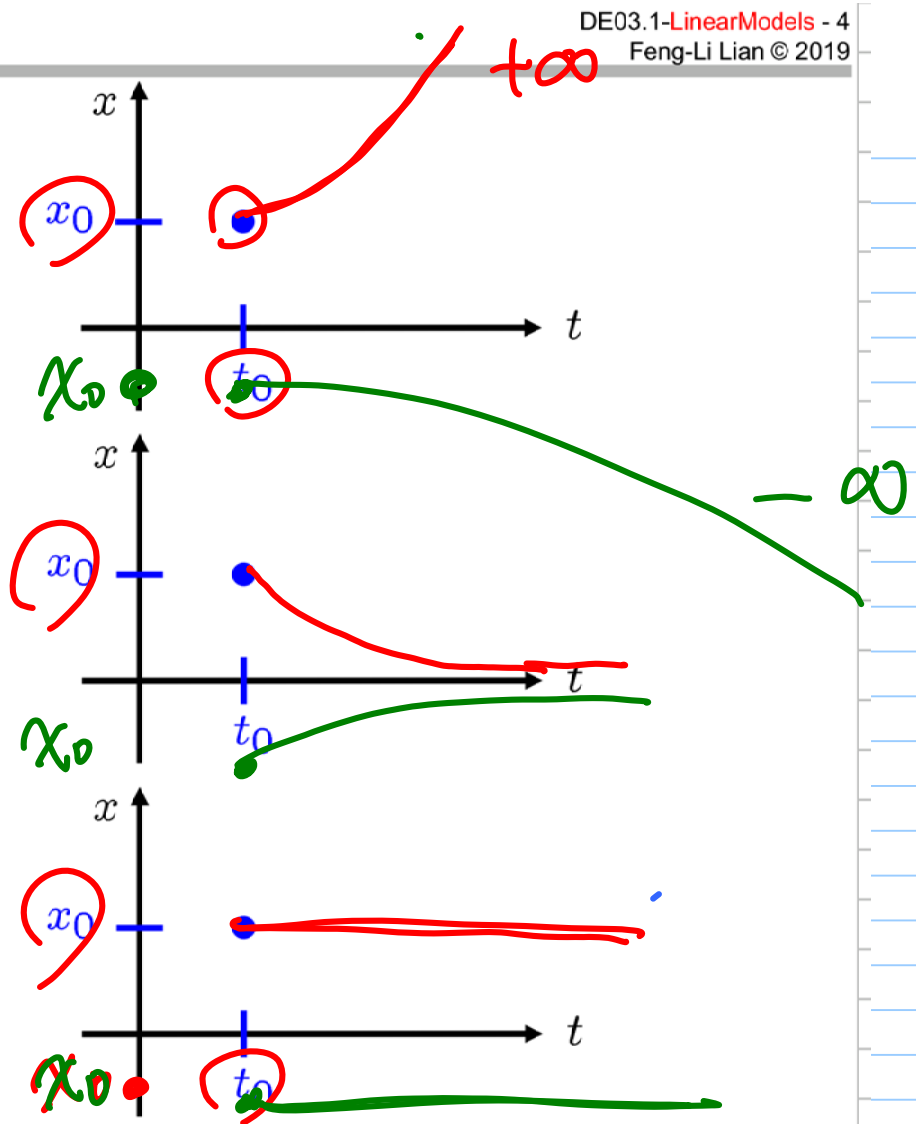
  or unstable

- decay: $k<0$

  or converge

  or stable

- constant: $k=0$



3.1: Example 1: Bacterial Growth

$P(t)$: number of bacteria in a culture

$t = 0 \qquad P(0) = P_0$

$t = 1h : \qquad P(1h) = \frac{3}{2} P_0$

$\Rightarrow t = ? \qquad P(t) = 3P_0$

- Law: $\frac{dP}{dt} \propto P(t)$

$\frac{dP}{dt} = k P(t)$

$P(t) = P_0 e^{k(t-0)}$

$P(1) = P_0 e^{k \cdot 1} = \frac{3}{2} P_0$

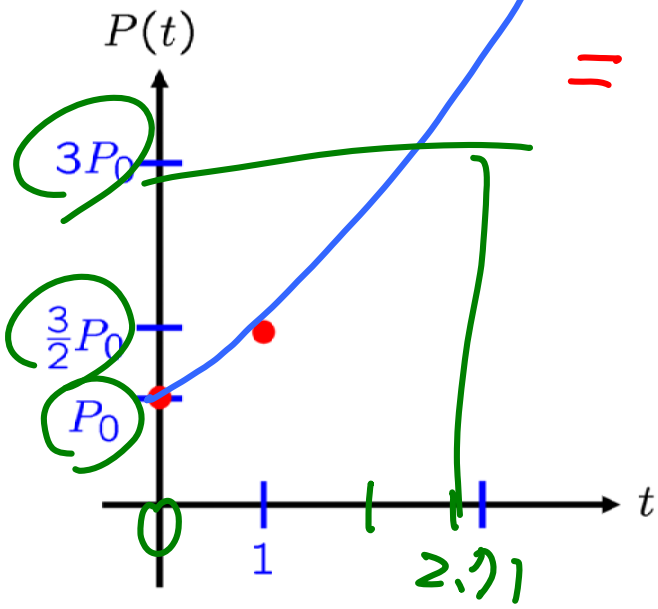
$$e^{k} = \frac{3}{2}$$

$$k = \ln\left(\frac{3}{2}\right) = 0.4055$$

$$P(t) = P_0 e^{0.4055t}$$

$$3\cancel{P_0} = \cancel{P_0} e^{0.4055t}$$

$$t = \frac{\ln 3}{0.4055} = \boxed{2.71}\ (h)$$



3.1: Example 2: Decay: Half Life of Plutonium

U-238 $\longrightarrow$ Pu-239

$A(t)$: amount of Pu-239 at time t

$t = 0$: $\boxed{A(0) = A_0}$

$t = 15yr$: 0.043% disintegrated

$\Rightarrow$ $\boxed{\text{half-life}}$

- Law: $\dfrac{dA}{dt} \propto A(t)$

$$\frac{dA}{dt} = k\,A(t)$$

$$\Rightarrow A(t) = A_0 e^{k(t)}$$

$$A(15) = A_0 e^{k\cdot 15} = \boxed{0.9995}\,A_0$$

$$e^{k \cdot 15} = 0.99957$$

$$k = \frac{\ln 0.99957}{15} = -0.0000286$$

$$A(t) = A_0 e^{-0.0000286\, t}$$

$$\frac{1}{2}\cancel{A_0} = \cancel{A_0}\, e^{-0.0000286\, t}$$

$$t = \frac{\ln\left(\frac{1}{2}\right) = -\ln 2}{-0.0000286} = \underline{24180} \text{ yrs}$$

3.1: Some Constants about exponential

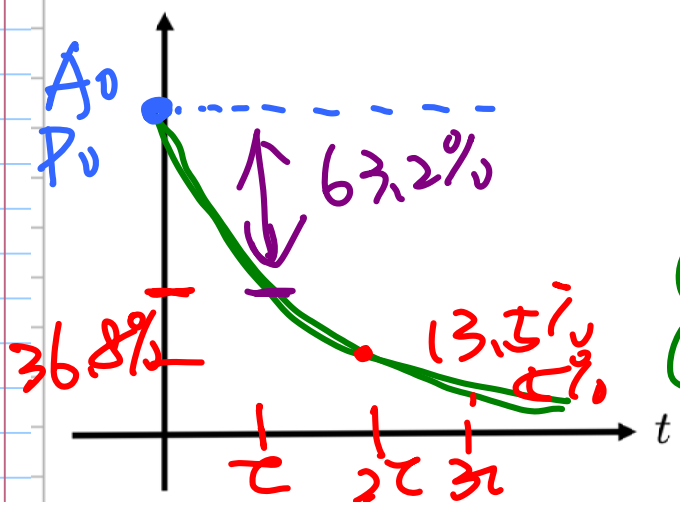
- for $k < 0$

$$\boxed{e^{kt}} = e^{-\frac{t}{\left(\frac{-1}{k}\right)}} = e^{-\frac{t}{\tau}}$$

$t = \tau$

$= 0.368$

$$\tau = -\frac{1}{k} = \frac{1}{|k|}$$



t	$e^{kt} = e^{-\frac{t}{\tau}}$		
0	e^0	1	
τ	e^{-1}	0.368	36%
2τ	e^{-2}	0.135	13.5%
3τ	e^{-3}	0.050	5%
4τ	e^{-4}	0.018	1%
5τ	e^{-5}	0.0067	~1%
∞	$e^{-\infty}$	0	

3.1: Transient and Steady-State

Given $\dfrac{dx(t)}{dt} = k\,(x(t) - b) = 0$ with $\boxed{x(0) = a}$

$x(t) = b$ $\quad x(t) \to b$

Solution $\dfrac{dx}{dt} - kx = -kb$ $\quad f(x)$

$$\boxed{x(t) = \underbrace{c\,e^{kt}}_{y_c} + \underbrace{b}_{y_p}}$$

$x(0) = c\,e^{0} + b = a \Rightarrow c = a - b$

$x(t) = (a-b)\,e^{kt} + b$

$k > 0$ $\quad x(t) \nearrow$

$k < 0$ $\quad x(t) \searrow$

3.1: Transient and Steady-State

$$\frac{dx(t)}{dt} = k\left(x(t) - b\right) = 0 \quad \text{with} \quad x(0) = a$$

equilibrium point:

	$k = 0$	$k > 0$	$k < 0$
$x(0) - b = 0$	$\frac{dx}{dt} = 0$	$\frac{dx}{dt} = 0$	$\frac{dx}{dt} = 0$
$x(0) - b > 0$	$\frac{dx}{dt} = 0$	$\frac{dx}{dt} > 0$	$\frac{dx}{dt} < 0$
$x(0) - b < 0$	$\frac{dx}{dt} = 0$	$\frac{dx}{dt} < 0$	$\frac{dx}{dt} > 0$

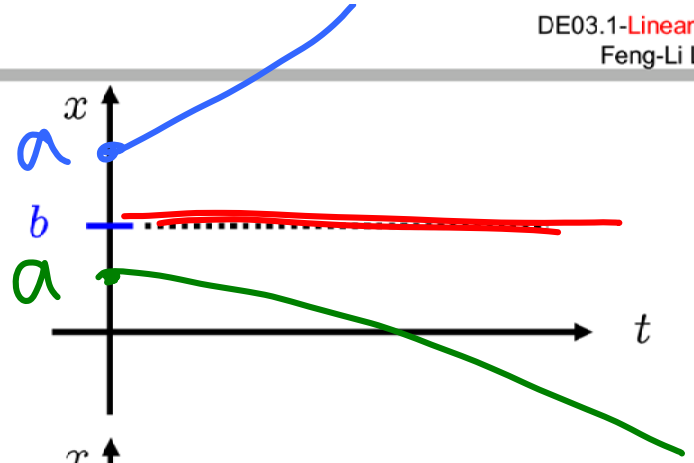
3.1: Transient and Steady-State

- $k > 0$

  growth

  or diverge

  or unstable

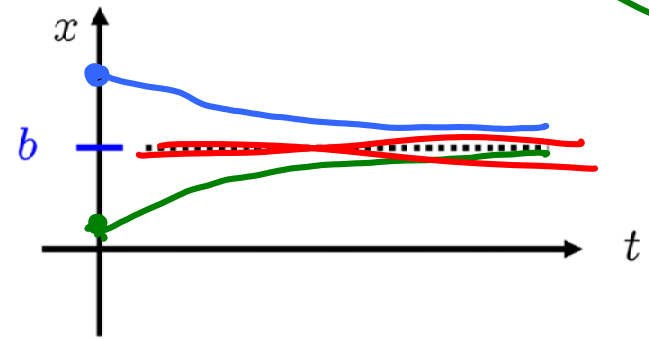


- $k < 0$

  decay

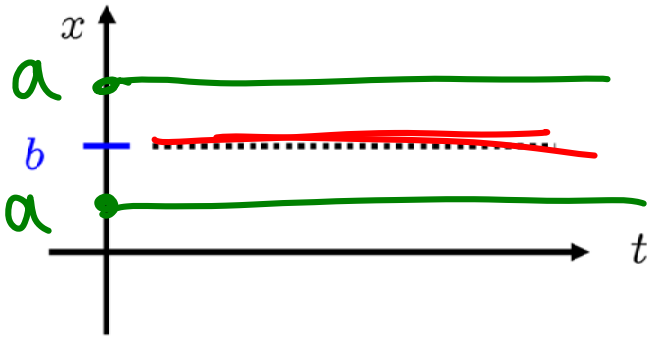
  or converge

  or stable



- $k = 0$

  constant



3.1: Example 4: Cooling of a Cake

$T(t)$: temperature of a cake at time t

$t = 0$: $T(0) = 150$

$t = 3$ mins : $T(3) = 90$

$T_m = 20$

- Law: $\frac{dT}{dt} \propto (T(t) - T_m)$

$$\frac{dT}{dt} = k(T - T_m)$$

$$T = 20 + c e^{kt}$$

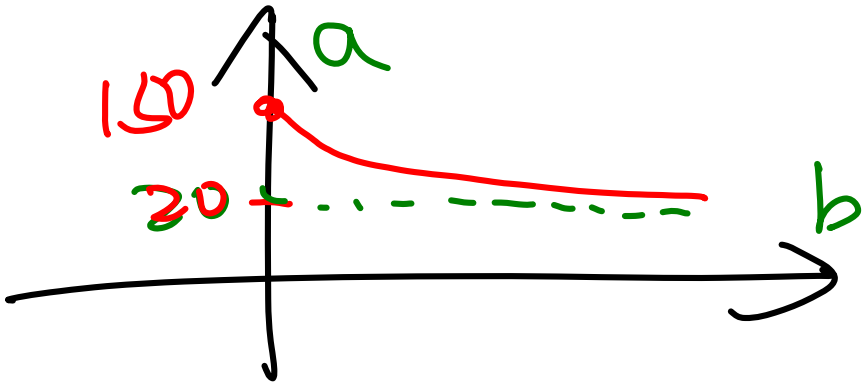
$t=0$, $150 = 20 + c e^{k \cdot 0}$ ($e^{k \cdot 0} \to 1$) $\Rightarrow c = 130$

$t=3$, $90 = 20 + 130 e^{k \cdot 3} \Rightarrow k = \frac{\ln\left(\frac{70}{130}\right)}{3}$

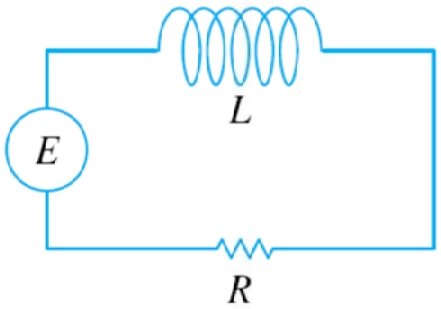
$= -0.20635$

$$T = 20 + 130\, \cancel{e^{-0.20635t}}^{\,0}$$

$t \to \infty \quad T(t) \to 20$



3.1: RL Series Circuits



$$E_0 = R\,i(t) + L\,\frac{di(t)}{dt}$$

with $i(0) = 0$

$$\Rightarrow \frac{di}{dt} + \frac{R}{L}\,i = \frac{E_0}{L}$$

$$\Rightarrow \text{I.F.}: \; e^{\int \frac{R}{L} dt} = e^{\frac{R}{L}t}$$

$$\Rightarrow e^{\frac{R}{L}t}\frac{di}{dt} + \frac{R}{L}\,i\,e^{\frac{R}{L}t} = \frac{E_0}{L}\,e^{\frac{R}{L}t}$$

$$\Rightarrow \frac{d}{dt}\left(e^{\frac{R}{L}t}\,i\right) = \frac{E_0}{L}\,e^{\frac{R}{L}t}$$

$$\Rightarrow \quad e^{\frac{R}{L}t}\, i = \frac{E_0}{R} e^{\frac{R}{L}t} + C$$

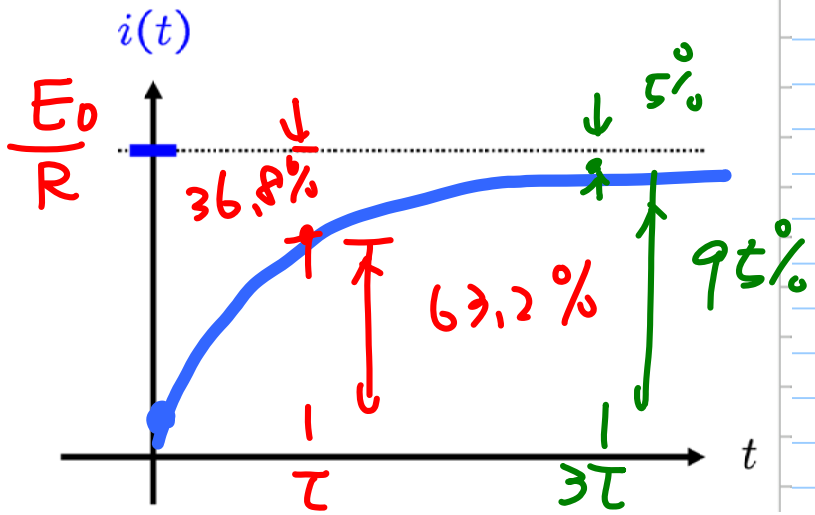
$$\Rightarrow \quad i(t) = C e^{-\frac{R}{L}t} + \frac{E_0}{R}$$

$$\Rightarrow \quad i(0) = 0 \quad \Rightarrow \quad C = -\frac{E_0}{R}$$

$$\Rightarrow \quad i(t) = \frac{E_0}{R} - \frac{E_0}{R} \boxed{e^{-\frac{R}{L}t}}$$

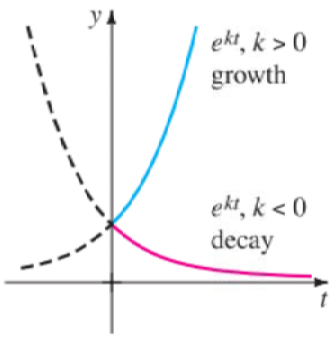
$$= \frac{E_0}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

$$\tau = -\frac{1}{\left(-\frac{R}{L}\right)} = \frac{L}{R}$$



Summary

Given $\dfrac{dx(t)}{dt} = k\, x(t)$ with $x(t_0) = x_0$

Solution $\Rightarrow$ $x(t) = C\, e^{kt} = (x_0\, e^{-kt_0})\, e^{kt}$

$$= x_0\, e^{k(t-t_0)}$$