

Fall 2019

微分方程 Differential Equations

Unit 02.5 Solutions by Substitutions

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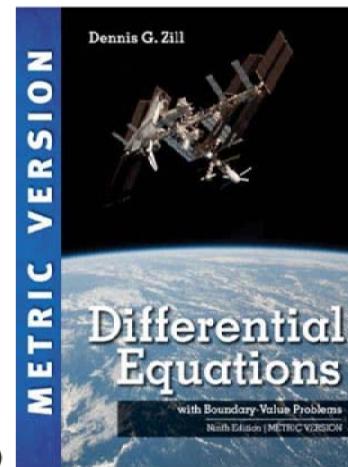
NTU-EE

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$$\frac{dy}{dx} = f(x, y)$$

$$\Rightarrow y(x) = g(x, u)$$

Figures and images used in these lecture notes are adopted from
[Differential Equations with Boundary-Value Problems](#), 9th Ed., D.G. Zill, 2018 (Metric Version)



- 2.1: Solution Curves without a Solution
 - 2.1.1: Direction Fields
 - 2.1.2: Autonomous First-Order DEs
- 2.2: Separable Equations
- 2.3: Linear Equations
- 2.4: Exact Equations
- **2.5: Solutions by Substitutions**
- 2.6: A Numerical Method

■ Transforming the DE into another DE

by means of a substitution.

$$\bullet \frac{dy}{dx} = f(x, y)$$

$$\Rightarrow y(x) = h(x)$$

Assume $y(x) = g(x, u(x))$

$$\Rightarrow \frac{dy}{dx} = g_x + g_u \frac{du}{dx} = f(x, g(x, u)) \Rightarrow \frac{du}{dx} = \frac{f - g_x}{g_u} = F(x, u)$$

1) Homoaeneous Equations2) Bernoulli's Equation3) Reduction to Separation of Variables

$$u^2 = \frac{1}{2x+5}$$

$$y(x) = u$$

$$\begin{array}{c} u \\ \downarrow \\ y \end{array} \quad \begin{array}{c} x \\ \downarrow \\ x \end{array}$$

$$\Rightarrow u(x)$$

$$\Rightarrow y(x) = g(x, u)$$

■ Homogeneous Functions of Degree a

$f(x, y)$ is a homogeneous functions of degree a

IF

$$f(t\underline{x}, t\underline{y}) = t^a f(x, y)$$

e.g., $f(x, y) = \underline{x^3} + 3\underline{y^3}$

 x^2 xy x^2y^2

$$f(\underline{t}x, \underline{t}y) = (\underline{t}x)^3 + 3(\underline{t}y)^3$$

$$= \underline{t^3}x^3 + 3\underline{t^3}y^3$$

$$= \underline{t^3}(x^3 + 3y^3)$$

$$= \underline{t^3}f(x, y)$$

■ Homogeneous Equations

$$M(x, y) dx + N(x, y) dy = 0$$

$M(x, y), N(x, y)$ Homogeneous Functions of Degree a

$$\Rightarrow \begin{cases} M(tx, ty) = t^a M(x, y) \\ N(tx, ty) = t^a N(x, y) \end{cases}$$

$$\begin{cases} y = u x \\ u = \frac{y}{x} \end{cases}$$

$$\Rightarrow \begin{cases} M(x, y) = x^a M(1, u) \\ N(x, y) = x^a N(1, u) \end{cases}$$

■ OR

$$\begin{cases} x = v y \\ v = \frac{x}{y} \end{cases}$$

$$\Rightarrow \begin{cases} M(x, y) = y^a M(v, 1) \\ N(x, y) = y^a N(v, 1) \end{cases}$$

■ Homogeneous Equations

$$\underline{M(x, y) dx} + \underline{N(x, y) dy} = 0$$

~~$x^a M(1, u) dx$~~ + ~~$x^a N(1, u) dy$~~ = 0 → du

$$M(1, u) dx + N(1, u) dy = 0$$

$$y = u x$$

$$dy = u dx + du x$$

$$M(1, u) dx + N(1, u) [u dx + x du] = 0$$

$$[M(1, u) + u N(1, u)] dx + x N(1, u) du = 0$$

$$\frac{dx}{x} + \frac{N(1, u) du}{M(1, u) + u N(1, u)} = 0$$

$u(x) \Rightarrow y(x)$

■ A Separable DE

Example 1: Solving a Homogeneous DE

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$$(x^2 + y^2) dx + (x^2 - xy) dy = 0$$

$$a=2$$

$$y = u \cdot x$$

$$(x^2 + (ux)^2) dx + (x^2 - x(ux)) dy = 0$$

$$(u dx + x du)$$

$$x^2 dx + u^2 x^2 dx + x^2 u dx - ux^2 dx + x^3 du - x^3 u du = 0$$

$$dx [x^2 + u^2 x^2] + du [x^3 - x^3 u] = 0$$

$$dx [1 + u^2] + du [x - xu] = 0$$

$$x \neq 0$$

$$\frac{\partial x^2}{\partial x}$$

$$\frac{d}{dx} x \left(\frac{dx}{x} + \frac{du}{u} \right) \left(\frac{(1-u)}{1+u^2} \right) > 0 \quad x \in D$$

$$\int \frac{1}{x} dx = \int \frac{-(-u)}{1+u^2} du$$

$$u(x) = \dots - x = g(x)$$

$$y = ux = g(x)x$$

$$y \leftarrow n \Rightarrow u$$

Bernoulli's Equation

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$$\frac{dy}{dx} + P(x)y = f(x)y^n \quad n \in \mathbb{R}$$

$n = 0$ or 1 : linear equations

$$\rightarrow n \neq 0 \text{ or } 1 \quad u = y^{1-n} = y^{-1} \quad u = \frac{1}{y}$$

$$\text{i.e., } y = g(x, u) = u^{\frac{1}{1-n}}$$

$$\Rightarrow \frac{du}{dx} = F(x, u) \quad \text{linear equations}$$

Example 2: Solving a Bernoulli DE

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$$x \frac{dy}{dx} + y = x^2 y^2$$

$y = u^{-1}$

$$u = y^{1-n} = y^{1-2} = \frac{1}{y}$$

$$\frac{dy}{dx} = -u^{-2} \frac{du}{dx}$$

$$\frac{dy}{dx} = (-1) u^{-2} \frac{du}{dx}$$

$$x (-1) u^{-2} \frac{du}{dx} + \frac{1}{u} = x^2 \frac{1}{u^2}$$

$$-\frac{x}{u^2}$$

$$\frac{du}{dx} + \left(-\frac{1}{x}\right) \frac{u^2}{1} = \cancel{x} \frac{-u^2}{\cancel{x}} x^2 \frac{1}{u^2}$$

$$\frac{dy}{dx} - \frac{1}{x} u = -x \quad \dots$$

$$\Rightarrow u(x) = -x^2 + cx \leftarrow$$

$$y = \underline{\underline{\frac{1}{u}}} = \frac{1}{-x^2 + cx}$$

Reduction to Separation of Variables

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$$\frac{dy}{dx} = f(Ax + By + C)$$

$$u = Ax + By + C, \quad B \neq 0$$

$$\frac{du}{dx} = A + B \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{B} \left(\frac{du}{dx} - A \right) = f(du)$$

$$\frac{du}{dx} - A = B f(du)$$

$$\frac{\frac{du}{dx}}{A + B f(du)} = A + B f(du)$$

Example 3: An IVP

$$\frac{dy}{dx} = (-2x + y)^2 - 7, \quad y(0) = 0$$

$$\boxed{\frac{dy}{dx}} = u^2 - 7$$

$$u = -2x + y$$

$$\frac{du}{dx} = -2 \Rightarrow +\boxed{\frac{du}{dx}}$$

$$\frac{du}{dx} + 2 = u^2 - 7$$

$$\frac{du}{dx} = u^2 - 9$$

$$\int \frac{du}{u^2 - 9} = \int dx$$

$$u(x) = \frac{3(1 + Ce^{6x})}{1(C - Ce^{6x})}$$

$$y = u + 2x$$

$$(u-3)(u+3)$$

$$\frac{dy}{dx} = f(x, y) \Rightarrow y(x) = g(x, u(x))$$
$$\Rightarrow \frac{du}{dx} = F(x, u) \quad \begin{matrix} 2,2 \\ 2,3 \end{matrix}$$
$$\Rightarrow u(x) = \dots$$
$$y(x) = \dots$$

e.g., $y = u x$

$$\overline{dy} = \overline{du}x + u\overline{dx}$$

$$dy = f(x, y)dx \Rightarrow du x + u dx = f(x, u x)dx$$