

Fall 2019

微分方程

Differential Equations

Unit 02.4

Exact Equations

Feng-Li Lian

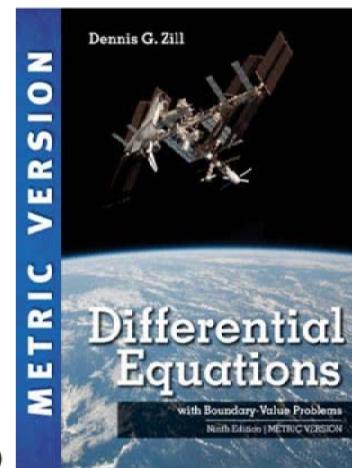
NTU-EE

$$M(x, y)dx + N(x, y)dy = 0$$

$$\Leftrightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Sep19 – Jan20

Figures and images used in these lecture notes are adopted from
[Differential Equations with Boundary-Value Problems](#), 9th Ed., D.G. Zill, 2018 (Metric Version)



- 2.1: Solution Curves without a Solution
 - 2.1.1: Direction Fields
 - 2.1.2: Autonomous First-Order DEs
- 2.2: Separable Equations
- 2.3: Linear Equations
- **2.4: Exact Equations**
- 2.5: Solutions by Substitutions
- 2.6: A Numerical Method

Exact Equation

DE02.4-ExactDE - 3
Feng-Li Lian © 2019

- $z = f(x, y)$

- continuous partial derivatives in \mathbb{R}

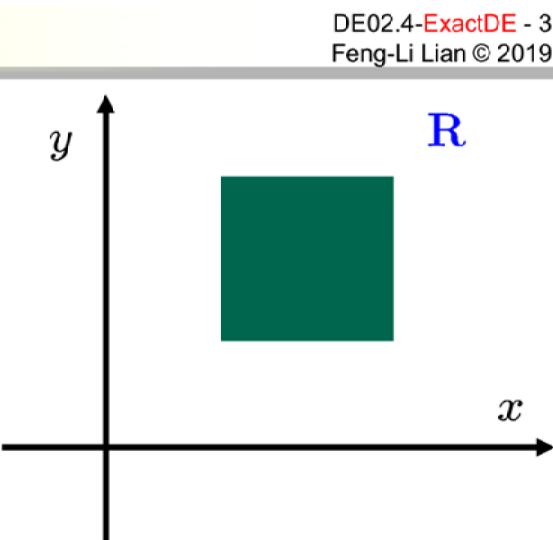
$$\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y}$$

- its differential:

$$\rightarrow dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

- IF: $f(x, y) = c$

$$\Rightarrow \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0$$



$$\frac{\partial f}{\partial x} \frac{\partial f}{\partial y}$$

- A differential expression:
 $M(x, y) dx + N(x, y) dy$
- is an **exact differential** in a region R of the xy -plane
- if it corresponds to the differential of some function $f(x, y)$ defined in R .

- A first-order DE of the form:

$$M(x, y) dx + N(x, y) dy = 0$$

- is said to be an **exact equation** if the expression on the left-hand side is an **exact differential**.

- Let $M(x, y)$ and $N(x, y)$ be continuous and have continuous first partial derivatives in a rectangular region R defined by $a < x < b$, $c < y < d$.

- Then a necessary and sufficient condition that $M(x, y) dx + N(x, y) dy$ be an exact differential is:

$$\left(\frac{\partial M(x, y)}{\partial y} \right) = \left(\frac{\partial N(x, y)}{\partial x} \right)$$

Example 1: Solving an Exact DE

DE02.4-ExactDE - 6
Feng-Li Lian © 2019

$$2xy \, dx + (x^2 - 1) \, dy = 0$$

M

$$\frac{\partial M}{\partial y} = \frac{\partial (2xy)}{\partial y} = 2x$$

$$\frac{\partial N}{\partial x} = \frac{\partial (x^2 - 1)}{\partial x} = 2x$$

$$M = 2xy = \frac{\partial f}{\partial x} \Rightarrow$$

$$N = x^2 - 1 = \frac{\partial f}{\partial y} \Rightarrow$$

$$(x^2 - 1) \frac{\partial y}{\partial x} + 2xy = 0$$

$$\begin{aligned} f &= \int 2xy \, dx = x^2y + h(y) \\ f &= \int (x^2 - 1) \, dy = \underline{x^2y} + g(x) \\ &\Rightarrow x^2y - y + C_1 = f = C_2 \\ &\Rightarrow x^2y - y + C = 0 \end{aligned}$$

Example 2: Solving an Exact DE

DE02.4-ExactDE - 7
Feng-Li Lian © 2019

$$(e^{2y} - y \cos(xy)) dx + (2x e^{2y} - x \cos(xy) + 2y) dy = 0$$

M

N

$$\frac{\partial M}{\partial y} = 2e^{2y} - \cos(xy) + xy \sin(xy)$$

$$\frac{\partial N}{\partial x} = 2e^{2y} - \cos(xy) + yx \sin(xy)$$

$$\frac{\partial f}{\partial x} = M = e^{2y} - y \cos(xy) \Rightarrow f = \underline{e^{2y} x} - \cancel{\sin(xy)} + h(y)$$

$$\frac{\partial f}{\partial y} = N = 2xe^{2y} - x \cos(xy) + 2y \Rightarrow f = \underline{xe^{2y}} - \cancel{\sin(xy)} + y^2 + g(x)$$

$$f(x, y) = xe^{2y} - \sin(xy) + y^2 + C_1$$

$$M(x, y) dx + N(x, y) dy = 0$$

$$\left[\frac{\partial M(x, y)}{\partial y} - \frac{\partial N(x, y)}{\partial x} \right] \neq 0$$

$\mu(x, y)$: Integrating Factor

$$\Rightarrow \mu(x, y) M(x, y) dx + \mu(x, y) N(x, y) dy = 0$$

Exact Equation !

$$\Rightarrow (\underline{\mu M})_y = (\underline{\mu N})_x$$

$$\Rightarrow \underline{\mu_y M} + \underline{\mu M_y} = \underline{\mu_x N} + \underline{\mu N_x}$$

$$\begin{aligned} \Rightarrow \cancel{\mu_x N} - \cancel{\mu_y M} &= -\mu N_x + \mu M_y \\ &= \mu(M_y - N_x) \end{aligned}$$

$$\bullet \text{ IF } \underline{\mu(x,y)} = \underline{\mu(x)} \quad \Rightarrow \quad \underline{\mu_x} = \underline{\frac{d\mu}{dx}}$$

$$\Rightarrow \underline{\frac{d\mu}{dx} N} = \underline{\mu} (\underline{M_y} - \underline{N_x})$$

$$\Rightarrow \boxed{\underline{\frac{d\mu}{dx}} = \frac{(\underline{M_y} - \underline{N_x})}{\underline{N}} \underline{\mu}}$$

$$\Rightarrow \underline{\left(\frac{d\mu}{\mu} \right)} = \frac{(\underline{M_y} - \underline{N_x})}{\underline{N}} \underline{dx}$$

$$\Rightarrow \underline{\ln \mu} = \int \frac{(\underline{M_y} - \underline{N_x})}{\underline{N}} \underline{dx}$$

$$\Rightarrow \underline{\mu(x)} = e^{\int \frac{(\underline{M_y} - \underline{N_x})}{\underline{N}} \underline{dx}}$$

$$\bullet \text{ IF } \mu(x, y) = \underline{\mu(y)} \Rightarrow \mu_y = \frac{d\mu}{dy}$$

$$\Rightarrow \underline{\frac{d\mu}{dy} M} = \mu (N_x - M_y)$$

$$\Rightarrow \underline{\underline{\frac{d\mu}{dy}}} = \frac{(N_x - M_y)}{M} \underline{\underline{\mu}}$$

$$\Rightarrow \underline{\underline{\frac{d\mu}{\mu}}} = \frac{(N_x - M_y)}{M} \underline{\underline{dy}}$$

$$\Rightarrow \ln \mu = \int \frac{(N_x - M_y)}{M} dy$$

$$\Rightarrow \underline{\underline{\mu(y)}} = e^{\int \frac{(N_x - M_y)}{M} dy}$$

Example 4: A Nonexact DE Made Exact

$$x^3 y^3 dx + y^3 (2x^2 + 3y^2 - 20) dy = 0$$

$$\frac{\partial M}{\partial y} = X \quad *$$

$$\frac{\partial N}{\partial x} = 4X$$

$$\frac{My - Nx}{N} = \frac{x - 4x}{2x^2 + 3y^2 - 20}$$

$$\frac{Nx - xMy}{M} = \frac{4x - x}{xy} = \frac{3}{y}$$

$$\mu(y) = e^{\int \frac{3}{y} dy} = e^{3 \ln y} = y^3$$

$$xy^4 dx + (2x^2 y^3 + 3y^5 - 20y^3) dy = 0$$

$$\frac{\partial M}{\partial y} = 4x y^3 \quad ||$$

$$\frac{\partial N}{\partial x} = 4x y^3$$

Summary

DE02.4-ExactDE - 12
Feng-Li Lian © 2019

$$\boxed{M(x,y)dx + N(x,y)dy = 0} \Rightarrow d(f(x,y)) = 0$$

$$\Leftrightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow f(x,y) = c$$

- Nonexact Equations: (2.4)

$$M(x,y)dx + N(x,y)dy = 0$$

$$\Rightarrow \boxed{\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0}$$

$$\Leftrightarrow (\mu M)_y = (\mu N)_x$$

$$\Rightarrow \mu(x) = e^{\int \frac{My-Nx}{N} dx}$$

$$\text{or } \mu(y) = e^{\int \frac{Nx-My}{M} dy}$$