

Fall 2019

微分方程 Differential Equations

Unit 02.3 Linear Equations

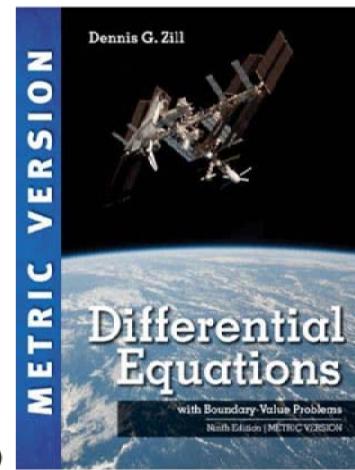
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NTU-EE

Sep19 – Jan20

$$y(x) = y_c(x) + y_p(x)$$

Figures and images used in these lecture notes are adopted from
[Differential Equations with Boundary-Value Problems](#), 9th Ed., D.G. Zill, 2018 (Metric Version)



- 2.1: Solution Curves without a Solution
 - 2.1.1: Direction Fields
 - 2.1.2: Autonomous First-Order DEs
- 2.2: Separable Equations
- 2.3: Linear Equations
- 2.4: Exact Equations
- 2.5: Solutions by Substitutions
- 2.6: A Numerical Method

Definition 2.3.1: Linear Equation

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- A first-order DE of the form:

$$a_1(x) \frac{dy}{dx} + a_0(x)y = g(x)$$

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- is said to be a **linear equation** in the variable y .

$$g(x) = 0 \quad \blacksquare \text{ homogeneous}$$

$$g(x) \neq 0 \quad \blacksquare \text{ nonhomogeneous}$$

- Standard form:

$$\frac{dy}{dx} + P(x)y = f(x)$$

Method of Solution

- Standard form:

$$\frac{dy}{dx} + P(x)y = f(x)$$

(a) $\frac{dy_c}{dx} + P(x)y_c = 0$: $y_c(x)$

■ Complementary function
homogeneous solution

(b) $\frac{dy_p}{dx} + P(x)y_p = f(x)$: $y_p(x)$

■ Particular solution of
nonhomogeneous equation

$$\Rightarrow y(x) = y_c(x) + y_p(x)$$

$$f(x, y) = g(x) h(y)$$

■ Standard form:

$$\frac{dy}{dx} + P(x) y = f(x)$$

$$\begin{aligned}\frac{dy}{dx} + P(x) y &= \frac{d(y_c + y_p)}{dx} + P(x)(y_c + y_p) \\ &= \frac{d(y_c)}{dx} + P(x)(y_c) \\ &\quad + \frac{d(y_p)}{dx} + P(x)(y_p) \\ &= 0 \quad + f(x) \\ &= f(x)\end{aligned}$$

Method of Solution: Homogeneous Solution

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$$(a) \frac{dy_c}{dx} + P(x) y_c = 0$$

$$\int \frac{1}{y_c} dy_c = \int -P(x) dx$$

$$\ln |y_c| = \left(\int -P(x) dx + C \right)$$

$$|y_c| = e^{\int -P(x) dx + C}$$

$$y_c = (\pm e^C) e^{-\int P(x) dx}$$

$$y_c = C e^{-\int P(x) dx}$$

$$y_c = C y_1$$

Method of Solution: Particular Solution

$$(b) \frac{dy_p}{dx} + P(x)y_p = f(x)$$

$$y_p = u(x)y_1(x)$$

$$\frac{d}{dx}(u y_1) + P(u y_1) = f$$

$$y_1 \frac{du}{dx} + u \frac{dy_1}{dx} + Puy_1 = f$$

$$y_1 \frac{du}{dx} + u \left[\frac{dy_1}{dx} + Py_1 \right] = f$$

$$u = \int \frac{f}{y_1} dx$$

$$y_p = \left(\int \frac{f}{y_1} dx \right) y_1 - (P(x)) y_1$$

$$y = y_p + y_c$$

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Example 1: Solving a Linear DE

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$$\frac{dy}{dx} - 3y = 0$$

$$\frac{dy}{dx} = 3y \quad \left| \int \frac{dy}{y} = \int 3 dx \right.$$

$$\text{I.F.} = e^{\int P(x)dx} = e^{\int (-3)dx} = e^{-3x}$$

$$y(x) = C y_1(x) = C e^{-\int P(x)dx} = C e^{-\int (-3)dx} = Ce^{3x}$$

$$\begin{aligned} \frac{dy}{dx} &= 3C e^{3x} \\ \rightarrow y &= -3C e^{3x} \end{aligned} = 0$$

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$$\ln|y| = 3x + C$$

$$Ce^{3x}$$

Example 2: Solving a Linear DE

$$\frac{dy}{dx} + 3y = 6$$

$y_c(x) = C e^{3x}$

$y_p(x) = u(x) e^{3x}$

$\frac{dy_p}{dx} = \frac{d}{dx} u(x) e^{3x} + u(x) 3e^{3x}$

$-3y_p = -3u(x)e^{3x}$

$\frac{d}{dx} u e^{3x} = 6$

$\frac{du}{dx} = 6e^{-3x}$

$du = 6e^{-3x} dx$

$u = \int 6e^{-3x} dx + C_1$

$u = -2e^{3x} + C_1$

$y = y_c + y_p$

$= C e^{3x} + (-2 + C_1 e^{3x})$

$= C_2 e^{3x} - 2$

Example 3: General Solution

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$$x \left[\frac{dy}{dx} \right] - 4y = x^6 e^x \quad P(x)$$

$$\Rightarrow \frac{1}{x} \frac{dy}{dx} - \frac{4}{x} y = x^5 e^x$$

$$\text{I.F.} = e^{\int P(x) dx} = e^{\int (-\frac{4}{x}) dx}$$

$$= e^{(-4) \ln x} = (e^{\ln x})^{-4} = (x)^{-4}$$

$$x^{-4} \frac{dy}{dx} - x^{-4} \cdot \frac{4}{x} y = x^{-4} x^5 e^x$$

$$\Rightarrow x^{-4} \frac{dy}{dx} - 4x^{-5} y = x e^x$$

$$\frac{d}{dx}[x^{-4} y] = x e^x$$

$$x^{-4} y = \int x e^x = x e^x - e^x + C$$

$$y = x^5 e^x - e^x + C x^4$$

Singular Points

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$$\underline{a_1(x) \frac{dy}{dx} + a_0(x)y = g(x)} \\ \equiv 0$$

$$\Rightarrow a_1(x) = 0 \quad \rightarrow x : \underline{\text{Singular Points}}$$

$$\Rightarrow P(x) = \frac{a_0(x)}{a_1(x)} \quad \rightarrow P(x) : \underline{\text{discontinuous at}} \quad x$$

Example 4: General Solution

$$(x^2 - 9) \frac{dy}{dx} + xy = 0$$

$$\div(x^2 - 9) \Rightarrow \frac{dy}{dx} + \frac{x}{x^2 - 9} y = 0$$

$$\begin{aligned} \text{I.F.} &= e^{\int P(x) dx} = e^{\int \frac{x}{x^2 - 9} dx} \\ \text{(1)} \quad (x^2 - 9 > 0) \quad &= e^{\frac{1}{2} \ln(x^2 - 9)} = (e^{\ln(x^2 - 9)})^{\frac{1}{2}} = \sqrt{x^2 - 9} \end{aligned}$$

$$\sqrt{x^2 - 9} \frac{dy}{dx} + \sqrt{x^2 - 9} \frac{x}{x^2 - 9} y = 0$$

$$\frac{d}{dx} [(\sqrt{x^2 - 9}) y] = 0 \Rightarrow \sqrt{x^2 - 9} y = C$$

$$y = \frac{C}{\sqrt{x^2 - 9}}$$

$$x^2 - 9 \neq 0$$

$$x \neq \pm 3$$

$$I = (-\infty, -3), (-3, 3), (3, +\infty)$$

①

②

③

$$\sqrt{x^2 - 9}$$

$x < -3$ ①
 $x > 3$ ③

$$y = \frac{C}{\sqrt{9 - x^2}} \quad -3 < x < 3 \quad \text{④}$$

Example 5: An IVP

$$\frac{dy}{dx} + Qy = x \quad y(0) = 4$$

$$P(x)=1 \\ I.F. = e^{\int P(x)dx} = e^{\int 1 dx} = e^x$$

$$\underbrace{e^x \frac{dy}{dx} + e^x y}_{\frac{d}{dx}[e^x y]} = e^x x$$

$$\frac{d}{dx}[e^x y] = e^x x$$

$$e^x y = \int e^x = xe^x - e^x + C$$

$$y(x) = x - 1 + Ce^{-x}$$

$$y(0)=4, \Rightarrow y(0)=0-1+Ce^0=4 \Rightarrow C=5$$

$$y(x) = \underline{x-1} + \underline{5e^{-x}}$$

$$\text{erf}(x) = \left[\frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \right]$$

■ Error function

$$\text{erfc}(x) = \left[\frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt \right]$$

■ Complementary Error function

$$\frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} dt = 1$$

$$\Rightarrow \begin{cases} \underbrace{\text{erf}(x)} + \underbrace{\text{erfc}(x)} = 1 \\ \underbrace{\text{erf}(0)} = 0 \\ \underbrace{\text{erfc}(0)} = 1 \end{cases}$$

Example 7: The Error Function

$$\frac{dy}{dx} - 2x y = 2$$

$$y(0) = 1$$

$$\text{I.F.} = e^{\int P(x) dx} = e^{\int -2x dx} = e^{-x^2}$$

$$e^{-x^2} \frac{dy}{dx} - 2x e^{-x^2} y = 2 e^{-x^2}$$

$$\frac{d}{dx}[e^{-x^2} y] = 2 e^{-x^2}$$

$$e^{-x^2} y = \int 2 e^{-x^2} dx + C = 2 \int e^{-x^2} dx + C$$

$$y(x) = 2 e^{x^2} \int_0^x e^{-t^2} dt + C e^{x^2}$$

$$= 2 e^{x^2} \frac{\sqrt{\pi}}{2} \operatorname{erf}(x) + C e^{x^2}$$

$$y(0) = 1, \quad y(0) = 2 e^{0^2} \frac{\sqrt{\pi}}{2} \operatorname{erf}(0) + C e^{0^2} = 1 \Rightarrow C = 1, \quad -e^{x^2} (\operatorname{erf}(x))$$

$x \rightarrow t$

$$= 2 \int_0^x e^{-t^2} dt + C$$

$$y(x) = e^{\frac{x^2 \pi}{2}} \operatorname{erf}(\frac{x}{\sqrt{2}}) + C e^{x^2}$$

Summary

$$\frac{dy}{dx} = f(x, y)$$

$$y +$$

$$I.F. = e^{\int P(x) dx}$$

$$y(x) = c e^{-\int P(x) dx} + e^{-\int P(x) dx} \int e^{\int P(x) dx} f(x) dx$$

$$\Rightarrow \frac{dy}{dx} + P(x)y = f(x)$$

$$\Rightarrow \frac{dy}{dx} + P(x)y = 0$$

$$\Rightarrow \frac{1}{y} dy = -P(x) dx$$

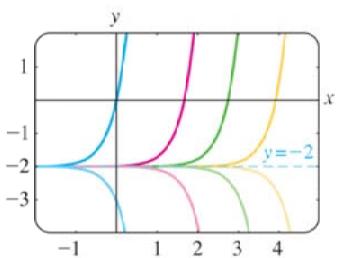
$$\Rightarrow y_1(x) = e^{-\int P(x) dx}$$

$$\Rightarrow \begin{cases} y_c(x) = c y_1(x) \\ y_p(x) = u(x) y_1(x) \end{cases}$$

Summary

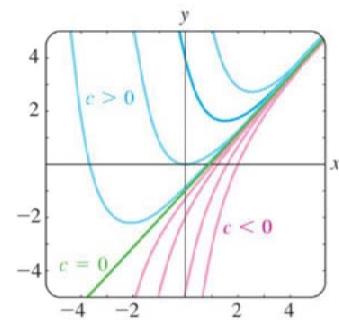
Example 2:

$$\frac{dy}{dx} - 3y = 6 \quad y = -2 + ce^{3x}$$



Example 5:

$$\frac{dy}{dx} + y = x \quad y = x - 1 + ce^{-x}$$
$$y(0) = 4$$



Example 7:

$$\frac{dy}{dx} - 2xy = 2$$
$$y(0) = 1 \quad y = e^{x^2} [1 + \sqrt{\pi} \operatorname{erf}(x)]$$

