

Fall 2019

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# 微分方程 Differential Equations

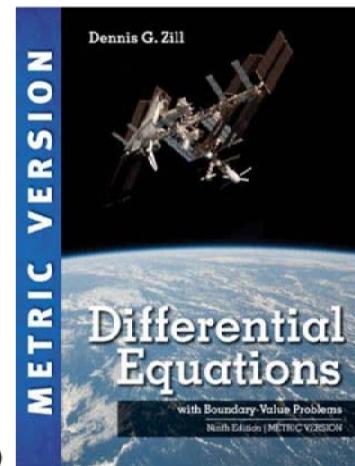
## Unit 02.2 Separable Equations

Feng-Li Lian

NTU-EE

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$$\frac{dy}{dx} = g(x) h(y)$$



Figures and images used in these lecture notes are adopted from  
[Differential Equations with Boundary-Value Problems](#), 9th Ed., D.G. Zill, 2018 (Metric Version)

- 2.1: Solution Curves without a Solution
  - 2.1.1: Direction Fields
  - 2.1.2: Autonomous First-Order DEs
- 2.2: Separable Equations
- 2.3: Linear Equations
- 2.4: Exact Equations
- 2.5: Solutions by Substitutions
- 2.6: A Numerical Method

## Solution by Integration

DE02.2-SeparableDE - 3  
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$$\bullet \boxed{\frac{dy}{dx}} = f(x, y) = \underline{\underline{g(x)}} \quad \text{continuous}$$

$$\rightarrow \boxed{\frac{dy}{dx}} = g(x)$$

$$\rightarrow \int \frac{dy}{dx} dx = \int g(x) dx$$

$$\rightarrow y(x) = G(x) + c$$

- A first-order DE of the form:

$$\frac{dy}{dx} = f(x, y) = \boxed{g(x)} \boxed{h(y)}$$

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- is said to be **separable** or to have **separable variables**.

IF  $\underline{h(y) \neq 0}, \forall y \in I$  ✓✓✓

$$\Rightarrow \int \frac{1}{h(y)} dy = \int g(x) dx$$

$$\Rightarrow H(y) = G(x) + C$$

$$\Rightarrow y = \phi(x)$$

### Definition 2.2.1: Separable Equations

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$$\Rightarrow \boxed{\frac{1}{h(y)}} dy = g(x) dx$$

$$\Rightarrow \underline{p(y)} dy = g(x) dx$$

IF  $y(x) = \phi(x)$  a solution of the DE

$$\Rightarrow dy = \phi'(x) dx$$

$$\Rightarrow p(\phi(x)) \phi'(x) dx = g(x) dx$$

$$\Rightarrow \int p(\phi(x)) \phi'(x) dx = \int g(x) dx$$

$$\Rightarrow \int p(y) dy = \int g(x) dx$$

$$\Rightarrow H(y) + C_1 = G(x) + C_2$$

$$\Rightarrow H(y) = G(x) + C$$

$$\Rightarrow y(x) = \phi(x)$$

## Example 1: Solving a Separable DE

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$$(1 + x) dy - y dx = 0$$

$$\frac{dy}{dx} = \frac{y}{(1+x)} = \frac{1}{1+x} y$$

$$\int \frac{1}{y} dy = \int \frac{1}{1+x} dx$$

$$\begin{aligned} & \frac{1}{y} \neq 0 \\ & 1/x \neq -1 \\ & y \neq 0 \end{aligned}$$

$$\ln|y| = \ln|1+x| + C_1$$

$$|y| = e^{\ln|1+x| + C_1} = e^{\ln|1+x|} e^{C_1} = |1+x| e^{C_1} = e^{C_1} |1+x|$$

$$y = \pm e^{C_1} (1+x)$$

$$y = C (1+x) \quad x \in (-\infty, -1), (-1, +\infty)$$

## Example 2: Solution Curve

$$\frac{dy}{dx} = -\frac{x}{y} \quad y(4) = -3$$

$$\frac{dy}{dx} = -(x)(\frac{1}{y})$$

$$\int y \, dy = \int x \, dx$$

$$\frac{1}{2}y^2 + C_1 = -\frac{1}{2}x^2 + C_2 \quad = C$$

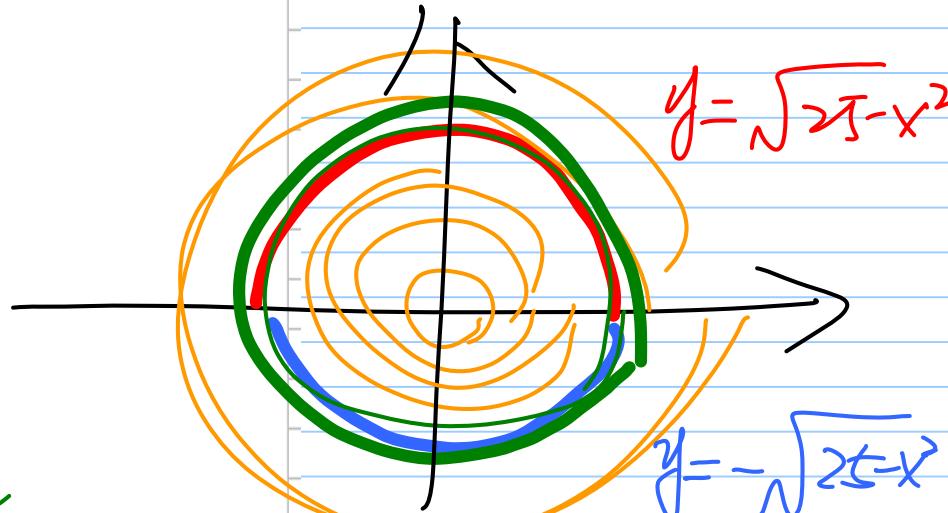
$$y^2 = -x^2 + [2C_2 - 2C_1]$$

$$y^2 + x^2 = C$$

$$y(4) = -3, \quad (-3)^2 + (4)^2 = C \Rightarrow C = 25$$

$$x^2 + y^2 = 25$$

$$x^2 + y^2 = C \geq 0$$



$$x^2 + y^2 = 25$$

### Example 3: Losing a Solution

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$$\frac{dy}{dx} = y^2 - 4 = 0 \quad y = \pm 2$$
$$= f(x, y) \quad y > 2, y < -2 \Rightarrow f > 0$$
$$-2 < y < 2 \Rightarrow f < 0$$

$$y^2 - 4 \neq 0$$
$$\int \frac{1}{y^2 - 4} dy = \int 1 dx$$
$$\Rightarrow \int \left( \frac{1}{y-2} - \frac{1}{y+2} \right) dy = x + C_1$$
$$\Rightarrow \frac{1}{4} \ln|y-2| - \frac{1}{4} \ln|y+2| + C_2 = x + C_1$$

$$\frac{1}{4} \ln \left| \frac{y-2}{y+2} \right| = X + \underline{\underline{C_1 - C_2}} \\ = C_3$$

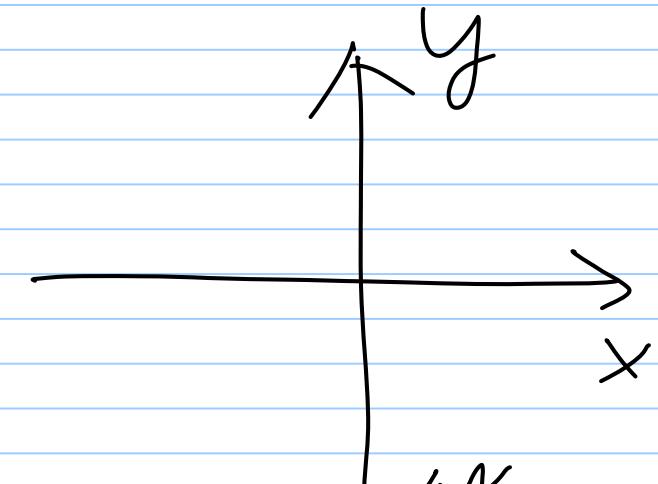
$$\ln \left| \frac{y-2}{y+2} \right| = 4X + 4C_3$$

$$\left| \frac{y-2}{y+2} \right| = e^{4X + 4C_3} = c$$

$$\frac{y-2}{y+2} = \underline{\pm e^{4C_3}} e^{4X}$$

$$\frac{y-2}{y+2} = ce^{4X}$$

$$\Rightarrow y = 2 \frac{1 + ce^{4X}}{1 - ce^{4X}}$$

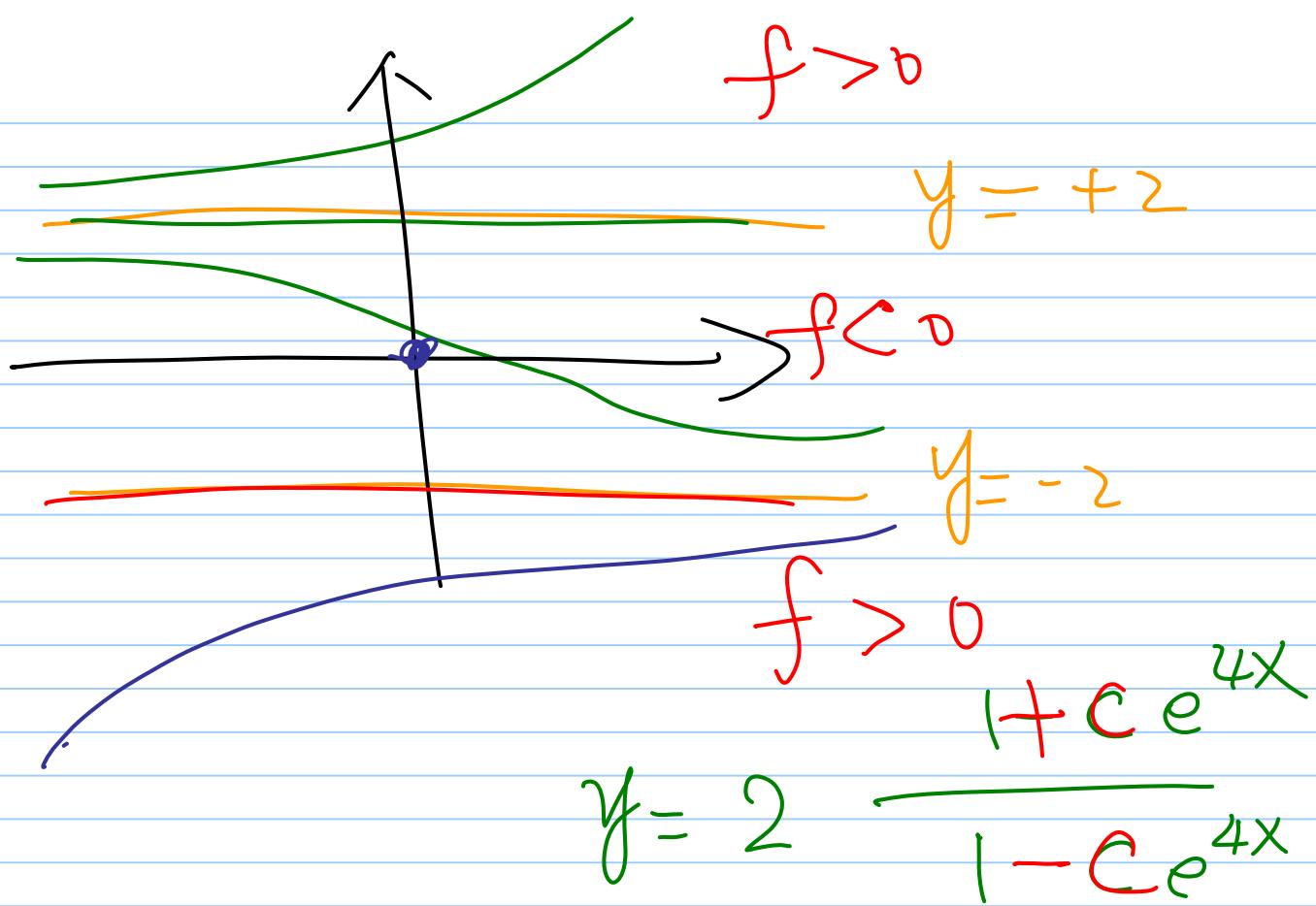


$$X \rightarrow -\infty \quad e^{4X} \rightarrow 0$$

$$\Rightarrow y \rightarrow 2$$

$$X \rightarrow +\infty \quad e^{-4X} \rightarrow 0$$

$$y = 2 \frac{e^{-4X} + c}{e^{-4X} - c} \Rightarrow -2$$



## Example 5: Solution by Integral-Defined Function

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$$\frac{dy}{dx} = e^{-x^2}$$

$y(3) = 5$

$$\int_3^x \frac{dy}{dt} dt = \int_0^x e^{-t^2} dt$$

$$x = 3 \rightarrow \infty$$

$$t = 3 \rightarrow \infty$$

$$y(x) - y(3) = \int_3^x e^{-t^2} dt$$

$$y(x) = y(3) + \int_3^x e^{-t^2} dt$$



$$\underline{\text{erf}(x)} = \left( \frac{2}{\sqrt{\pi}} \right) \int_0^x e^{-t^2} dt$$

$$y = \frac{1}{2} \cancel{\frac{2}{\sqrt{\pi}}} \text{erf}(x) + C$$

$$Q - x^2$$

$$y(3) = 5$$

$$5 = \frac{\sqrt{\pi}}{2} \text{erf}(3) + C$$

$$y = \frac{\sqrt{\pi}}{2} \text{erf}(x) + \frac{\sqrt{\pi}}{2} \text{erf}(3) + 5$$

$$= \frac{\sqrt{\pi}}{2} \text{erf}(x)$$

## Summary

$$\frac{dy}{dx} = f(x, y)$$

$$\Rightarrow \frac{dy}{dx} = g(x) h(y)$$

$$\Rightarrow \left\{ \frac{1}{h(y)} dy \right\} = g(x) dx$$

$$\Rightarrow H(y) = G(x) + C$$

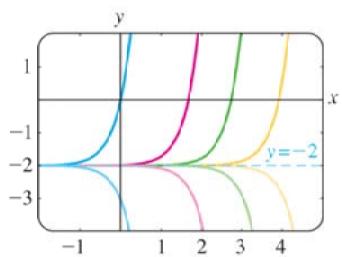
$$\Rightarrow y = \underline{\underline{\Phi(x)}}$$

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## Summary

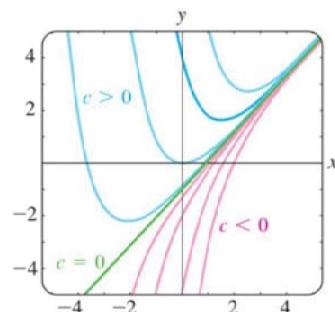
Example 2:

$$\frac{dy}{dx} - 3y = 6 \quad y = -2 + c e^{3x}$$



Example 5:

$$\frac{dy}{dx} + y = x \quad y = x - 1 + c e^{-x}$$
$$y(0) = 4$$



Example 7:

$$\frac{dy}{dx} - 2xy = 2$$
$$y(0) = 1 \quad y = e^{x^2} [1 + \sqrt{\pi} \operatorname{erf}(x)]$$

