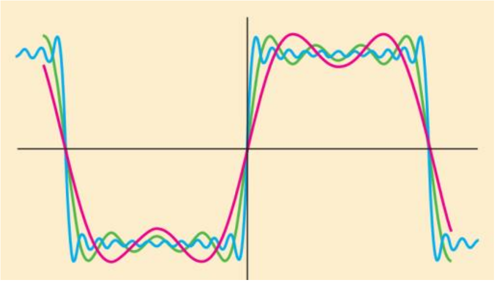

Fall 2019

微分方程
Differential Equations

Unit 11.3
Fourier Cosine and Sine Series

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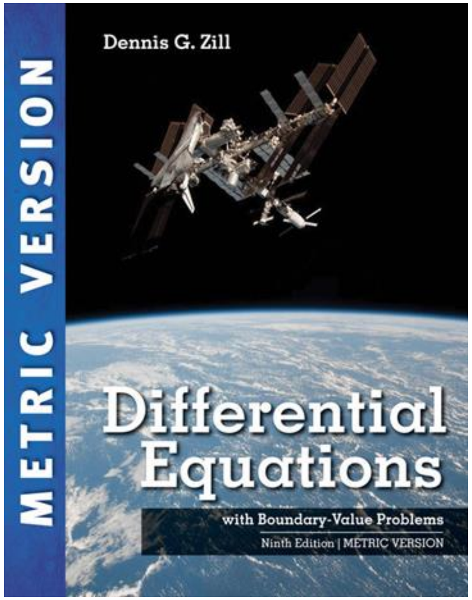
NTU-EE

Sep19 – Jan20

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos \frac{n\pi}{p} x$$

$$f(x) = \sum_{n=1}^{\infty} b_n \cdot \sin \frac{n\pi}{p} x$$



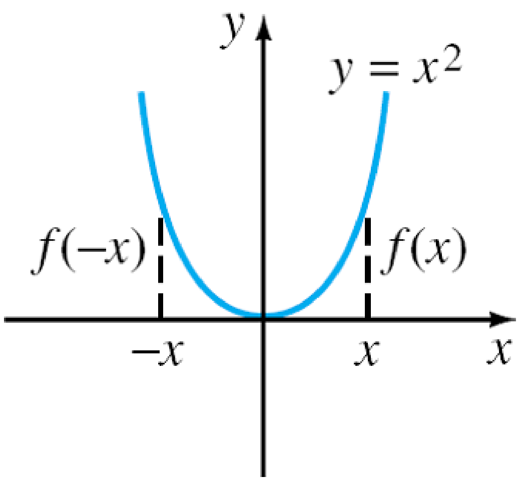
Figures and images used in these lecture notes are adopted from
Differential Equations with Boundary-Value Problems, 9th Ed., D.G. Zill, 2018 (Metric Version)

Outline

- Even Functions:

$$f(-x) = f(x)$$

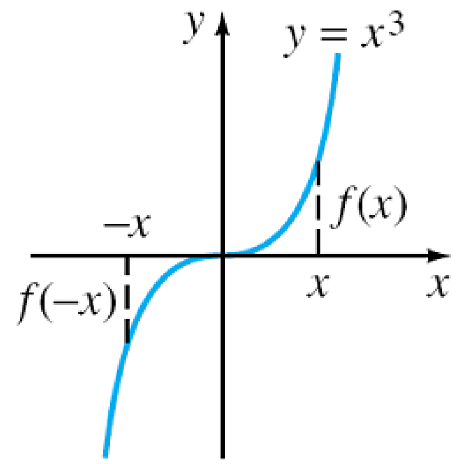
symmetric about y-axis



- Odd Functions:

$$f(-x) = -f(x)$$

symmetric about the origin



- Example:

$$\cos x, \quad \cos nx, \quad \cos \frac{n\pi}{p} x,$$

$$\sin x, \quad \sin nx, \quad \sin \frac{n\pi}{p} x,$$

$$e^{x}, \quad e^{-x}$$

11.3: Theorem 11.3.1: Properties of Even/Odd Functions

(a) $f_e(x) \cdot g_e(x)$

(b) $f_o(x) \cdot g_o(x)$

(c) $f_o(x) \cdot g_e(x)$

(d) $f_e(x) \pm g_e(x)$

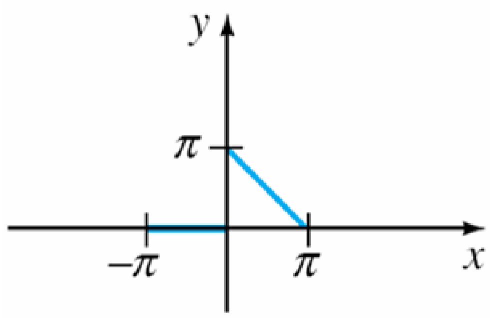
(e) $f_o(x) \pm g_o(x)$

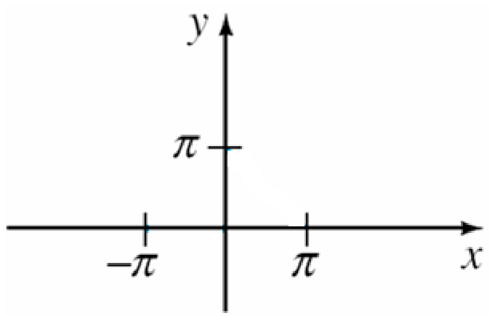
(f) $\int_{-a}^{a} f_e(x)\, dx =$

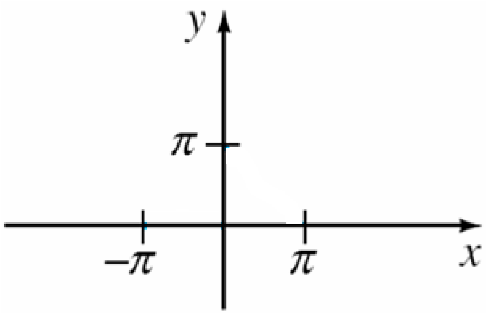
(g) $\int_{-a}^{a} f_o(x)\, dx =$

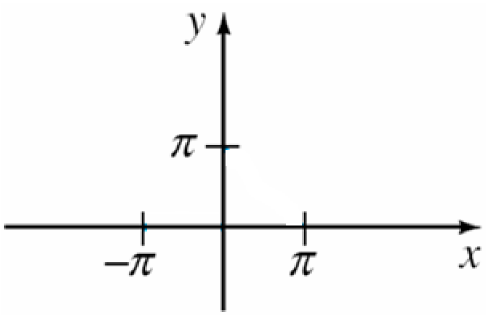
11.3: Even-Odd Decomposition

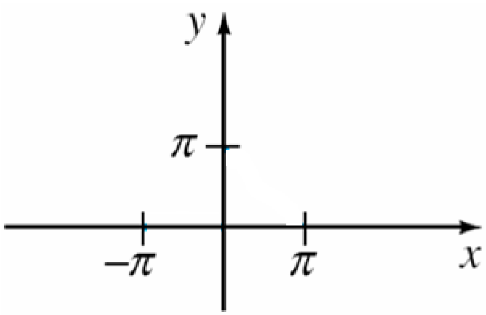
$f(x) \quad =$











11.3: Def 11.6: Fourier Cosine and Sine Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cdot \cos \frac{n\pi}{p} x + b_n \cdot \sin \frac{n\pi}{p} x \right)$$

- IF $f(x)$: even function on $(-p, p)$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos \frac{n\pi}{p} x$$

$$a_0 = \frac{2}{p} \int_0^p f(x)\, dx$$

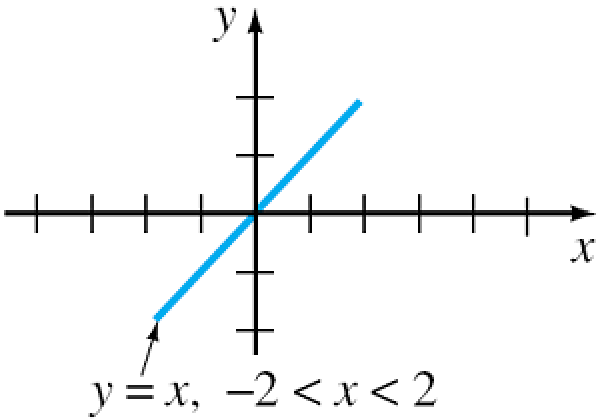
$$a_n = \frac{2}{p} \int_0^p f(x) \cdot \cos \frac{n\pi}{p} x \, dx$$

- IF $f(x)$: odd function on $(-p, p)$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} b_n \cdot \sin \frac{n\pi}{p} x \qquad b_n = \frac{2}{p} \int_0^p f(x) \cdot \sin \frac{n\pi}{p} x \, dx$$

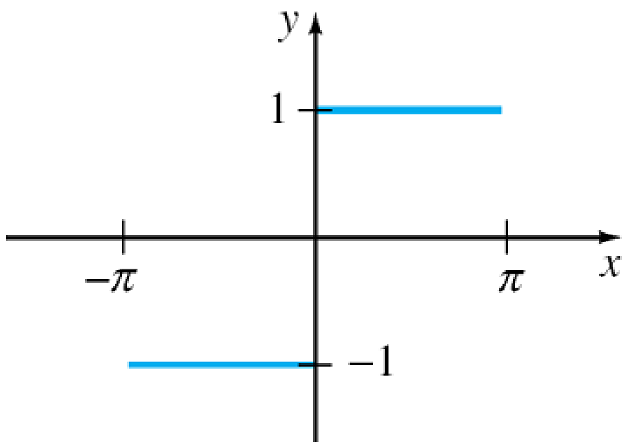
11.3: Example 1

$$f(x) = x, \quad -2 < x < 2$$

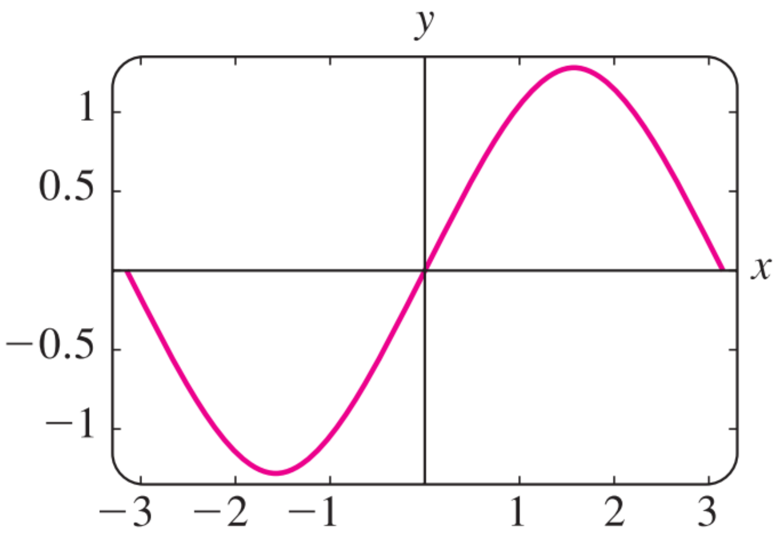


11.3: Example 2

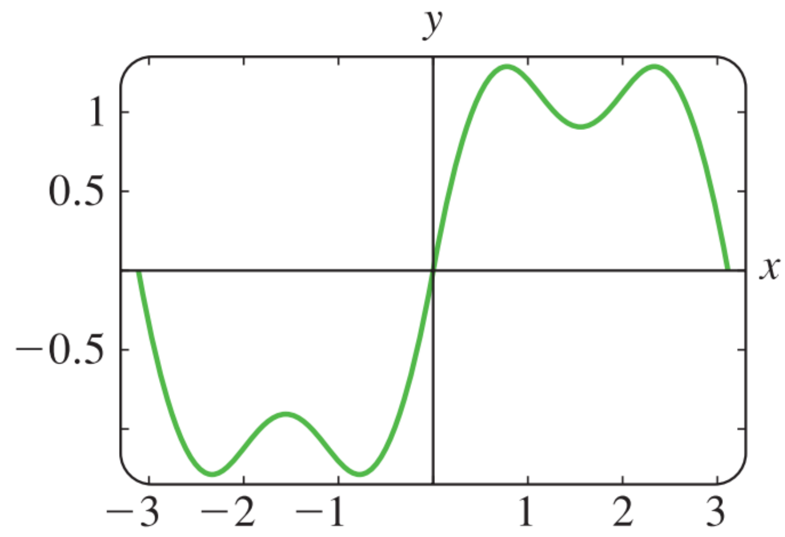
$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 \leq x < \pi \end{cases}$$



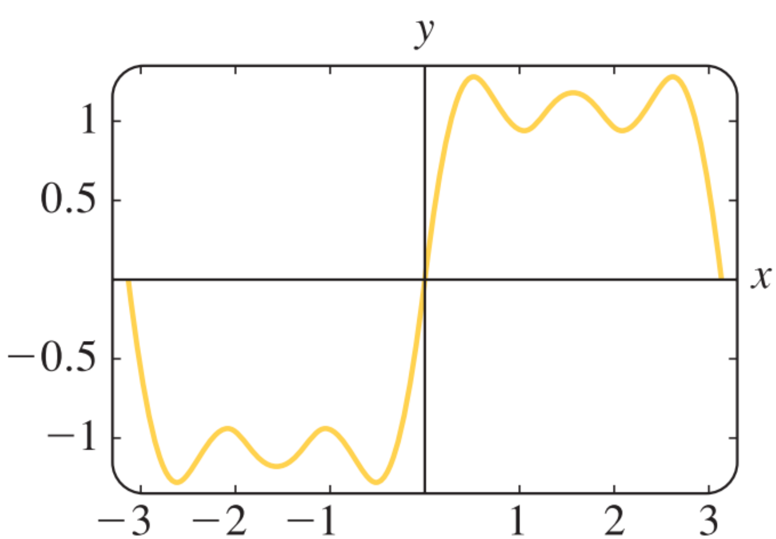
11.3: Gibbs Phenomenon



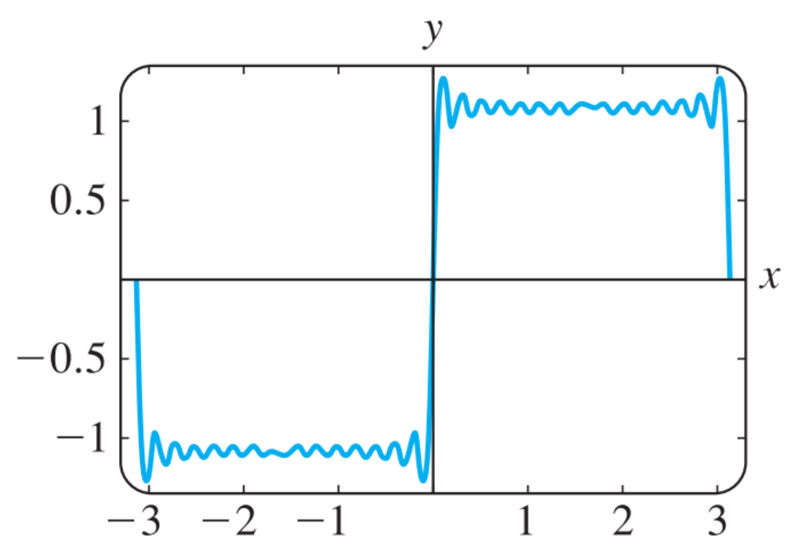
(a) $S_1(x)$



(b) $S_2(x)$

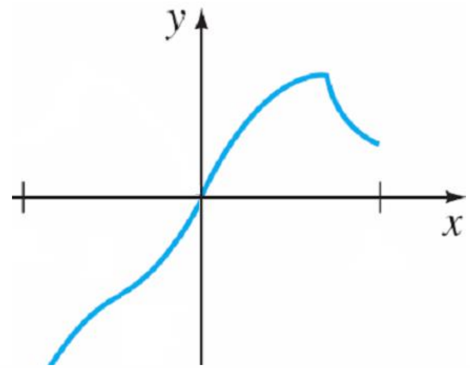


(c) $S_3(x)$

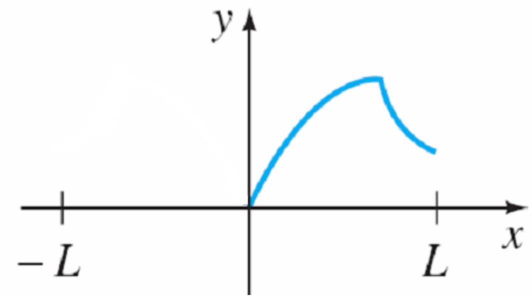


(d) $S_{15}(x)$

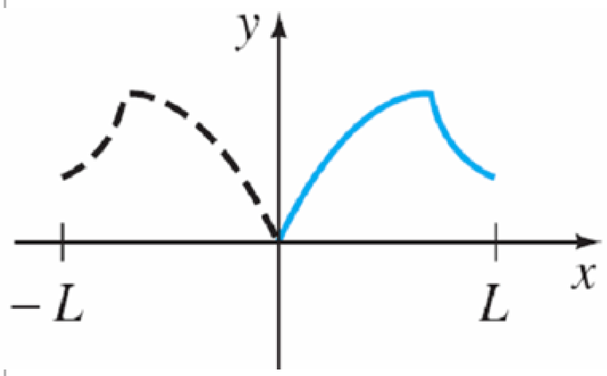
$y = f(x)$ on $x \in (-p, p)$



How about $y = f(x)$ on $x \in (0, L)$

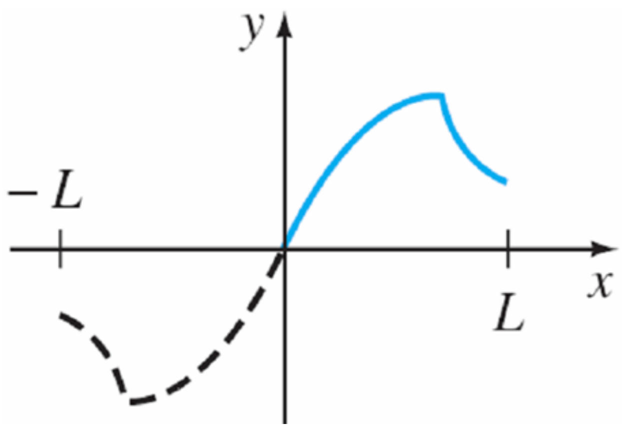


even reflection



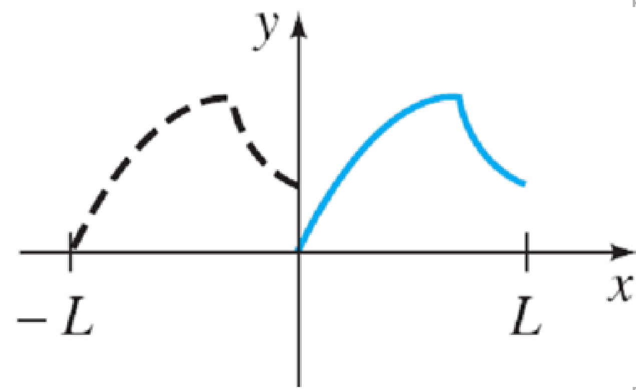
→ cosine series

odd reflection



→ sine series

identity reflection



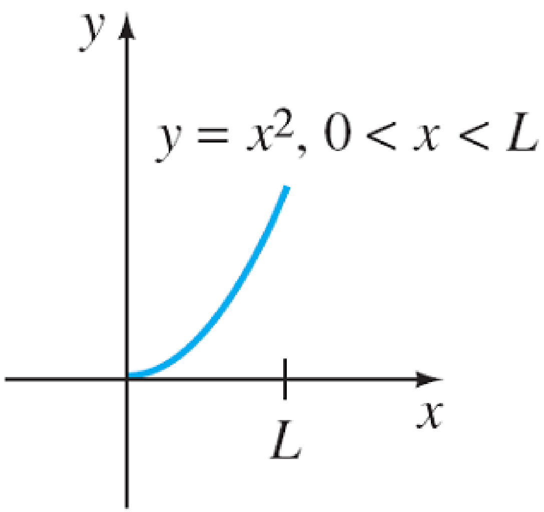
→ Fourier series

Expand $f(x) = x^2, \quad 0 < x < L,$

(a) in a cosine series

(b) in a sine series

(c) in a Fourier series



$$a_0 = \frac{2}{L} \int_0^L f(x)\, dx$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cdot \cos \frac{n\pi}{L} x\, dx$$

$$f(x) = \frac{L^2}{3} + \frac{4L^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi}{L} x$$

11.3: Example 3

$$b_n = \frac{2}{L} \int_0^L f(x) \cdot \sin \frac{n\pi}{L} x \, dx$$

$$f(x) = \frac{2L^2}{\pi} \sum_{n=1}^{\infty} \left\{ \frac{(-1)^{n+1}}{n} + \frac{2}{n^3 \pi^2} \left[(-1)^n - 1 \right] \right\} \sin \frac{n\pi}{L} x$$

$\underline{p = L/2}$

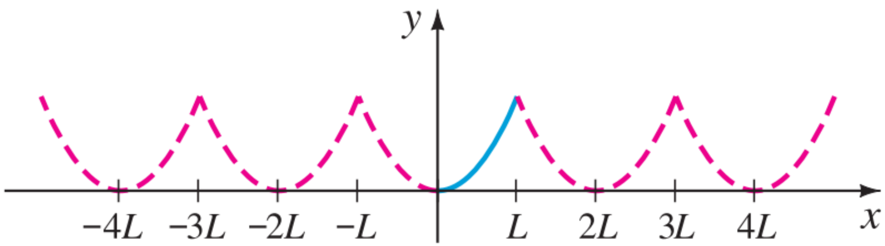
$$a_0 = \frac{2}{L} \int_0^L f(x) \, dx$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cdot \cos \frac{2n\pi}{L} x \, dx$$

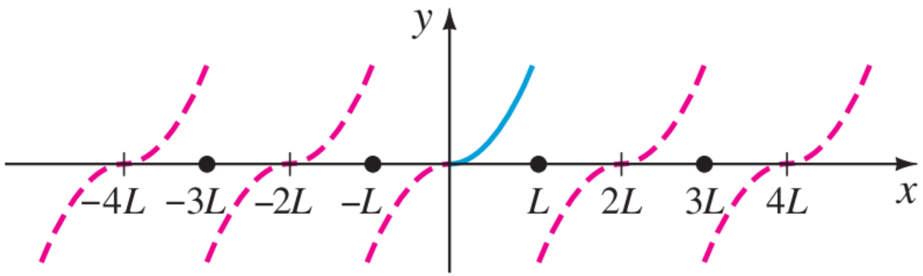
$$b_n = \frac{2}{L} \int_0^L f(x) \cdot \sin \frac{2n\pi}{L} x \, dx$$

11.3: Example 3

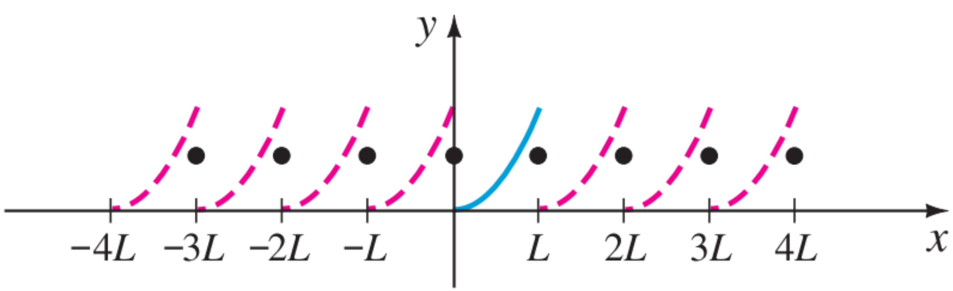
$$f(x) = \frac{L^2}{3} + \frac{L^2}{\pi} \sum_{n=1}^{\infty} \left\{ \frac{1}{n^2\pi} \cos \frac{2n\pi}{L} x - \frac{1}{n} \sin \frac{2n\pi}{L} x \right\}$$



(a) Cosine series



(b) Sine series



(c) Fourier series

Fourier Series $\Rightarrow$ determine a particular solution of a DE

when the driving force $f(t)$ is periodic

- Example: $$m \frac{dx^2(t)}{dt^2} + k\, x(t) = y(t)$$

$$f(x) = \frac{a_0}{2} \cdot 1 + \sum_{n=1}^{\infty} \left(a_n \cdot \cos \frac{n\pi}{T} t + b_n \cdot \sin \frac{n\pi}{T} t \right)$$

$\Rightarrow$ $f_0(t)$: particular solution of $y(t) = 1$

$g_n(t)$: particular solution of $y(t) = \cos \frac{n\pi}{T} t$

$h_n(t)$: particular solution of $y(t) = \sin \frac{n\pi}{T} t$

$$\Rightarrow \; x_p(t) = \frac{a_0}{2} \cdot f_0(t) + \sum_{n=1}^{\infty} \Big(a_n \cdot g_n(t) + b_n \cdot h_n(t) \Big)$$

is the particular solution of $y(t) = f(t)$

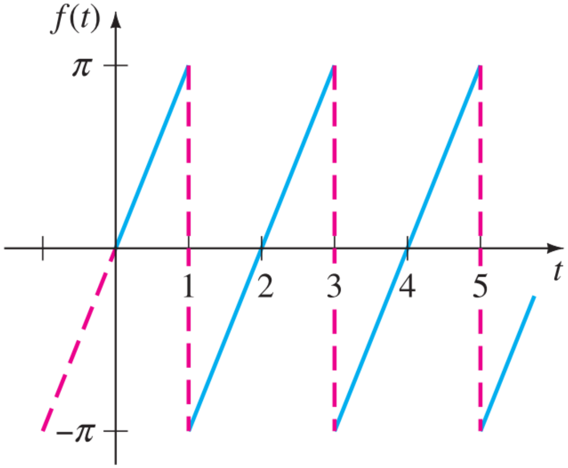
11.3: Example 4

$$1 \frac{dx^2(t)}{dt^2} + 60\, x(t) = f(t)$$

- f(t) is an odd function

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} b_n \cdot \sin \frac{n\pi}{p} x$$

$$b_n = \frac{2}{p} \int_0^p f(x) \cdot \sin \frac{n\pi}{p} x \, dx$$



$$f(t) = \pi t, 0 < t < 1$$

Summary - 11.3: Fourier Cosine and Sine Series

- Even Functions: $f(-x) = f(x)$
- Odd Functions: $f(-x) = -f(x)$
- Even-Odd Decomposition: $f(x) = f_e(x) + f_o(x)$
- Fourier Cosine and Sine Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cdot \cos \frac{n\pi}{p} x + b_n \cdot \sin \frac{n\pi}{p} x \right)$$

- IF $f(x)$: even function on $(-p, p)$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos \frac{n\pi}{p} x$$

- IF $f(x)$: odd function on $(-p, p)$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} b_n \cdot \sin \frac{n\pi}{p} x$$