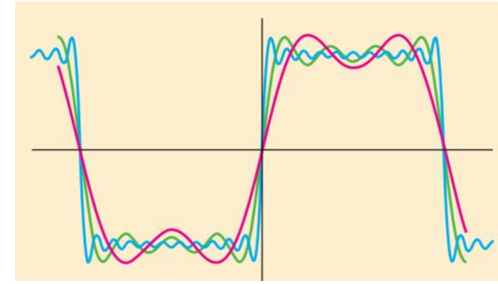

Fall 2019

微分方程
Differential Equations

Unit 11.2
Fourier Series

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NTU-EE

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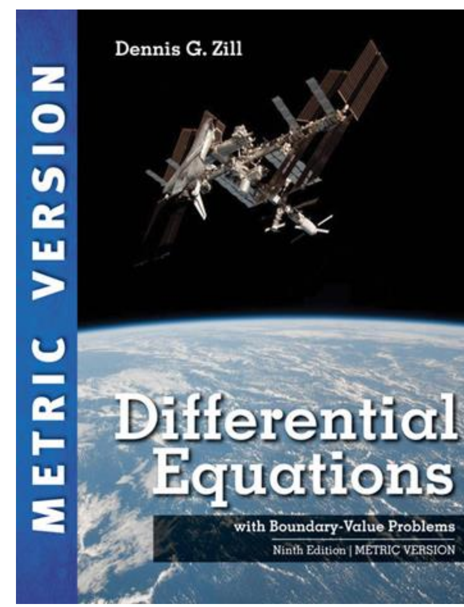
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cdot \cos \frac{n\pi}{p} x + b_n \cdot \sin \frac{n\pi}{p} x \right)$$

Figures and images used in these lecture notes are adopted from
Differential Equations with Boundary-Value Problems, 9th Ed., D.G. Zill, 2018 (Metric Version)



Outline

- Fact:

$$\left\{ 1, \quad \cos\frac{\pi}{p}x, \quad \cos\frac{2\pi}{p}x, \quad \cos\frac{3\pi}{p}x, \quad \cdots \right.$$
$$\left. \sin\frac{\pi}{p}x, \quad \sin\frac{2\pi}{p}x, \quad \sin\frac{3\pi}{p}x, \quad \cdots \right\} \quad \text{on } I = [-p, p]$$

is a complete orthogonal set of trigonometric functions

- A Trigonometric Series for $f(x)$ on $I = [-p, p]$

$$\Rightarrow f(x) = \frac{a_0}{2}\cdot 1 + a_1\cdot\cos\frac{\pi}{p}x + a_2\cdot\cos\frac{2\pi}{p}x + \cdots + a_n\cdot\cos\frac{n\pi}{p}x \cdots$$
$$+ b_1\cdot\sin\frac{\pi}{p}x + b_2\cdot\sin\frac{2\pi}{p}x + \cdots + b_n\cdot\sin\frac{n\pi}{p}x \cdots$$
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cdot\cos\frac{n\pi}{p}x + b_n\cdot\sin\frac{n\pi}{p}x \right)$$

- What are a_n, b_n:

- $n = 0$

$$\left(f(x),\ 1 \right) = \int_{-p}^{p} f(x), \cdot 1 \, dx$$

- $n = m$

$$\left(f(x),\ \cos\frac{m\pi}{p}x \right) = \int_{-p}^{p} f(x), \cdot \cos\frac{m\pi}{p}x \;\, dx$$

$f(x)$ on $I = [\,-p\,,\,p\,]$

$$\Rightarrow\ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cdot \cos\frac{n\pi}{p}x + b_n \cdot \sin\frac{n\pi}{p}x \right)$$

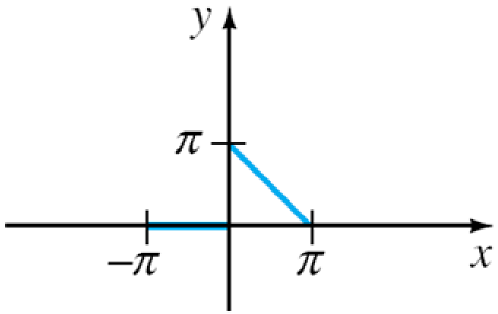
$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x) \cdot \cos\frac{n\pi}{p}x\,dx$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x) \cdot \sin\frac{n\pi}{p}x\,dx$$

11.2: Example 1: Expansion in a Fourier Series

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases}$$



$$p = \pi \qquad \Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cdot \cos \frac{n\pi}{\pi} x + b_n \cdot \sin \frac{n\pi}{\pi} x \right)$$

- f, f' : piecewise continuous on $I = (-p, p)$

  $\rightarrow$ may have discontinuities at a finite number of points

THEN,

$\rightarrow$ at a point of continuity

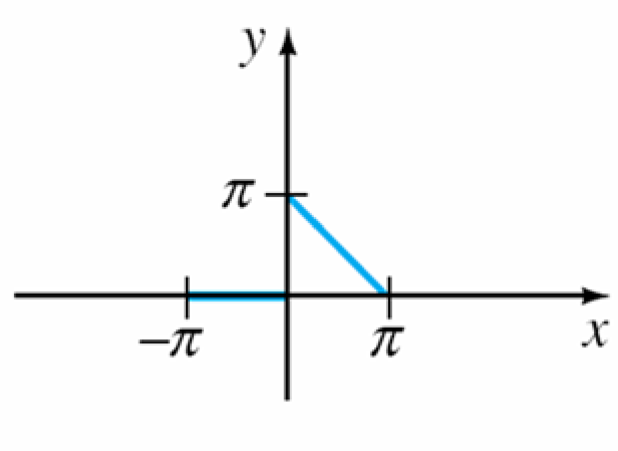
the fourier series of $f(x)$ converges to $f(x)$ at the point

$\rightarrow$ at a point of discontinuity

the fourier series of $f(x)$ converges to the average

$$\frac{f(x^+) + f(x^-)}{2}$$

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases}$$



$$\Rightarrow \frac{f(x^+) + f(x^-)}{2} =$$

$$\Rightarrow f(x)\Big|_{x=0} = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left(\frac{1-(-1)^n}{n^2\,\pi} \cos nx + \frac{1}{n} \sin nx \right)\Bigg|_{x=0}$$

11.2: Periodic Extension

on $I = [\,-p\ ,\ p\,]$

$$\left\{ 1, \quad \cos\frac{\pi}{p}x, \quad \cos\frac{2\pi}{p}x, \quad \cos\frac{3\pi}{p}x, \quad \cdots \cos\frac{n\pi}{p}x, \quad \cdots \right.$$

$$\left. \sin\frac{\pi}{p}x, \quad \sin\frac{2\pi}{p}x, \quad \sin\frac{3\pi}{p}x, \quad \cdots \quad \sin\frac{n\pi}{p}x, \quad \cdots \right\}$$

$\Rightarrow$ Period $=$

$\Rightarrow$ Fundamental Period $=$

- $f(x)$ on $I = [\,-p\,,\,p\,]$

  How about on

  $$[\,-5p\,,\,-3p\,],\quad [\,-3p\,,\,-p\,],\quad [\,p\,,\,3p\,],\quad [\,3p\,,\,5p\,],\quad \cdots$$

$$\Rightarrow\ f(x\qquad\qquad) = \frac{a_0}{2}$$

$$+ \sum_{n=1}^{\infty}\left(a_n \cdot \cos\frac{n\pi}{p}(x\qquad\qquad) + b_n \cdot \sin\frac{n\pi}{p}(x\qquad\qquad)\right)$$

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cdot \cos \frac{n\pi}{p} x + b_n \cdot \sin \frac{n\pi}{p} x \right)$$

$$S_0(x) =$$

- In Example 1:

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left(\frac{1 - (-1)^n}{n^2 \pi} \cos nx + \frac{1}{n} \sin nx \right)$$

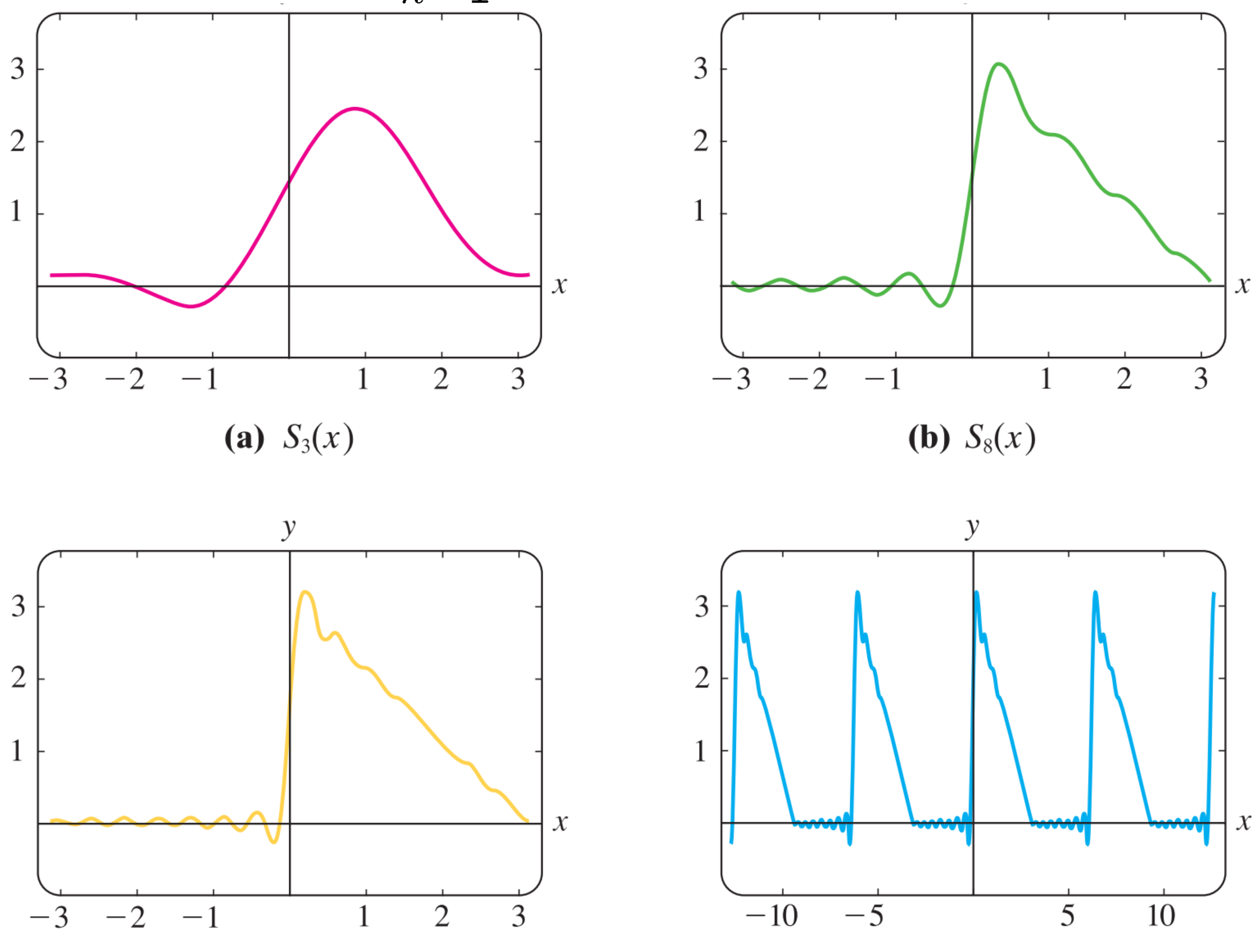
$$S_0(x) = \frac{\pi}{4}$$

$$S_1(x) = \frac{\pi}{4} + \left(\frac{2}{\pi} \cdot \cos x + \sin x \right)$$

$$S_2(x) = \frac{\pi}{4} + \left(\frac{2}{\pi} \cdot \cos x + \sin x \right) + \left(\frac{1}{2} \sin 2x \right)$$

11.2: Sequence of Partial Sums

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cdot \cos \frac{n\pi}{p} x + b_n \cdot \sin \frac{n\pi}{p} x \right)$$



(a) $S_3(x)$ **(b)** $S_8(x)$

(c) $S_{15}(x)$ **(d)** $S_{15}(x)$

- Fourier Series $\quad f(x)$ on $I = [-p, p]$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cdot \cos \frac{n\pi}{p} x + b_n \cdot \sin \frac{n\pi}{p} x \right)$$

$$a_0 = \frac{1}{p} \int_{-p}^{p} f(x)\, dx$$

$$a_n = \frac{1}{p} \int_{-p}^{p} f(x) \cdot \cos \frac{n\pi}{p} x \, dx$$

$$b_n = \frac{1}{p} \int_{-p}^{p} f(x) \cdot \sin \frac{n\pi}{p} x \, dx$$

- Sequence of Partial Sums

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cdot \cos \frac{n\pi}{p} x + b_n \cdot \sin \frac{n\pi}{p} x \right)$$

$$S_0(x) = \frac{a_0}{2}$$